Integral Calculus

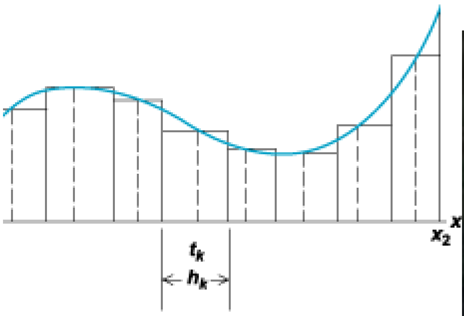
& Antidifferentiation

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

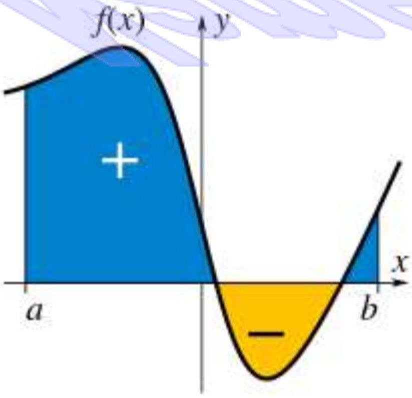
Integration Formulas:

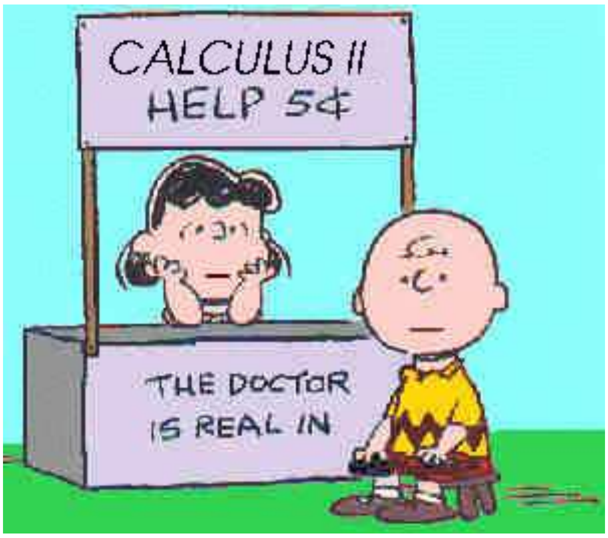
1. $\int 1\,dx = x + C$
2. $\int a\,dx = ax + C$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x(\tan x)\,dx = \sec x + C$
9. $\int \csc x(\cot x)\,dx = -\csc x + C$
10. $\int \frac{1}{x}dx = \ln|x| + C$
11. $\int e^x\,dx = e^x + C$
12. $\int a^x\,dx = \frac{a^x}{\ln a} + C\ a > 0,\ a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1} x + C$



1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b c\,dx = c(b-a)$, where c is any constant
4. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant
6. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
7. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

Name:



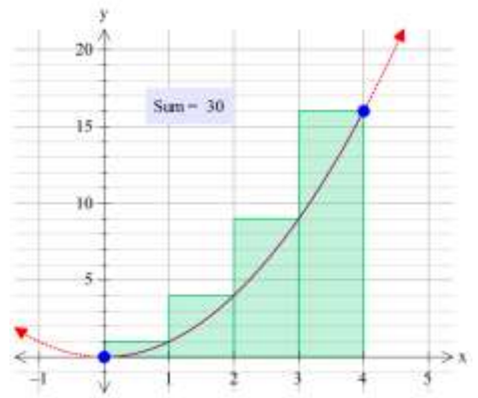
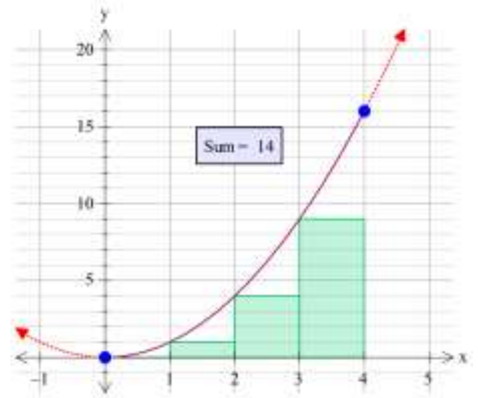


Student Notes for Integration

Estimating The Area Under a Curve

(You need a computer that has GeoGebra loaded or a Ti-nSpire CX CAS calculator)

1. Open up the files from the S: drive (**S:\Maths\Yr 12 Methods\7. Integration**)
   - **a. Area by rectangles.ggb**
   - **b. upper_lower_rectangles.ggb**
   - **c. OR Area Approximations (RAM, Trap, Simp).tns**
2. Using the file, **upper_lower_rectangles.ggb (or Ti file)**, type in the box for the rule, x^2 (type: x ^ 2)
   - **a.** Make the lower bound 0 (use the scroll bar up the top left).
   - **b.** Make the upper bound 4.
   - **c.** Make the number of divisions, n, 4 (this will make rectangles of width 1 unit).
   - **d.** Click on the Green checkbox (Show Upper Rectangles). Upper rectangles are made by using the upper (higher) *y* value for each division.
   - **e.** Draw this on the graph.

Rule	Upper Rectangles	Lower Rectangles
$y = x^2$		

   - **f.** Calculate the area of each rectangle (A = L x W).

$y = x^2$

x	*y*
0	0
1	1
2	4
3	9
4	16
5	25

1. A = 1 x 1 = 1
2. A = 4 x 1 = 4
3. A = 9 x 1 = 9
4. A = 16 x 1 = 16

   - **g.** What is the total area? 30 sq. units
   - **h.** Is this greater or less than the actual area under the curve from 0 to 4? greater
   - **i.** Using another colour and the applet repeat for lower rectangles. (Lower rectangles are made by using the lower *y* value for each division

A = 0 + 1 + 4 + 9 = 14 sq. units

   - **j.** The actual area will be between these two answers. Calculate it by using calculus.

$$\int_0^4 x^2 dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{4^3}{3} - 0 = \frac{64}{3} = 21\frac{1}{3}$$

k. Now change the value of ***n*** and complete the table:

n	**Width**	**Upper Rectangles**	**Lower Rectangles**	**Actual Area**
4	1	30	14	$21\frac{1}{3}$
8	0.5	25.5	17.5	$21\frac{1}{3}$
20	0.2	22.96	19.76	$21\frac{1}{3}$
40	0.1	22.14	20.54	$21\frac{1}{3}$
100	0.04	21.65	21.01	$21\frac{1}{3}$

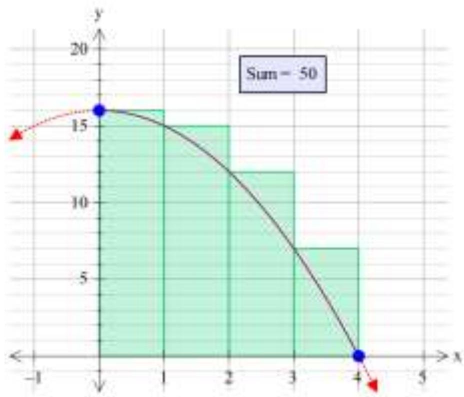
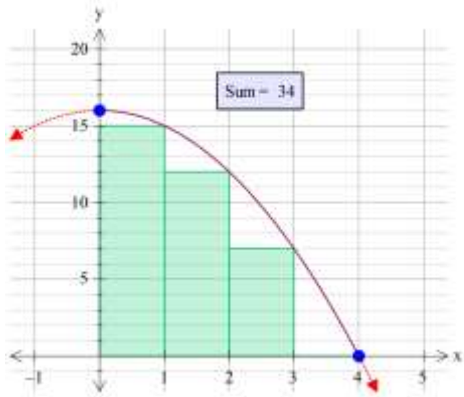
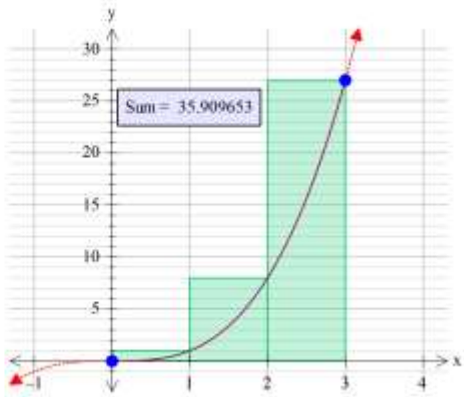
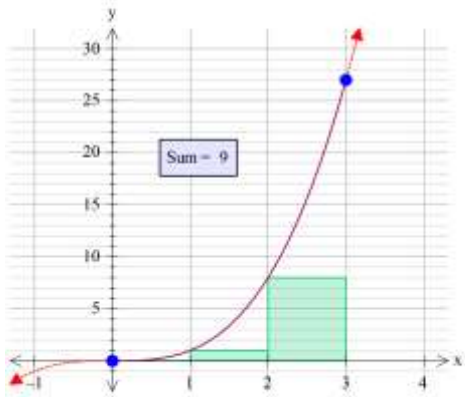
l. What do you notices as the number of divisions increases?

As *n* increases the approximations getting closer and closer to the actual area.

3. Repeat the process for the following functions:

a. $f(x) = 16 - x^2$

b. $f(x) = x^3$; $0 \le x \le 3$ (use 3 divisions initially, then 6, 15, 30, 100)

<table>
<tr><td>Rule</td><td>Upper Rectangles</td><td>Lower Rectangles</td></tr>
<tr><td>$y = 16 - x^2$</td><td>
</td><td>
</td></tr>
<tr><td>$y = x^3$</td><td>
</td><td>
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In Summary:

The area under a curve can be estimated by using _Upper_ and _Lower_ rectangles. As the _number_ of the rectangles _increases_ the approximation is _better_.

4. Using the file, **Areas by rectangles.ggb** , type in the rule, $f(x) = 25 - x^2$ (type: 25-x ^ 2)

a. Make the lower bound 0 and the upper bound 5 and divisions 5.

b. Complete:

n	width	Upper	Lower	Left	Right	Actual
5	1	95	70	95	70	83⅓
10	0.5	89.38	76.88	89.38	76.88	83⅓
20	0.25	86.41	80.16	86.41	80.16	83⅓
50	0.1	84.58	82.08	84.58	82.08	83⅓
100	0.05	83.96	82.71	83.96	82.71	83⅓

➢ Type in the rule, $f(x) = 5x - x^2$ (type: 5x - x ^ 2)

a. Make the lower bound 0 and the upper bound 5 and divisions 5.

b. Complete:

n	width	Upper	Lower	Left	Right	Actual
5	1	26.25	14	20	20	20.83
10	0.5	23.75	17.5	20.63	20.63	20.83
20	0.25	22.34	19.22	20.78	20.78	20.83
50	0.1	21.45	20.2	20.83	20.83	20.83
100	0.05	21.14	20.52	20.83	20.83	20.83

c. Are the Upper Rectangles always equal to the Right rectangles? No

d. Explain the difference between Upper, Lower, Left and Right rectangles.

