Homework: Linear Transformations from Geometry, Part I

Linh Tran

March 15, 2017

To find a single matrix which can rotate a vector by $150°$ and then reflect the result over the line $2x + 3y = 0$, we need to first find a matrix which can perform those transformations separately, then multiply them to find the combined transformation matrix.

For the rotation, we can use the standard 2D rotation matrix:

$$M_{rot} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

And for the reflection matrix, we can use the following matrix, where m is the slope of the line of reflection:

$$M_{ref} = \frac{1}{1+m^2} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$ (derived below)

To obtain a single matrix matrix which can do both of the above transformations, we multiply the two matrices together, but the order matters. The following would carry out the transformations in the desired order (rotate, then reflect):

$$\vec{u} = M_{ref} M_{rot} \vec{v}$$

So the combined transformation matrix would just be the product of the reflection and rotation matrices:

$$M_{combined} = M_{ref} M_{rot}$$

1 Derivation of the reflection matrix

To derive the above reflection matrix, we need to look at an arbitrary 2-vector $\vec{v}$ and an arbitrary line with slope m as the line of reflection.

$$\vec{l} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1}$$

the unit vector in the direction of the line of reflection

$$\vec{p} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{2}$$

the unit vector perpendicular to the line of reflection

$$\vec{v} = a\vec{l} + b\vec{p} \tag{3}$$

$\vec{v}$ expressed as a linear combination of $\vec{l}$ and $\vec{p}$

$$\vec{u} = a\vec{l} - b\vec{p} \tag{4}$$

We can then determine the reflected vector by inverting the sign of b

$$\vec{v} = a \begin{pmatrix} 1 \\ m \end{pmatrix} + b \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{5}$$

$$\vec{v} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{6}$$

$$\vec{u} = a \begin{pmatrix} 1 \\ m \end{pmatrix} - b \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{7}$$

$$\vec{u} = \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{8}$$

$$M_{ref} \vec{v} = \vec{u} \tag{9}$$

M_{ref} is the reflection matrix we wish to find

$$M_{ref}\begin{pmatrix}1 & -m\\ m & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}1 & m\\ m & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} \tag{10}$$

$$M_{ref}\begin{pmatrix}1 & -m\\ m & 1\end{pmatrix} = \begin{pmatrix}1 & m\\ m & -1\end{pmatrix} \tag{11}$$

$$M_{ref} = \begin{pmatrix}1 & m\\ m & -1\end{pmatrix}\begin{pmatrix}1 & -m\\ m & 1\end{pmatrix}^{-1} \tag{12}$$

$$M_{ref} = \begin{pmatrix}1 & m\\ m & -1\end{pmatrix}\frac{1}{1+m^2}\begin{pmatrix}1 & m\\ -m & 1\end{pmatrix} \tag{13}$$

$$M_{ref} = \frac{1}{1+m^2}\begin{pmatrix}1-m^2 & 2m\\ 2m & m^2-1\end{pmatrix} \tag{14}$$