**Physical interpretation and visualization.**

- **Active viewpoint (upper picture).**
  $T$ is a linear transformation of the plane that maps the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ onto another basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.
  Its matrix $A = [T]$ is the same in either basis.
  Every linear combination of the red vectors $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ is mapped to $T\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

- **Passive viewpoint (bottom picture).**
  Every vector $\mathbf{v}$ has two sets of coordinates:
  $$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = y_1\mathbf{u}_1 + y_2\mathbf{u}_2.$$
  Matrix $A$ describes a change of coordinates:
  $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
  We regard this as a linear transformation $T^*$ on the space of linear functionals that transforms the green coordinate functionals to the red ones:
  $$T^*(X_1) = Y_1 = aX_1 + cX_2, \;\; T^*(X_2) = Y_2 = bX_1 + dX_2, \;\; \text{where } A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$
  Given an inner product $\langle\cdot|\cdot\rangle$ on the plane, each linear functional $F$ corresponds to a unique vector $\nabla F$ such that $F(\cdot) = \langle\nabla F|\cdot\rangle$.
  Thus, we can regard $T^*$ as the *(adjoint)* transformation of the plane.
  If we choose an inner product such that the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is orthonormal, then $\nabla X_i = \mathbf{v}_i$, and the matrix of $T^*$ is $A^T$.

- Intuitively, if $T$ represent a movement of a physical system (of a vector or family of vectors), the bottom picture describes the movement from the point of view of a "traveling" observer associated with the green basis.
  That observer perceives the old (red) basis as moving backwards: $\mathbf{u}_i = T^{-1}\mathbf{v}_i$, while the change of the coordinate system (linear functionals and their gradients) is described by the adjoint transformation $T^*$.

**Mathematical Details.**

- Let $\mathcal{B} = \{\mathbf{u_1}, \mathbf{u_2}\}$ be a basis in the vector space $V = \mathbb{R}^2$.
  Denote by $Y_1, Y_2$ the coordinate functionals $\mathbb{R}^2 \to \mathbb{R}$ defined as follows: if $\mathbf{v} = y_1\mathbf{u}_1 + y_2\mathbf{u}_2$, then $Y_1(\mathbf{v}) = y_1$ and $Y_2(\mathbf{v}) = y_2$. We visualize $Y_1$ and $Y_2$ via their level curves $Y_i(\mathbf{x}) = const$. These lines form a coordinate grid, shown in red.
- Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ be an invertible matrix. The vectors $\mathbf{v}_1 = a\mathbf{u}_1 + b\mathbf{u}_2$ and $\mathbf{v}_2 = c\mathbf{u}_1 + d\mathbf{u}_2$ form a basis $\mathcal{C}$ in $\mathbb{R}^2$. The relation between the $\mathcal{B}$- and the $\mathcal{C}$-bases can be written in matrix form as

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

- Let $T$ be the linear transformation of $V$, such that $T\mathbf{u}_1 = \mathbf{v}_1$ and $T\mathbf{u}_2 = \mathbf{v}_2$.
  Clearly, the $\mathcal{B}$-matrix of $T$ is $A$.
  Moreover, the $\mathcal{C}$-matrix of $T$ is also $A$, since

$$[T]_{\mathcal{C}} = A\,[T]_{\mathcal{B}}\,A^{-1} = A\left(AA^{-1}\right) = AI = A.$$

- If $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ is any vector, then its (red) $\mathcal{B}$-coordinates are:

$$\begin{aligned} Y_1(\mathbf{v}) &= x_1 Y_1(\mathbf{v}_1) + x_2 Y_1(\mathbf{v}_2) = x_1 a + x_2 c = \begin{bmatrix} a & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ Y_2(\mathbf{v}) &= x_1 b + x_2 d = \begin{bmatrix} b & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

- Let $\langle\cdot|\cdot\rangle$ be the inner product in $V$, such that the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is orthonormal. Then,

$$Y_1(\mathbf{v}) = (a\mathbf{v}_1 + c\mathbf{v}_2) \cdot \mathbf{v}$$

  and

$$Y_2(\mathbf{v}) = (b\mathbf{v}_1 + d\mathbf{v}_2) \cdot \mathbf{v}.$$

  This implies that vectors $\mathbf{u}^1 = a\mathbf{v}_1 + c\mathbf{v}_2$ and $\mathbf{u}^2 = b\mathbf{v}_1 + d\mathbf{v}_2$ satisfy the following relations:

$$\left\langle \mathbf{u}^i | \mathbf{u}_j \right\rangle = Y_i(\mathbf{u}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

  Moreover, for any vector $\mathbf{x}$, we have$Y_i(\mathbf{x}) = \left\langle \mathbf{u}^i | \mathbf{x} \right\rangle$.
- Denote by $T^*$ the linear transformation, such that $T^*\mathbf{v}_i = \mathbf{u}^i$. Then the $\mathcal{C}-$matrix of $T^*$ is $A^T$.