5. Using the file, **Areas by rectangles.ggb** , type in the rule, $f(x) = e^{0.5x}$ (type: e ^ (0.5x))

a. Make the lower bound 0 and the upper bound 5 and divisions 5.

b. Complete:

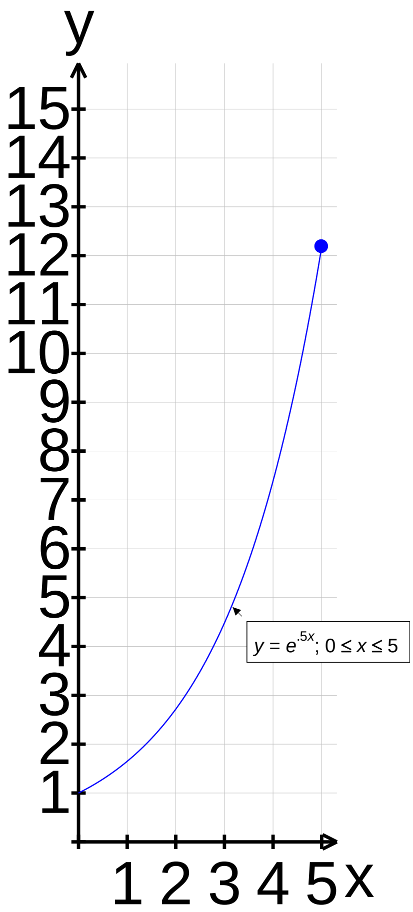
n	width	Upper	Lower	Left	Right	Actual
5	1	28.42	17.42	17.42	28.42	22.36
10	0.5	25.28	19.69	19.69	25.28	22.36
20	0.25	23.79	21	21	23.79	22.36
50	0.1	22.93	21.81	21.81	22.93	22.36
100	0.05	22.65	22.09	22.09	22.65	22.36

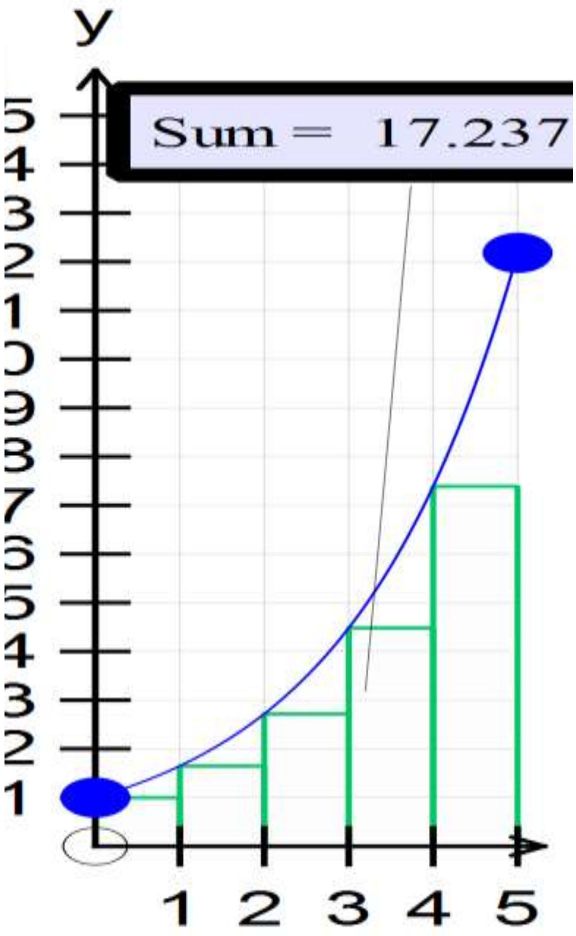
c. What is the area enclosed by $y = e^{0.5x}$, the x-axis and the lines $x = 0$ and $x = 5$?(width 1 unit)

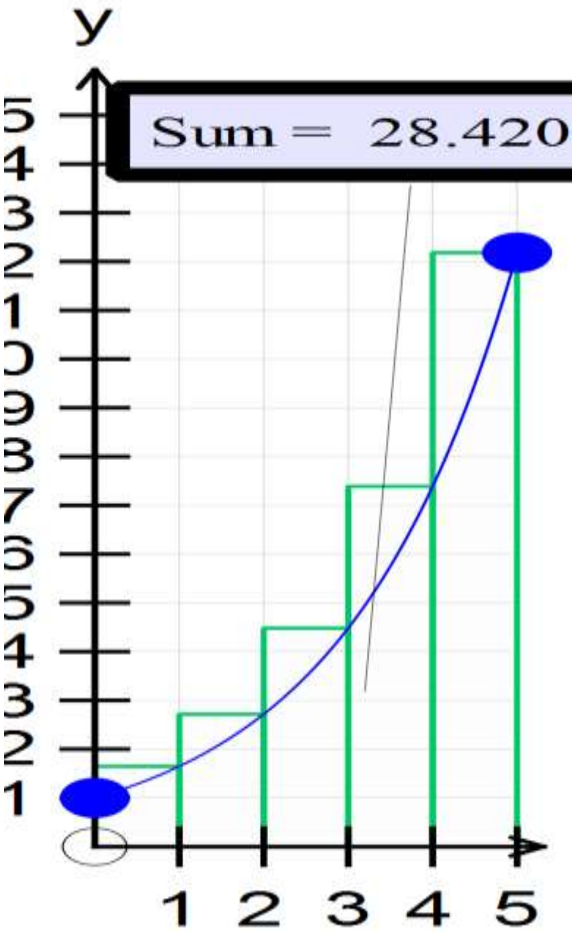
i. Counting Squares; (_20.5_ sq. unit)
ii. Left – endpoint rectangles; (_17.42_ sq. unit)
iii. Right – endpoint rectangles;(_28.42_ sq. unit)
iv. Calculus (_ 22.36_ sq. unit).

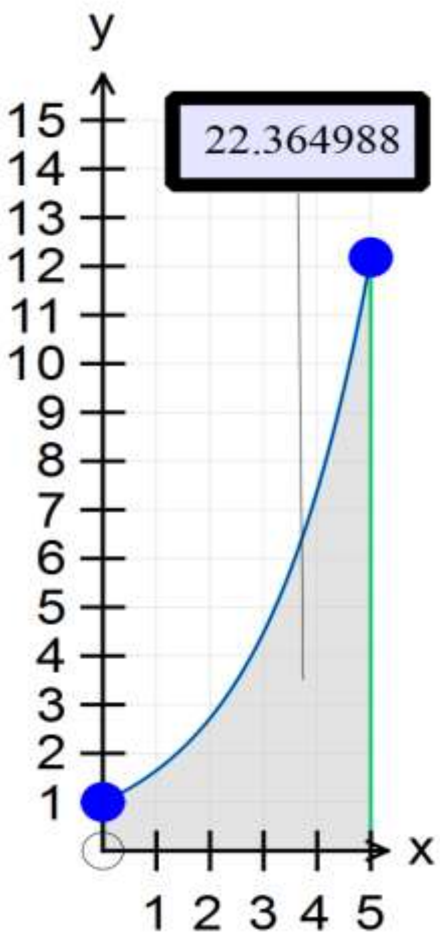
These can be drawn on the next page of your notes booklet.

- **Ex11A** 1, 2, 3, 6, 8


y
x
y = e^.5x; 0 ≤ x ≤ 5


y
Sum = 17.237


y
Sum = 28.420


y
x
22.364988

Antidiffentiation

Terminology

The indefinite integral: (or the anti-derivative) is of the form: F(x) + c where F(x) = $\int f(x)dx$.

An anti-derivative is when the value of "c" is known.

Example: **the** anti-derivative of $3x^2 - 5x + 2$ is $x^3 - \frac{5x^2}{2} + 2x + c$, however

An anti-derivative is $x^3 - \frac{5x^2}{2} + 2x + 11.2$ or $x^3 - \frac{5x^2}{2} + 2x - 12$ etc..

Existing Rule: $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$

New Rule: $\int (ax+b)^n dx) = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

Examples: Find the anti-derivative of:

(a) $\frac{dy}{dx} = x^2(3x+4)$ (b) $f'(x) = (2x+5)^3$ (c) $\int \frac{1}{(2x-5)^3} dx$

(d) $\int \sqrt{(5x+3)} dx$

Solutions:

(a)
$$\frac{dy}{dx} = x^2(3x+4)$$
$$\frac{dy}{dx} = 3x^3 + 4x^2$$
$$y = \int \frac{dy}{dx} dx$$
$$y = \int (3x^3 + 4x^2) dx$$
$$y = \frac{3x^4}{4} + \frac{4x^3}{3} + c$$

(b)
$$f'(x) = (2x+5)^3$$
$$f(x) = \int (2x+5)^3 dx$$
$$f(x) = \frac{(2x+5)^4}{2 \times 4} + c$$
$$f(x) = \frac{(2x+5)^4}{8} + c$$

(c)
$$\int \frac{1}{(2x-5)^3} dx$$
$$= \int (2x-5)^{-3} dx$$
$$= \frac{(2x-5)^{-2}}{2 \times -2} + c$$
$$= \frac{(2x-5)^{-2}}{-4} + c$$
$$= \frac{-1}{4(2x-5)^2} + c$$

(d)
$$\int \sqrt{(5x+3)} dx$$
$$= \int (5x+3)^{\frac{1}{2}} dx$$
$$= \frac{(5x+3)^{\frac{3}{2}}}{5 \times \frac{3}{2}} + c$$
$$= \frac{(5x+3)^{\frac{3}{2}}}{\frac{15}{2}} + c$$
$$= \frac{2(5x+3)^{\frac{3}{2}}}{15} + c$$

Ex11B 1, 2, 3, 5; **Ex11C** 1

Finding the original function from the gradient function

Example: At all points on a certain curve $\frac{dy}{dx} = 4x - 6$. The curve passes through the point (2, 4). Find the equation of the curve.

Soultion:

$$y = \int \frac{dy}{dx}\, dx$$

$$y = \int 4x - 6\, dx$$

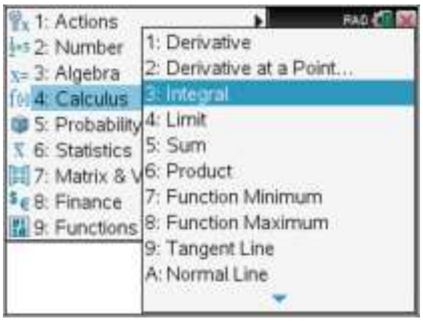
$$y = \frac{4x^2}{2} - 6x + c = 2x^2 - 6x + c$$

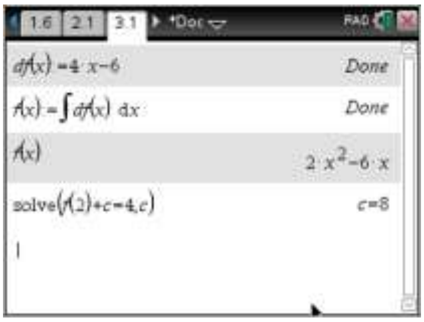
$$(2,4) \therefore \quad 4 = 2(2)^2 - 6(2) + c$$

$$4 = 8 - 12 + c$$

$$8 = c$$

$$\Rightarrow y = 2x^2 - 6x + 8$$





- **Ex11B** 4, 6, 7, 8

Evaluating the Definite Integral

$$\int_a^b f(x)\,dx$$
$$= \left[F(x)\right]_a^b \quad \text{where } F(x) \text{ is the anti-derivative of } f(x)$$
$$= F(b) - F(a) \quad \text{This is known as the FUNDAMENTAL THEOREM OF CALCULUS}$$

Example: Evaluate $\int_{-1}^{2}(-2x-4)\,dx$

Solution:

$$\int_{-1}^{2}(-2x-4)\,dx = \left[-x^2-4x\right]_{-1}^{2}$$
$$=\left(-(2)^2-4(2)\right)-\left(-(-1)^2-4(-1)\right)$$
$$=(-4-8)-(-1+4)$$
$$=-12-3$$
$$=-15$$

Properties of the definite integral

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = \int_c^b f(x)dx + \int_a^c f(x)dx,\ a < c < b$
3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Note that this is a definition.)

Examples:

(1) Given that $\int_1^4 f(x)dx = 10$, find:

a. $\int_1^4 3f(x)dx$ b. $\int_1^4 (1-f(x)dx)$ c. $\int_1^4 \left(\frac{1}{2}f(x)+x\right)dx$

Solutions:

a. $\int_1^4 3f(x)dx = 3\int_1^4 f(x)dx = 3\times 10 = 30$

b. $\int_1^4 (1-f(x)dx) = \int_1^4 1\,dx - \int_1^4 f(x)\,dx = [x]_1^4 - 10 = 4-1-10 = -7$

c. $\int_1^4 \left(\frac{1}{2}f(x)+x\right)dx = \frac{1}{2}\int_1^4 f(x)\,dx + \int_1^4 x\,dx = \frac{1}{2}\times 10 + \left[\frac{x^2}{2}\right]_1^4 = 5 + \left[\frac{16}{2}\right] - \left[\frac{1}{2}\right] = 5+8-\frac{1}{2} = 12\frac{1}{2}$

(2) If $\int_a^b g(x)\,dx = k$ then find:

a. $\int_b^a g(x)\,dx$ b. $\int_a^b \left(\frac{1}{k}g(x)-1\right)dx$ c. $\int_b^a (2-g(x))dx$

Solutions:

a. $\boxed{\int_b^a g(x)\,dx = -\int_a^b g(x)\,dx = -k}$

b. $\boxed{\int_a^b \left(\frac{1}{k}g(x)-1\right)dx = \frac{1}{k}\int_a^b g(x)\,dx - \int_a^b 1\,dx = \frac{1}{k}\times k - [x]_a^b = 1-(b-a) = 1-b+a}$

c. $\boxed{\int_b^a (2-g(x))dx = \int_b^a 2\,dx - \int_b^a g(x)\,dx = [2x]_b^a - (-k) = 2a-2b+k}$

(3) If $\int_1^k (k+x)dx = 32, \quad k>1$ find k.

Solution:

$$\int_1^k (k+x)dx = 32$$

$$\left[kx+\frac{x^2}{2}\right]_1^k = 32$$

$$\left(k^2+\frac{k^2}{2}\right)-\left(k+\frac{1}{2}\right) = 32$$

$$\frac{3k^2}{2} - k + \frac{1}{2} = 32$$

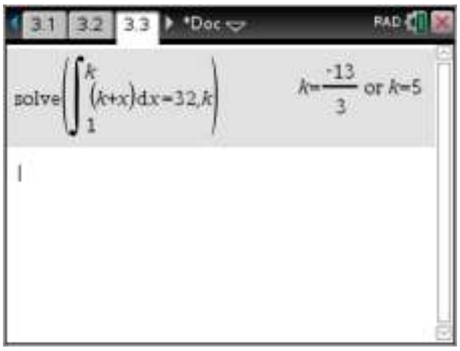
$$3k^2 - 2k + 1 = 64$$

$$3k^2 - 2k - 65 = 0$$

$$(3k+13)(k-5) = 0$$

$$k = \frac{-13}{5} \quad or \quad k = 5$$

$$reject \quad as\, k>1$$



- **Ex11E** 1, 2; **Ex11E** 4

Integration of e^{kx}

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + c \quad or \quad \frac{e^{kx}}{k}, \quad k \neq 0$$

Example: Find the general anti-derivative of each of the following:

(a) e^{4x} (b) $e^{5x} + 6x$ (c) $e^{-x} + e^{x}$

Solutions:

Solutions:

(a) $$\int e^{4x}\, dx = \frac{1}{4} e^{4x} + c \quad or \quad \frac{e^{4x}}{4} + c$$

(b) $$\int e^{5x} + 6x\, dx = \frac{1}{5} e^{5x} + \frac{6x^2}{2} + c = \frac{e^{5x}}{5} + 3x^2 + c$$

(c) $$\int \left(e^{-x} + e^{x}\right) dx = -e^{-x} + e^{x} + c$$

- **Ex11D** 1, 2, 3, 4, 5; **Ex11E** 3

Integration of $\dfrac{1}{(ax+b)}$ **or** $(ax+b)^{-1}$

$$\int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\log_e|ax+b|+c$$

Examples:

(a) Find the anti-derivative of $\dfrac{2}{3x-2}$

(b) Given $\dfrac{dy}{dx}=\dfrac{3}{x}$ and $y=10$ when $x=1$, find an expression for y in terms of x.

Solution:

(a)

$$\int \frac{2}{3x-2}\,dx$$
$$= 2\int \frac{1}{3x-2}\,dx$$
$$= 2\times\frac{1}{3}\log_e|3x-2|+c$$
$$= \frac{2\log_e|3x-2|}{3}+c$$

(b)

$$y=\int \frac{3}{x}\,dx = 3\int \frac{1}{x}\,dx = 3\log_e|x|+c$$
$$(1,10)\Rightarrow 10 = 3\log_e|1|+c$$
$$10 = 0+c$$
$$10 = c$$
$$\therefore y = 3\log_e|x|+10$$

Examples:

(a) $\displaystyle\int \frac{x^2-7}{x}\,dx$

Solution:

$$\int \frac{x^2-7}{x}\,dx = \int\left(x-\frac{7}{x}\right)dx = \frac{x^2}{2}-7\log_e|x|+c$$

(b) Given $\displaystyle\int_1^k \frac{1}{1+2x}\,dx = \log_e 3$, find k, if $k>1$.

Solution:

$$\left[\frac{\log_e|1+2x|}{2}\right]_1^k = \log_e 3$$
$$\frac{1}{2}\left(\log_e|1+2k| - \log_e|3|\right) = \log_e 3$$
$$\log_e|1+2k| - \log_e|3| = 2\log_e 3$$
$$\log_e|1+2k| = \log_e 9 + \log_e 3$$
$$\log_e|1+2k| = \log_e 27$$
$$\therefore 1+2k = 27$$
$$2k = 26$$
$$k = 13$$

Remember:

$$\frac{x}{x+1} = \frac{1(x+1)-1}{x+1} = 1-\frac{1}{x+1}$$

or

```
-1| 1    0
  |↓  __-1____
    1   -1
```

- **Ex11C** 2, 3, 4, 5, 6, 7, 8; **Ex11E** 5

Integration of Circular Functions

$$\int \sin(nx+b)\,dx = -\frac{1}{n}\cos(nx+b)+c$$
$$\int \cos(nx+b)\,dx = \frac{1}{n}\sin(nx+b)+c$$

Examples: find the anti-derivatives of:

(a) $\sin\left(3x+\frac{\pi}{4}\right)$ (b) $3\cos\left(2x-\frac{\pi}{6}\right)$

Solutions:

- **Solutions:**
-
- (a) $\int \sin\left(3x+\frac{\pi}{4}\right)dx$ $= -\frac{1}{3}\cos\left(3x+\frac{\pi}{4}\right)+c$ $= \frac{-\cos\left(3x+\frac{\pi}{4}\right)}{3}+c$

  (b) $\int 3\cos\left(2x-\frac{\pi}{6}\right)dx$ $= \frac{3}{2}\sin\left(2x-\frac{\pi}{6}\right)+c$

- **Example:** Find the exact value of each of the following:
- (a) $\int_0^{\frac{\pi}{4}} \sin 2x\,dx$ (b) $\int_0^{\frac{\pi}{2}} (2\cos x+1)dx$
- **Solutions:**
- (a) $\int_0^{\frac{\pi}{4}} \sin 2x\,dx$ $= \left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{-\cos\left(2\left(\frac{\pi}{4}\right)\right)}{2}\right) - \left(\frac{-\cos(2(0))}{2}\right)$ $= \left(\frac{-\cos\left(\frac{\pi}{2}\right)}{2}\right) - \left(\frac{-\cos(0)}{2}\right)$ $= 0 - \frac{-1}{2} = \frac{1}{2}$

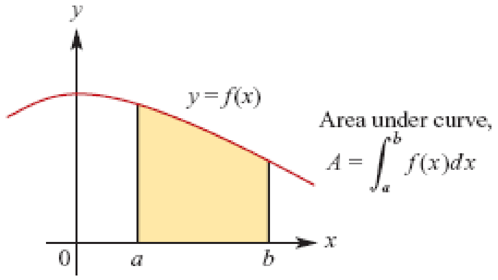
  (b) $\int_0^{\frac{\pi}{2}} (2\cos x+1)dx$ $= [2\sin x + x]_0^{\frac{\pi}{2}}$ $= \left(2\sin\left(\frac{\pi}{2}\right)+\frac{\pi}{2}\right) - (2\sin(0)+0)$ $= \left(2(1)+\frac{\pi}{2}\right) - 0$ $= 2+\frac{\pi}{2}$

- **Ex11G** 1, 2, 5

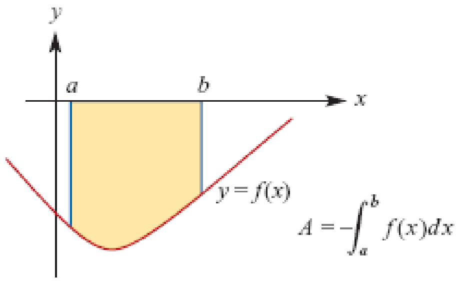
Area bounded by a curve

Finding the area of a region

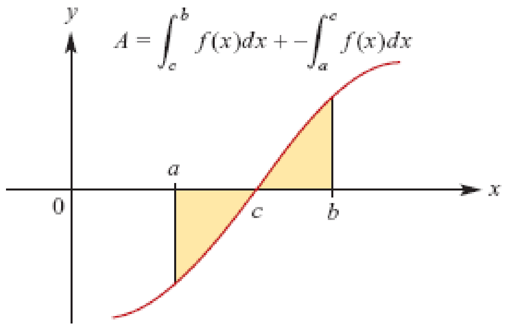
- If $f(x) \geq 0$ for all $x \in [a, b]$, the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)dx = G(b) - G(a)$ where G is an antiderivative of f.



- If $f(x) \leq 0$ for all $x \in [a, b]$, the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $A = -\int_a^b f(x)dx$



- If $c \in (a, b)$, $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area A of the shaded region is given by $A = \int_c^b f(x)dx + -\int_a^c f(x)dx$



Note: In determining the areas 'under' curves, the sign of $f(x)$ in the given interval is the critical factor.

Examples:

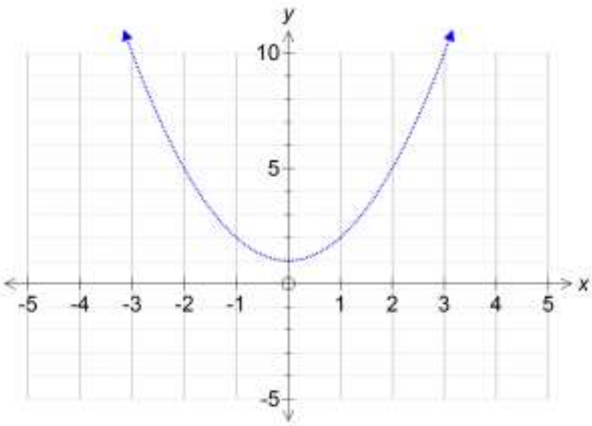
(1) Find the area between the curve $f(x) = x^2 + 1$, the x-axis and the ordinates $x = -2$ and $x = 3$.

Solution:

1. First sketch the graph and shade the area required.
2. Write the integral:

$$Area = \int_{-2}^{3} (x^2 + 1)dx$$

3. Solve:



$$Area = \int_{-2}^{3} (x^2 + 1) dx$$

$$Area = \left[\frac{x^3}{3} + x \right]_{-2}^{3}$$

$$Area = \left(\frac{(3)^3}{3} + (3) \right) - \left(\frac{(-2)^3}{3} + (-2) \right)$$

$$Area = (9+3) - \left(\frac{-8}{3} - 2 \right)$$

$$Area = 12 - \left(\frac{-14}{3} \right)$$

$$Area = 16\frac{2}{3} \text{ square units}$$

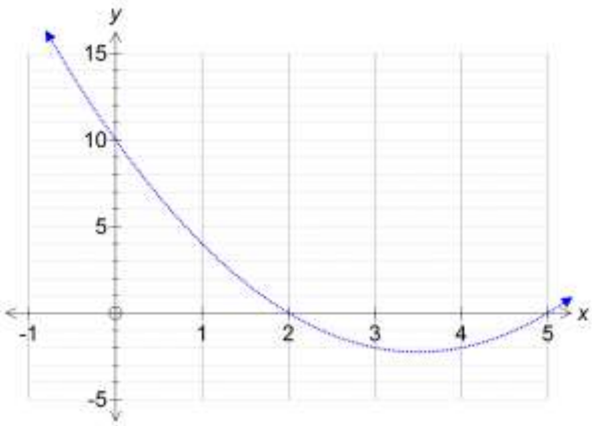
(2) Find the area between the curve $y = x^2 - 7x + 10$, the x-axis and the ordinates $x = 3$ and $x = 4$.

$$Area = -\int_{3}^{4} (x^2 - 7x + 10) dx = \int_{4}^{3} (x^2 - 7x + 10) dx$$

$$Area = \left[\frac{x^3}{3} - \frac{7x^2}{2} + 10x \right]_{4}^{3}$$

$$Area = \left(\frac{27}{3} - \frac{63}{2} + 30 \right) - \left(\frac{64}{3} - \frac{112}{2} + 40 \right)$$

$$Area = 2\frac{1}{6} \text{ sq. units}$$



(3) Find the area bounded by the graph of $y = x^3 - x^2 - 2x$ and the x-axis.
Solution:

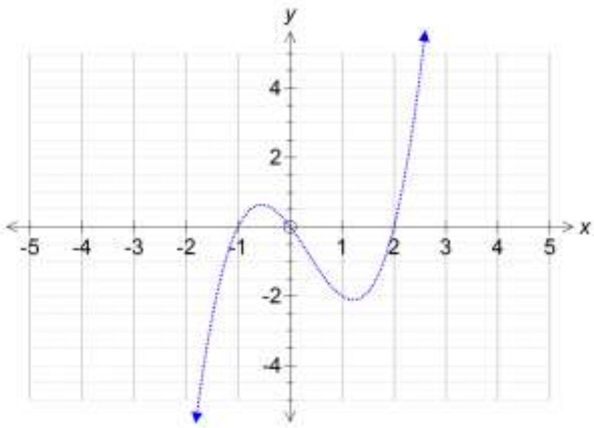
$$Area = \int_{-1}^{0} (x^3 - x^2 - 2x) dx + \int_{2}^{0} (x^3 - x^2 - 2x) dx$$

$$Area = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^{0} + \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{2}^{0}$$

$$Area = \left[(0) - \left(\frac{1}{4} - \frac{1}{3} - 1 \right) \right] + \left[(0) - \left(\frac{16}{4} - \frac{8}{3} - 4 \right) \right]$$

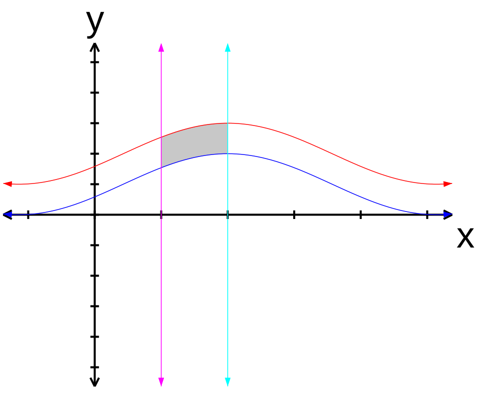
$$Area = \left[0 - \left(-\frac{5}{12} \right) \right] + \left[0 - \left(-\frac{8}{3} \right) \right]$$

$$Area = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} = 3\frac{1}{12} \text{ sq. units}$$

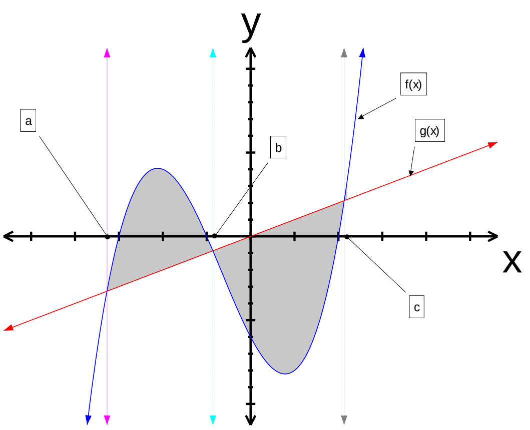


- **Ex11F** 2 acf, 3, 4, 5, 6, 9 10;
- **Ex11G** 3 4ace, 6

The area between two curves



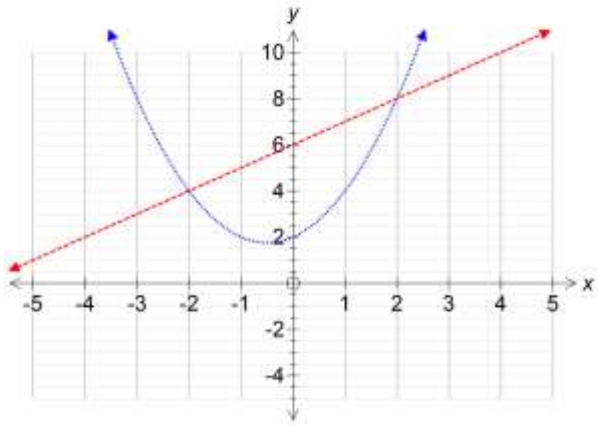
$$Area = \int (Top\ \ function - bottom\ \ function)dx$$



$$Area = \int_a^b (f(x) - g(x))dx + \int_b^c (g(x) - f(x))dx$$

Example 1

Find the area enclosed by the line with equation $y = x + 6$ and the parabola with equation $y = x^2 + x + 2$.



Solution:

1. Sketch.
2. Find points of intersections.

$$x + 6 = x^2 + x + 2$$
$$x^2 - 4 = 0$$
$$x = \pm 2$$

(no need to find Y coordinate)

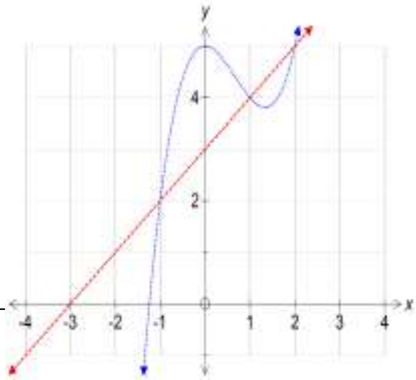
3. Find area.

$$Area = \int_{-2}^{2} (top\ \ function - bottom\ \ function)dx$$
$$Area = \int_{-2}^{2} \left(x + 6 - (x^2 + x + 2)\right)dx$$
$$Area = \int_{-2}^{2} (-x^2 + 4)dx$$
$$Area = \left[\frac{-x^3}{3} + 4x\right]_{-2}^{2}$$
$$Area = \left[\frac{-8}{3} + 8\right] - \left[\frac{8}{3} - 8\right]$$
$$Area = \frac{16}{3} - \frac{-16}{3} = \frac{32}{3} = 10\frac{2}{3}\ sq.units$$

Example 2

Find the area enclosed by the graphs with equations $y = x + 3 \quad and \quad y = x^3 - 2x^2 + 5$

Solution:



$$x^3 - 2x^2 + 5 = x + 3$$
$$x^3 - 2x^2 - x + 2 = 0$$
$$(x+1)(x-1)(x-2) = 0$$
$$x = \pm 1, 2$$

$$Area = \int_{-1}^{1} \left(\left(x^3 - 2x^2 + 5\right) - (x+3)\right)dx + \int_{1}^{2} \left((x+3) - \left(x^3 - 2x^2 + 5\right)\right)dx$$

$$Area = \int_{-1}^{1} \left(x^3 - 2x^2 - x + 2\right)dx + \int_{1}^{2} \left(-x^3 + 2x^2 + x - 2\right)dx$$

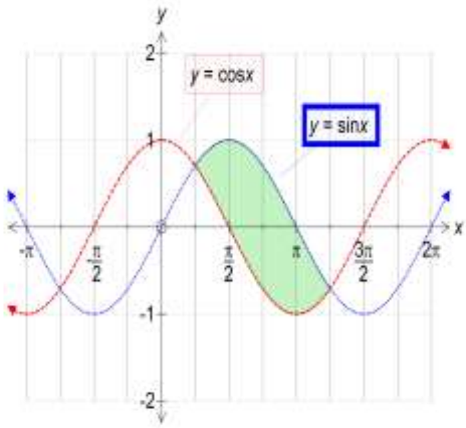
$$Area = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x\right]_{-1}^{1} + \left[-\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} - 2x\right]_{1}^{2}$$

$$Area = \left[\left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) - \left(\frac{1}{4} - \frac{-2}{3} - \frac{1}{2} - 2\right)\right] + \left[\left(-\frac{16}{4} + \frac{16}{3} + \frac{4}{2} - 4\right) - \left(-\frac{1}{4} + \frac{2}{3} + \frac{1}{2} - 2\right)\right]$$

$$Area = \left[\left(\frac{13}{12}\right) - \left(-\frac{19}{12}\right)\right] + \left[\left(\frac{-2}{3}\right) - \left(\frac{-13}{12}\right)\right] = \left[\frac{32}{12}\right] + \left[\frac{5}{12}\right] = \frac{37}{12} = 3\frac{1}{12} \ sq.units$$

Example 3: Find the area of the shaded region of the graph:

Solution: Points of intersection:



$$\sin x = \cos x$$
$$\tan x = 1$$
$$x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$$

$$Area = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx$$

$$Area = \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$Area = \left[-\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right)\right] - \left[-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right]$$

$$Area = \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$Area = \frac{4}{\sqrt{2}} = 2\sqrt{2} \ sq.units$$

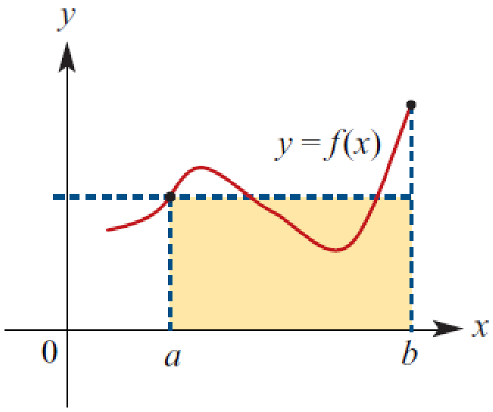
- **Ex11I** 1, 2, 3, 4, 5, 6, 7, 8

Average Value of a function

The average value of a function f with rule $y = f(x)$ for an interval $[a, b]$ is defined as:

$$\text{Average value of a function} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

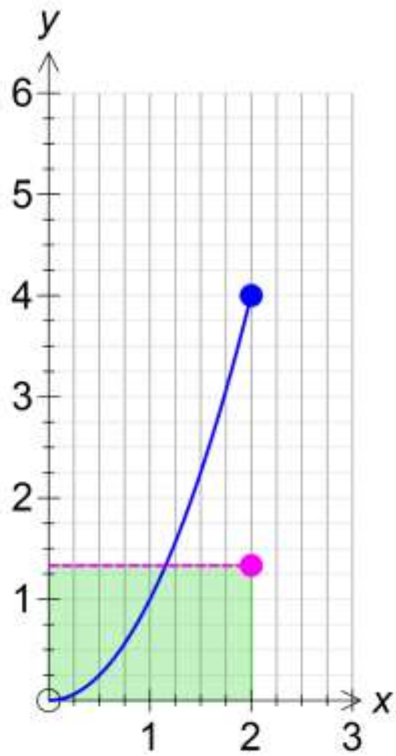
In terms of the graph of $y = f(x)$, the average is the height of a rectangle having the same area as the area under the graph for the interval $[a, b]$.



Example: Find the average value of $f(x) = x^2$ for the interval [0, 2]. Illustrate with a horizontal line determined by this value.

Solution:

$$\begin{aligned} Average &= \frac{1}{2-0}\int_0^2 x^2\,dx \\ &= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 \\ &= \frac{1}{2}\times\frac{8}{3} \\ &= \frac{4}{3} \end{aligned}$$



Ex11J 1, 2, 3, 4, 5, 6

Motion & Rates

Motion

If we are given a rule for acceleration, by antidifferentiation we can obtain the rules for velocity and displacement.

Example 1: A body starts from *O* and moves in a straight line. After *t* seconds ($t \geq 0$) its velocity, *v* m/s, is given by $v = 2t - 4$. Find:

a its position *x* in terms of *t*;
b its position after 3 seconds;
c its average velocity in the first 3 seconds;
d the distance travelled in the first 3 seconds;
e its average speed in the first 3 seconds.

Solution:

a

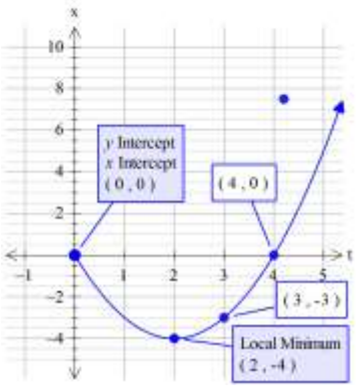
$$x = \int v\,dt = \int 2t - 4\,dt = t^2 - 4t + c$$
$$t = 0, x = 0 \Rightarrow c = 0$$
$$\therefore x = t^2 - 4t$$

b

$$x = t^2 - 4t,\ t = 3$$
$$x = (3)^2 - 4(3) = -3m$$

c

$$t = 0, x = 0$$
$$t = 3, x = -3$$
$$Avg.vel = \frac{-3-0}{3-0} = -1m/s$$

d 

goes down 4 then up 1 = total of 5m.

e $\frac{5}{3} m/s$

Example 2: A particle starts from rest 3 metres from a fixed point and moves in a straight line with an acceleration of $a = 6t + 8$. Find its position and velocity at time *t* seconds.

Solution:

$$v = \int a\,dt = \int 6t + 8\,dt = 3t^2 + 8t + c_1$$
$$at\ rest: t = 0, v = 0 \Rightarrow c_1 = 0$$

$$x = \int v\,dt = \int 3t^2 + 8t\,dt = t^3 + 4t^2 + c_2$$
$$t = 0, x = 3 \Rightarrow c_2 = 3$$
$$\therefore x = t^3 + 4t^2 + 3$$

Rates of Change

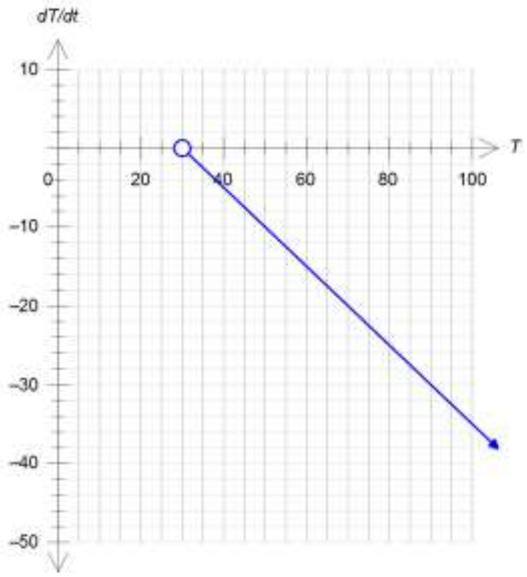
Example: The rate of change of temperature with respect to time of a liquid which has been boiled and then allowed to cool is given by $\frac{dT}{dt} = -0.5(T-30)$, where T is the temperature (°C) at time t (minutes).

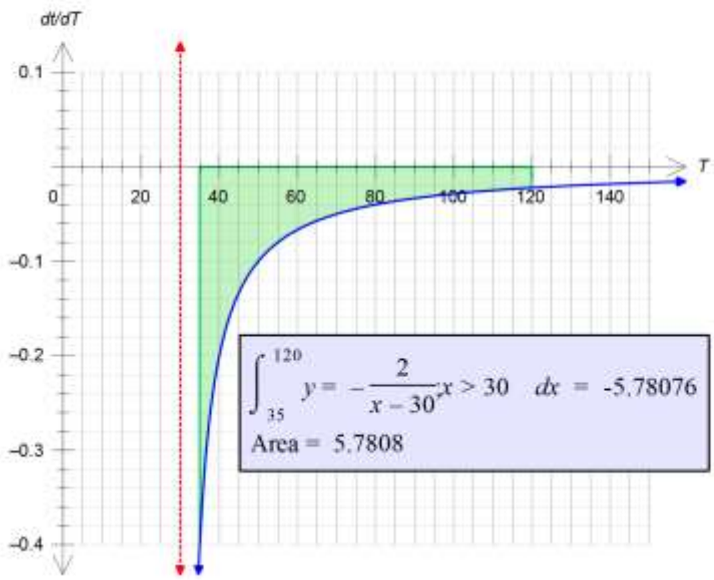
a Sketch the graph of $\frac{dT}{dt}$ against T for $T > 30$.

b Sketch the graph of $\frac{dt}{dT}$ against T for $T > 30$.

c **i** Find the area of the region enclosed by the graph of **b**, the x-axis and the lines $T = 35$ and $T = 120$. Give your answer correct to 2 decimal places.
ii What does this area represent?

Solution





$$Area = -\int_{35}^{120} \frac{dt}{dT}\,dT = -\int_{35}^{120} \frac{-2}{T-30}\,dT = -\left[-2\ln(T-30)\right]_{35}^{120} = -(-2\ln(90)+2\ln(5)) = 2\ln(18) \approx 5.78$$

The area represents the time it takes to cool from 120 °C to 35 °C.

- **Ex11J** 7, 8, 12, 13, 15, 18, 19

Find $\frac{d}{dx}(\ldots)$ hence … (or Integration by Recognition)

- If $\frac{d}{dx}\left(x^2\right)=2x$ then $\int 2x\,dx = x^2 + c$

Example 1: Find the derivative of $(x^2+1)^3$ hence find:

a. $\int 6x(x^2+1)^2 dx$; b. $\int x(x^2+1)^2\,dx$.

Solution:

$$\frac{d}{dx}\left((x^2+1)^3\right)=3(x^2+1)^2(2x)=6x(x^2+1)^2$$

a. If $\frac{d}{dx}\left((x^2+1)^3\right)=6x(x^2+1)^2$ **then** $\int 6x(x^2+1)^2\,dx=\left(x^2+1\right)^3+c$

b. $\int x(x^2+1)^2\,dx=\frac{1}{6}\int 6x(x^2+1)^2\,dx=\frac{1}{6}(x^2+1)^3+c$

Example 2: Find the derivative of $y=\log_e(8x^2-1)$ hence find $\int \frac{x}{8x^2-1}dx$.

Solution:

$$y=\log_e(8x^2-1)$$

$$\frac{dy}{dx}=\frac{16x}{8x^2-1}$$

$$\therefore \int \frac{x}{8x^2-1}dx=\frac{1}{16}\int \frac{16x}{8x^2-1}dx=\frac{1}{16}\log_e(8x^2-1)+c$$

Example 3: find the derivative of $y=x\log_e x-1$, hence find $\int_1^e 2\log_e x\,dx$.

Solution:

$$y=x\log_e x-1$$

$$\frac{dy}{dx}=(1)\log_e x+x\times\frac{1}{x}-0=\log_e x+1$$

$$\therefore \int\left(\log_e x+1\right)dx = x\log_e x-1$$

$$\Rightarrow \frac{1}{2}\int 2(\log_e x+1)dx = x\log_e x-1$$

$$\int (2\log_e x+2)dx=2(x\log_e x-1)$$

$$\int (2\log_e x)dx+\int 2\,dx = 2(x\log_e x-1)$$

$$\int_1^e 2\log_e x\,dx+\int_1^e 2\,dx=\left[2(x\log_e x-1)\right]_1^e$$

$$\int_1^e 2\log_e x\,dx=\left[2(x\log_e x-1)\right]_1^e-\int_1^e 2\,dx$$

$$\int_1^e 2\log_e x\,dx=\left[2(x\log_e x-1)\right]_1^e-\left[2x\right]_1^e$$

$$\int_1^e 2\log_e x\,dx=\left[2(e\log_e e-1)-2(\log_e 1-1)\right]-\left[2e-2\right]$$

$$\int_1^e 2\log_e x\,dx=\left[2(e-1)-2(-1)\right]-\left[2e-2\right]$$

$$\int_1^e 2\log_e x\,dx=2e-2+2-2e+2=2$$

Example 4: Find $\frac{d}{dx}(x\sin 3x)$ and hence evaluate $\int_0^{\frac{\pi}{6}} x\cos 3x \, dx$.

Solution:

$$\frac{d}{dx}(x\sin 3x) = \sin 3x + 3x\cos 3x \quad \Rightarrow \int (\sin 3x + 3x\cos 3x)dx = x\sin 3x + c$$

$$\int \sin 3x \, dx + 3\int x\cos 3x \, dx = x\sin 3x$$

$$\int_0^{\frac{\pi}{6}} \sin 3x + 3\int_0^{\frac{\pi}{6}} x\cos 3x = [x\sin 3x]_0^{\frac{\pi}{6}}$$

$$3\int_0^{\frac{\pi}{6}} x\cos 3x = [x\sin 3x]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin 3x$$

$$3\int_0^{\frac{\pi}{6}} x\cos 3x = [x\sin 3x]_0^{\frac{\pi}{6}} - \left[-\frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{6}}$$

$$3\int_0^{\frac{\pi}{6}} x\cos 3x = \left[\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\right)\right) - 0\right] - \left[-\frac{1}{3}\cos\left(\frac{\pi}{2}\right) - \left(-\frac{1}{3}\cos(0)\right)\right]$$

$$3\int_0^{\frac{\pi}{6}} x\cos 3x = \left[\left(\frac{\pi}{6}\right) - 0\right] - \left[0 - \left(-\frac{1}{3}\right)\right]$$

$$3\int_0^{\frac{\pi}{6}} x\cos 3x = \left[\left(\frac{\pi}{6}\right) - \frac{1}{3}\right] = \frac{\pi}{6} - \frac{1}{3}$$

$$\int_0^{\frac{\pi}{6}} x\cos 3x = \frac{1}{3}\left(\frac{\pi}{6} - \frac{1}{3}\right) = \frac{\pi}{18} - \frac{1}{9} = \frac{\pi - 2}{18}$$

- **Ex11H** 3, 4, 5, 8
- **Worksheet "#1"**
- **Worksheet "#2"**

Worksheet #1

1. Find the derivative of $e^{(x^2+2)}$ and hence find $\int 2xe^{(x^2+2)}dx$.

2. Find the derivative of $\log_e(x^3-5)$ and hence find a) $\int \frac{3x^2}{x^3-5}dx$; b) $\int \frac{6x^2}{x^3-5}dx$.

3. Find the derivative of $\cos(x^2)$ and hence find a) $\int -2x\sin(x^2)dx$; b) $\int x\sin(x^2)dx$.

4. Find the derivative of $\sin^3 x$ and hence find a) $\int 3\sin^2 x\cos x\, dx$; b) $\int \sin^2 x\cos x\, dx$.

Answers

1. $e^{(x^2+2)}+c$ **2.** $\frac{3x^2}{x^3-5}$ $a)\log_e(x^3-5)+c; b)2\log_e(x^3-5)+c$

3. $-2x\sin(x^2); a)\cos(x^2)+c; b)\frac{-\cos(x^2)}{2}+c$

4. $3\sin^2 x\cos x;\quad a)\sin^3 x+c;\quad b)\frac{\sin^3 x}{3}+c$

Worksheet #2

1. **a)** Find the derivative of $\log_e(2x^2+1)$.

b) Hence find $\int \frac{4x}{2x^2+1}dx$.

2. **a)** Find the derivative of $\log_e(2-3x)$.

b) Hence find (i) $\int \frac{-3}{2-3x}dx$; (ii) $\int \frac{2}{2-3x}dx$.

3. **a)** Show that $\frac{x+3}{x-1}=1+\frac{4}{x-1}$.

b) Hence find $\int \frac{x+3}{x-1}dx$.

4. Find the derivative of $(x^3+1)^5$ and hence find $\int 15x^2(x^3+1)^4 dx$.

5. Find the derivative of $(x^2+2)^4$. Hence find (i) $\int 2x(x^2+2)^3 dx$; (ii) $\int 5x(x^2+2)^3 dx$

Answers

1. a) $\frac{4x}{2x^2+1}$; b) $\log_e(2x^2+1)+c$

2. a) $\frac{-3}{2-3x}$ b) (i) $\log_e(|2-3x|)+c$ (ii) $\frac{-2}{3}\log_e(|2-3x|)+c$

3. b) $x+4\log_e(|x-1|)+c$

4. $15x^2(x^3+1)^4$; $(x^3+1)^5+c$

5. $8x(x^2+2)^3; (i)\frac{(x^2+2)^4}{4}+c; (ii)\frac{5(x^2+2)^4}{8}+c$

Exam Questions - Integration

2008 Exam 1

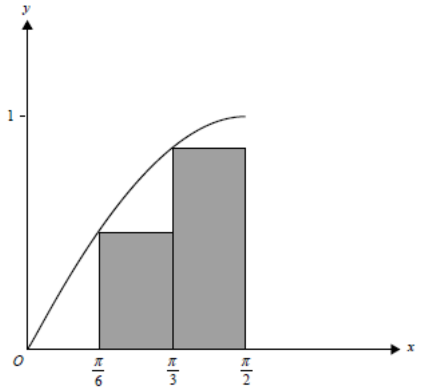
Question 5

The area of the region bounded by the y-axis, the x-axis, the curve $y = e^{2x}$ and the line $x = C$, where C is a positive real constant, is $\frac{5}{2}$. Find C.

3 marks

2008 Exam 2

Question 1



The area under the curve $y = \sin(x)$ between $x = 0$ and $x = \frac{\pi}{2}$ is approximated by two rectangles as shown. This approximation to the area is

A. 1

B. $\frac{\pi}{2}$

C. $\frac{(\sqrt{3}+1)\pi}{12}$

D. 0.5

E. $\frac{(\sqrt{3}+1)\pi}{6}$

Question 3

The average value of the function with rule $f(x) = \log_e(3x + 1)$ over the interval $[0, 2]$ is

A. $\frac{\log_e(7)}{2}$

B. $\log_e(7)$

C. $\frac{7\log_e(7)}{3} - 2$

D. $\frac{7\log_e(7) - 6}{6}$

E. $\frac{35\log_e(7) - 12}{18}$

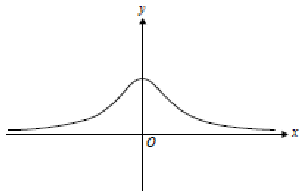
Question 4

If $\int_1^3 f(x)\,dx = 5$, then $\int_1^3 (2f(x) - 3)\,dx$ is equal to

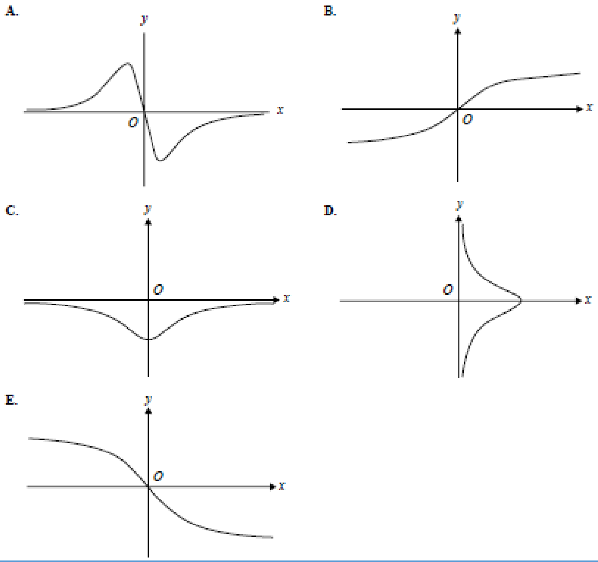
A. 4
B. 5
C. 7
D. 10
E. 16

Question 19

The graph of a function f is shown below.

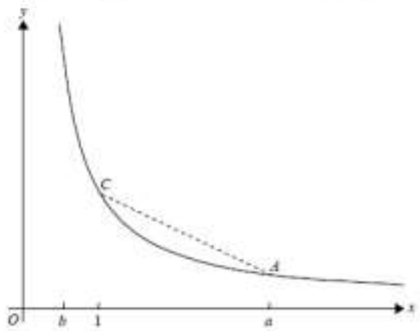


The graph of an **antiderivative** of f could be



Question 2

The diagram below shows part of the graph of the function $f: R^+ \to R$, $f(x) = \frac{7}{x}$.



The line segment CA is drawn from the point $C(1, f(1))$ to the point $A(a, f(a))$ where $a > 1$.

a. i. Calculate the gradient of CA in terms of a.

ii. At what value of x between 1 and a does the tangent to the graph of f have the same gradient as CA?

b. i. Calculate $\int_1^{e} f(x)\,dx$.

ii. Let b be a positive real number less than one. Find the exact value of b such that $\int_b^{1} f(x)\,dx$ is equal to 7.

1 + 2 = 3 marks

c. i. Express the area of the region bounded by the line segment CA, the x-axis, the line $x = 1$ and the line $x = a$ in terms of a.

ii. For what exact value of a does this area equal 7?

iii. Using the value for a determined in c.ii., explain in words, without evaluating the integral, why $\int_1^{a} f(x)\,dx < 7$.

Use this result to explain why $a < e$.

2 + 2 + 1 = 5 marks

d. Find the exact values of m and n such that $\int_1^{m} f(x)\,dx = 3$ and $\int_1^{\frac{n}{2}} f(x)\,dx = 2$.

2 marks

Total 13 marks

2009 Exam 1

Question 2

a. Find an anti-derivative of $\frac{1}{1-2x}$ with respect to x.

2 marks

b. Evaluate $\int_1^{4} \left(\sqrt{x}+1\right)dx$.

3 marks

2009 Exam 2

Question 18

The average value of the function $f: R \setminus \left\{-\frac{1}{2}\right\} \to R,\ f(x) = \frac{1}{2x+1}$ over the interval $[0, k]$ is $\frac{1}{6}\log_e(7)$.

The value of k is

A. $\frac{-6}{\log_e(7)} - \frac{1}{2}$

B. 3

C. e^3

D. $\frac{-\log_e(7)}{2(\log_e(7)+6)}$

E. 171

Question 22

Consider the region bounded by the x-axis, the y-axis, the line with equation $y = 3$ and the curve with equation $y = \log_e(x-1)$.

The exact value of the area of this region is

A. $e^{-3} - 1$

B. $16 + 3\log_e(2)$

C. $3e^3 - e^{-3} + 2$

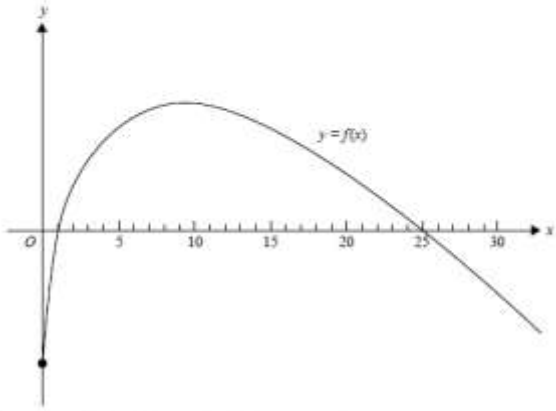
D. $e^3 + 2$

E. $3e^2$

Question 1

Let $f: R^{+} \cup \{0\} \to R,\ f(x) = 6\sqrt{x} - x - 5$.

The graph of $y = f(x)$ is shown below.



a. State the interval for which the graph of f is strictly decreasing.

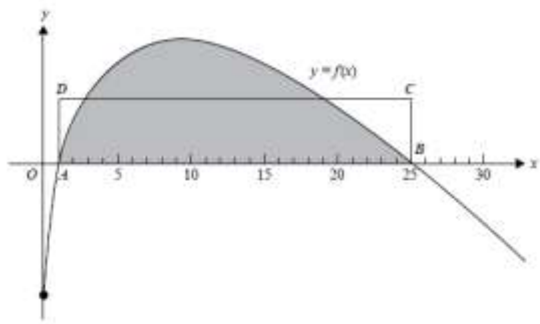
2 marks

b. On the set of axes above, sketch the graph of $y = |f(x)|$.

2 marks

c. Points A and B are the points of intersection of $y = f(x)$ with the x-axis. Point A has coordinates $(1, 0)$ and point B has coordinates $(25, 0)$.

Find the length of AD such that the area of rectangle $ABCD$ is equal to the area of the shaded region.



2010 Exam 1

Question 2

a. Find an antiderivative of $\cos(2x + 1)$ with respect to x.

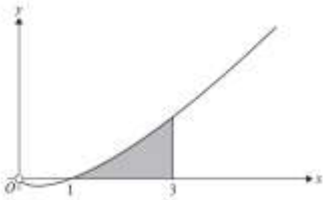
1 m

b. Find p given that $\int_2^3 \frac{1}{1-x}\,dx = \log_e(p)$.

3 ma

Question 9

Part of the graph of $f: R^+ \to R, f(x) = x\log_e(x)$ is shown below.



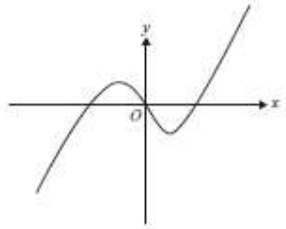
a. Find the derivative of $x^2\log_e(x)$.

1 mark

b. Use your answer to part a. to find the area of the shaded region in the form $a\log_e(b) + c$ where a, b and c are non-zero real constants.

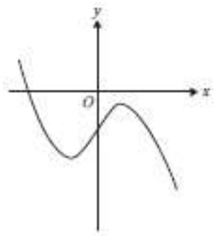
Question 19

The graph of the gradient function $y = f'(x)$ is shown below.

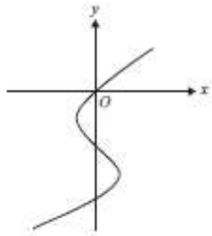


Which of the following could represent the graph of the function $f(x)$?

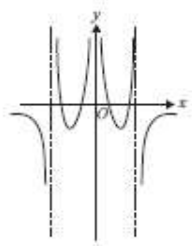
A.



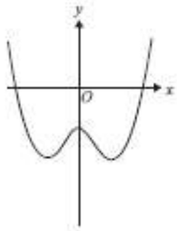
B.



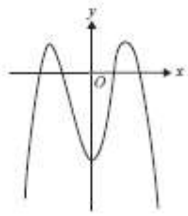
C.



D.



E.



2010 Exam 2

Question 10

The average value of the function $f(x) = e^{2x}\cos(3x)$ for $0 \le x \le \pi$ is closest to

A. −82.5
B. 26.3
C. −26.3
D. −274.7
E. π

Question 20

Let f be a differentiable function defined for all real x, where $f(x) \ge 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)dx = a$, then $2\int_0^{5a}\left[f\left(\frac{x}{5}\right) + 3\right]dx$ is equal to

A. $2a + 6$
B. $10a + 6$
C. $20a$
D. $40a$
E. $50a$

Question 22

Let f be a differentiable function defined for $x > 2$ such that

$$\int_3^{ab+2} f(x)dx = \int_3^{a+2} f(x)dx + \int_3^{b+2} f(x)dx \text{ where } a > 1 \text{ and } b > 1.$$

The rule for $f(x)$ is

A. $\sqrt{x-2}$
B. $\log_e(x-2)$
C. $\sqrt{2x-4}$
D. $\log_e|2x-4|$
E. $\frac{1}{x-2}$

2011 Exam 1

Question 2

a. Find an antiderivative of $\frac{1}{3x-4}$ with respect to x.

1 mark

b. Solve the equation $4^x - 15 \times 2^x = 16$ for x.

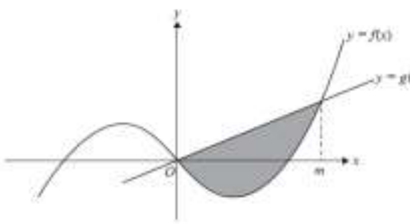
Question 9

Parts of the graphs of the functions

$$f: R \to R, f(x) = x^3 - ax \qquad a > 0$$
$$g: R \to R, g(x) = ax \qquad a > 0$$

are shown in the diagram below.

The graphs intersect when $x = 0$ and when $x = m$.



The area of the shaded region is 64.

Find the value of a and the value of m.

2011 Exam 2

Question 11

The average value of the function with rule $f(x) = \log_e(x + 2)$ over the interval $[0, 3]$ is

A. $\log_e(2)$

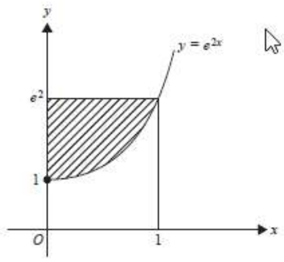
B. $\frac{1}{3}\log_e(6)$

C. $\log_e\left(\frac{3125}{4}\right) - 3$

D. $\frac{1}{3}\log_e\left(\frac{3125}{4}\right) - 3$

E. $\frac{5\log_e(5) - 2\log_e(2) - 3}{3}$

Question 14



To find the area of the shaded region in the diagram shown, four different students proposed the following calculations.

i. $\int_0^1 e^{2x}\,dx$

ii. $e^2 - \int_0^1 e^{2x}\,dx$

iii. $\int_1^{e^2} e^{2y}\,dy$

iv. $\int_1^{e^2} \frac{\log_e(x)}{2}\,dx$

Which of the following is correct?

A. ii. only

B. ii. and iii. only

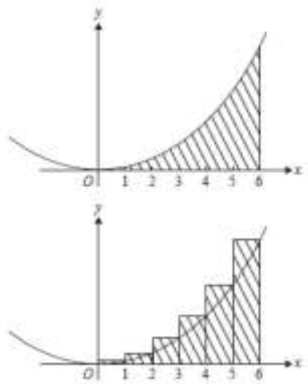
C. i., ii., iii. and iv.

D. ii. and iv. only

E. i. and iv. only

Question 19

A part of the graph of $f: R \to R, f(x) = x^2$ is shown below. Zoe finds the approximate area of the shaded region by drawing rectangles as shown in the second diagram.



Zoe's approximation is p% more than the exact value of the area.

The value of p is closest to

A. 10

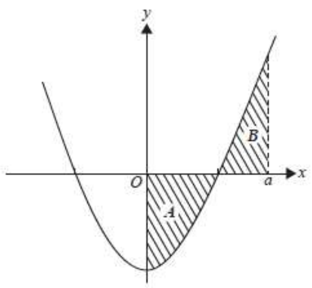
B. 15

C. 20

D. 25

Question 20

A part of the graph of $g: R \to R, g(x) = x^2 - 4$ is shown below.



The area of the region marked A is the same as the area of the region marked B.

The exact value of a is

A. 0

B. 6

C. $\sqrt{6}$

D. 12

E. $2\sqrt{3}$

2012 Exam 1

Question 2

Find an anti-derivative of $\frac{1}{(2x-1)^3}$ with respect to x.

2 mark

Question 9

a. Let $f: R \to R$, $f(x) = x\sin(x)$.

Find $f'(x)$.

1 mark

b. Use the result of part a. to find the value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos(x)\,dx$ in the form $a\pi + b$.

2012 Exam 2

Question 10

The average value of the function $f: [0, 2\pi] \to R$, $f(x) = \sin^2(x)$ over the interval $[0, a]$ is 0.4.

The value of a, to three decimal places, is

A. 0.850

B. 1.164

C. 1.298

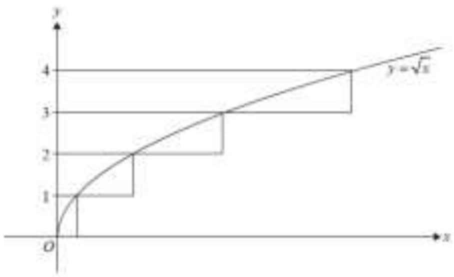
D. 1.339

E. 4.046

Question 14

The graph of $f: R^+ \cup \{0\} \to R$, $f(x) = \sqrt{x}$ as shown below.

In order to find an approximation to the area of the region bounded by the graph of f, the y-axis and the line $y = 4$, Zoe draws four rectangles, as shown, and calculates their total area.



Zoe's approximation to the area of the region is

A. 14

B. 21

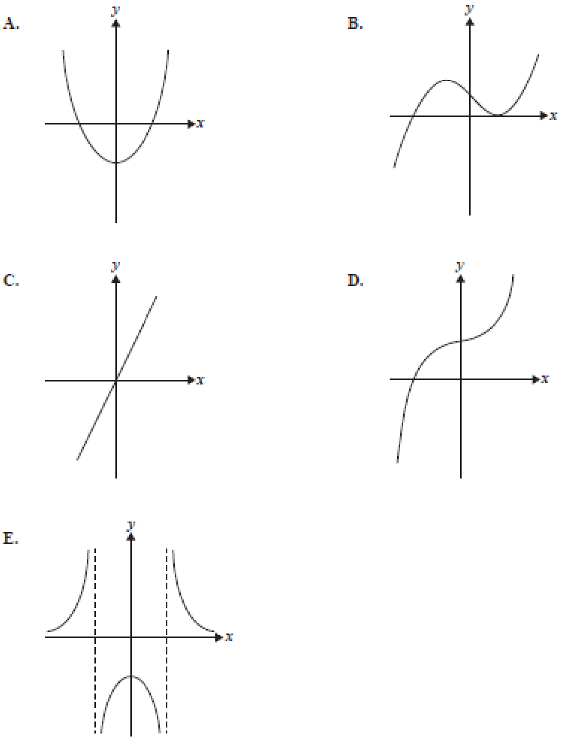
C. 29

D. 30

E. $\frac{64}{3}$

Question 15

If $f'(x) = 3x^2 - 4$, which one of the following graphs could represent the graph of $y = f(x)$?

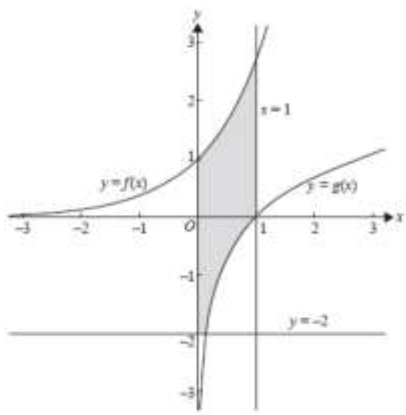


Question 5

The shaded region in the diagram below is the plan of a mine site for the Black Possum mining company. All distances are in kilometres.

Two of the boundaries of the mine site are in the shape of the graphs of the functions

$$f: R \to R, f(x) = e^x \text{ and } g: R^+ \to R, g(x) = \log_e(x).$$



a. i. Evaluate $\int_{[illegible]}^{[illegible]} f(x)dx$.

ii. Hence, or otherwise, find the area of the region bounded by the graph of g, the x and y axes, and the line $y = -2$.

iii. Find the total area of the shaded region.

1 + 1 + 1 = 3 marks

b. The mining engineer, Victoria, decides that a better site for the mine is the region bounded by the graph of g and that of a new function $k: (-\infty, a) \to R$, $k(x) = -\log_e(a - x)$, where a is a positive real number.

i. Find, in terms of a, the x-coordinates of the points of intersection of the graphs of g and k.

ii. Hence, find the set of values of a, for which the graphs of g and k have two distinct points of intersection.

2013 Exam 1

Question 2 (2 marks)

Find an anti-derivative of $(4 - 2x)^{-3}$ with respect to x.

Question 3 (2 marks)

The function with rule $g(x)$ has derivative $g'(x) = \sin(2\pi x)$.

Given that $g(1) = \frac{1}{\pi}$, find $g(x)$.

Question 6 (3 marks)

Let $g: R \to R$, $g(x) = (a - x)^2$, where a is a real constant.

The average value of g on the interval $[-1, 1]$ is $\frac{31}{12}$.

Find all possible values of a.

2013 Exam 2

Question 6

For the function $f(x) = \sin(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\frac{1}{4}, 5\right]$ is

A. 0

B. $\frac{34}{19}$

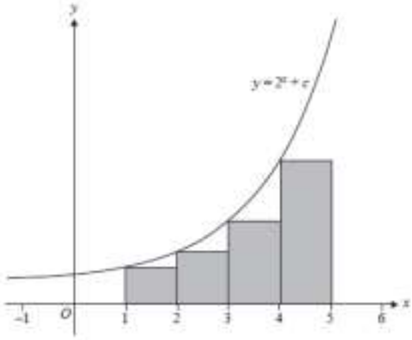
C. $\frac{7}{2}$

D. $\frac{2\pi + 10}{4}$

E. $\frac{23}{4}$

Question 14

Consider the graph of $y = 2^x + c$, where c is a real number. The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x-axis and the lines $x = 1$ and $x = 5$.



If the total area of the shaded rectangles is 44, then the value of c is

A. 14

B. -4

C. $\frac{14}{5}$

D. $\frac{7}{2}$

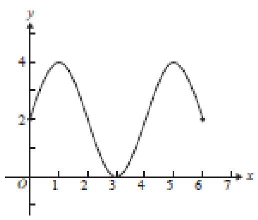
E. $-\frac{16}{5}$

Question 15

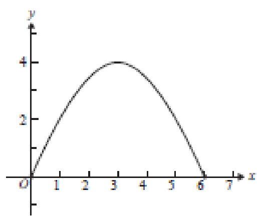
Let h be a function with an average value of 2 over the interval $[0, 6]$.

The graph of h over this interval could be

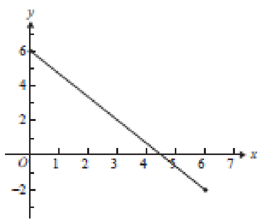
A.



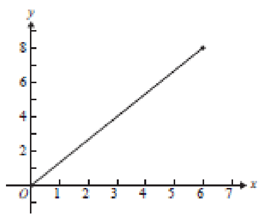
B.



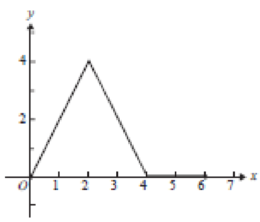
C.



D.

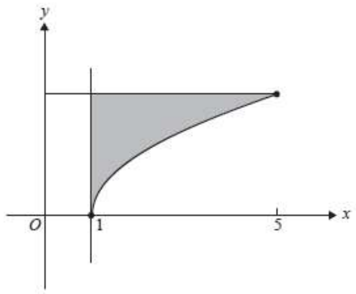


E.



Question 16

The graph of $f: [1, 5] \to R, f(x) = \sqrt{x-1}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_1^5 \left(\sqrt{x-1}\right)dx$

B. $\int_0^2 \left(\sqrt{x-1}\right)dx$

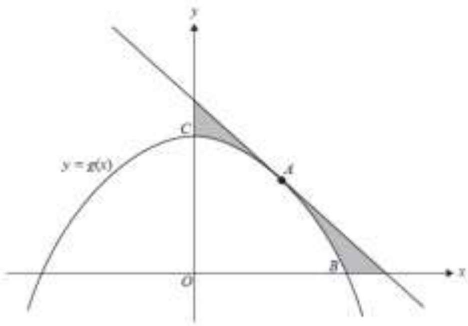
C. $\int_0^5 \left(2-\sqrt{x-1}\right)dx$

D. $\int_0^2 \left(x^2+1\right)dx$

E. $\int_0^2 \left(x^2\right)dx$

Question 4 (16 marks)

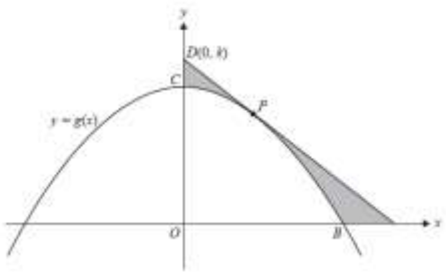
Part of the graph of a function $g: R \to R, g(x) = \frac{16-x^2}{4}$ is shown below.



a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram above. The tangent to the graph of g at the point A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A. 2 marks

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y-axis at point $D(0, k)$, where $5 \le k \le 8$.



c. Find the gradient of the tangent in terms of k. 2 marks

ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the point A, and the x-axis and y-axis.

Evaluate the area of this shaded region. 3 marks

d. i. Find the rule $A(k)$ for the function of k that gives the area of the shaded region. 2 marks

ii. Find the **maximum** area of the shaded region and the value of k for which this occurs. 2 marks

iii. Find the **minimum** area of the shaded region and the value of k for which this occurs. 2 marks

2014 Exam 1

Question 2 (2 marks)

Let $\int_0^4 \frac{1}{2x+1}dx = \log_e(b)$.

Find the value of b.

Question 7 (3 marks)

If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

2014 Exam 2

Question 3

The area of the region enclosed by the graph of $y = x(x+2)(x-4)$ and the x-axis is

A. $\frac{128}{3}$

B. $\frac{20}{3}$

C. $\frac{236}{3}$

D. $\frac{148}{3}$

E. 36

Question 8

If $\int_1^4 f(x)dx = 6$, then $\int_1^4 (5-2f(x))dx$ is equal to

A. 3

B. 4

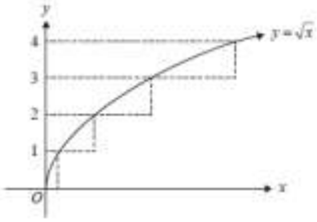
C. 5

D. 6

E. 16

Question 19

Jake and Anita are calculating the area between the graph of $y=\sqrt{x}$ and the y-axis between $y=0$ and $y=4$. Jake uses a partitioning, shown in the diagram below, while Anita uses a definite integral to find the exact area.



The difference between the results obtained by Jake and Anita is

A. 0

B. $\frac{22}{3}$

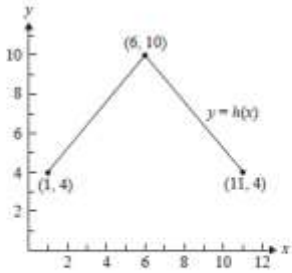
C. $\frac{26}{3}$

D. 14

E. 35

Question 20

The graph of a function, h, is shown below.



The average value of h is

A. 4

B. 5

C. 6

D. 7

E. 10

2015 Exam 1

Question 2 (3 marks)

Let $f'(x)=1-\frac{3}{x}$, where $x \neq 0$.

Given that $f(e)=-2$, find $f(x)$.

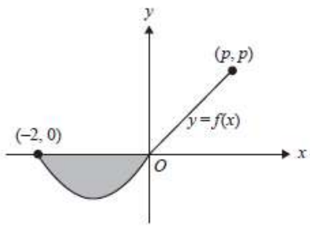
Question 3 (2 marks)

Evaluate $\int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx$.

2015 Exam 2

Question 8

The graph of a function $f: [-2, p] \to R$ is shown below.



The average value of f over the interval $[-2, p]$ is zero.

The area of the shaded region is $\frac{25}{8}$.

If the graph is a straight line, for $0 \le x \le p$, then the value of p is

A. 2

B. 5

C. $\frac{5}{4}$

D. $\frac{5}{2}$

E. $\frac{25}{4}$

Question 15

If $\int_0^5 g(x)dx=20$ and $\int_0^5 (2g(x)+ax)dx=90$, then the value of a is

A. 0

B. 4

C. 2

D. -3

E. 1

Question 16

Let $f(x)=ax^m$ and $g(x)=bx^n$, where a, b, m and n are positive integers. The domain of f = domain of g = R.

If $f'(x)$ is an antiderivative of $g(x)$, then which one of the following must be true?

A. $\frac{m}{n}$ is an integer

B. $\frac{n}{m}$ is an integer

C. $\frac{a}{b}$ is an integer

D. $\frac{b}{a}$ is an integer

E. $n-m=2$

Question 19

If $f(x)=\int_0^x \left(\sqrt{t^2+4}\right) dt$, then $f'(-2)$ is equal to

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. $2\sqrt{2}$

D. $-2\sqrt{2}$

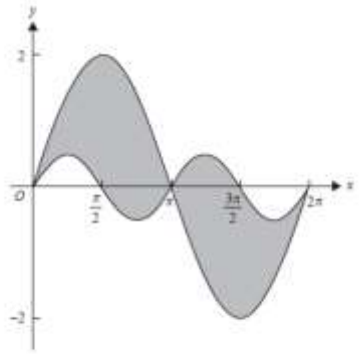
E. $4\sqrt{2}$

Question 4 (9 marks)

An electronics company is designing a new logo, based initially on the graphs of the functions

$f(x) = 2\sin(x)$ and $g(x) = \frac{1}{2}\sin(2x)$, for $0 \le x \le 2\pi$.

These graphs are shown in the diagram below, in which the measurements in the x and y directions are in metres.



The logo is to be painted onto a large sign, with the area enclosed by the graphs of the two functions (shaded in the diagram) to be painted red.

a. The total area of the shaded regions, in square metres, can be calculated as $a\int_0^{\pi} \sin(x)\,dx$.

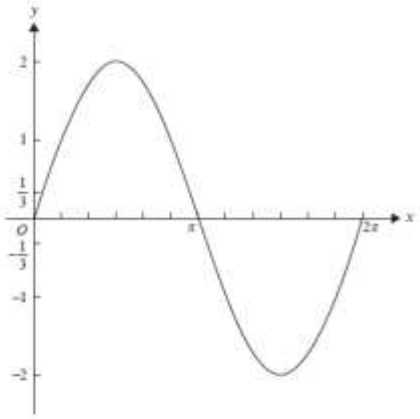
What is the value of a? 1 mark

The electronics company considers changing the circular functions used in the design of the logo.

Its next attempt uses the graphs of the functions $f(x) = 2\sin(x)$ and $h(x) = \frac{1}{3}\sin(3x)$, for $0 \le x \le 2\pi$.

b. On the axes below, the graph of $y = f(x)$ has been drawn.

On the same axes, draw the graph of $y = h(x)$. 2 marks



c. State a sequence of two transformations that maps the graph of $y = f(x)$ to the graph of $y = h(x)$. 2 marks

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The electronics company now considers using the graphs of the functions $k(x) = m\sin(x)$ and $q(x) = \frac{1}{n}\sin(nx)$, where m and n are positive integers with $m \ge 2$ and $0 \le x \le 2\pi$.

d. i. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if n is even**.

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers. 2 n

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ii. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if n is odd**.

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers. 2 n

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