MY FAVORITE lesson

Kent Thele

Conic Connections with Polar Functions

When I first studied conic sections, I learned about eccentricities, foci, and directrices. I did not understand, however, why such diverse curves shared these characteristics. That remained a mystery to me until years later. Now, I help my precalculus students see those connections—first while studying conics using dynamic geometry (Thele 2008), and then, surprisingly and somewhat elegantly, in one of my favorite lessons on polar functions, which I describe here.

The polar lesson begins as students graph functions of the form $r = 1 - b\cos\theta$, creating lemaçons (see **fig. 1a**). I then ask my students to predict the graphs of the polar family $r = 1/(1 - b\cos\theta)$. We start with $b = 0.5$, and they are surprised when they see it creates an ellipse. After seeing that $b = 1$ generates a parabola, most students correctly guess that $b = 2$ will create a hyperbola (see **fig. 1b**), but they are surprised when they learn that b is the conic-section eccentricity. When I ask my students what is special about the pole of the polar graph, they are surprised yet again: The pole is a focus of the conic section.

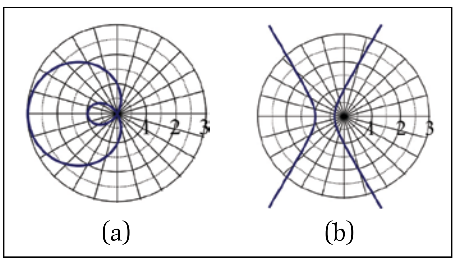


Fig. 1 Polar graphs show (a) a lemaçon, $r = 1 - 2\cos\theta$ and (b) a hyperbola, $r = 1/(1 - 2\cos\theta)$.

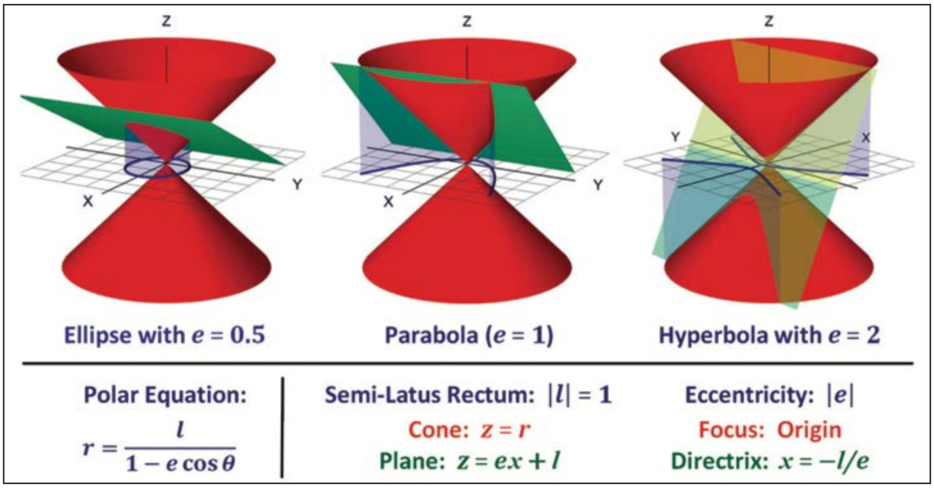


Fig. 2 Conic sections use a 45° double-napped cone created using www.teraplot.com.

We investigate these relationships using a 45° double-napped cone (see **fig. 2**). Here, the conic section of interest is, not the actual intersection of the plane and cone, but rather the vertical projection of that intersection onto the horizontal xy plane whose origin is at the vertex of the cone. In this setting, we observe the following relationships (Kendig 2005):

- The vertex of the cone is a focus of the projected conic.
- The z-intercept of the plane is the semi-latus rectum (positive y-intercept) of the projected conic, the size parameter in the polar equation.
- The slope of the plane is the eccentricity of the projected conic, the shape parameter in the polar equation.
- The intersection of the plane with the xy plane is a directrix of the projected conic.

What elegance! Both parameters in the polar equation have physical significance in the 3D model, and secondary equations are straightforward.

These connections help demystify and unify conic sections. I encourage you to consider incorporating them into your lessons.

REFERENCES

Kendig, Keith. 2005. *Conics.* Dolciani Mathematical Expositions #29. Washington, DC: Mathematical Association of America.

Thele, Kent. 2008. Geometer's Sketchpad file. http://sketchexchange.keypress.com/sketch/view/576/conic-connections

The Back Page provides a forum for readers to share a favorite lesson, not exceeding 600 words. Lessons should be submitted to **http://mt.msubmit.net**.

Department editor
Roger Day, day@ilstu.edu, Illinois State University, Normal, Illinois

KENT THELE, kentthele@johnpauliihs.org, teaches geometry and precalculus at John Paul II High School in Plano, Texas. He enjoys exploring mathematical concepts using graphing and dynamic geometry software.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

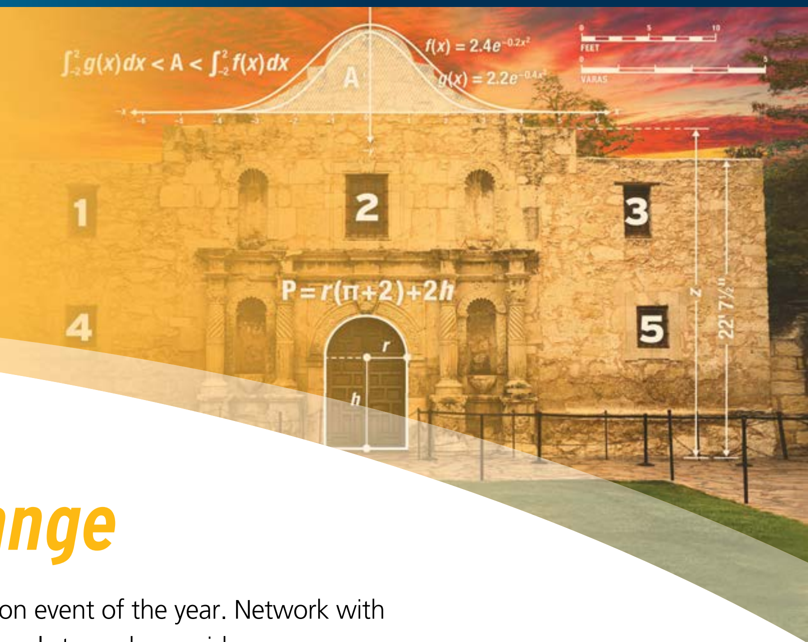
NCTM

Save the Date!

PREMIER MATH EDUCATION EVENT

NCTM ANNUAL MEETING & EXPOSITION 2017

April 5–8 | San Antonio



Creating Communities and Cultivating Change

It's never too early to plan ahead for the leading math education event of the year. Network with thousands of your peers and fellow math education professionals to exchange ideas, engage with innovation in the field and discover new learning practices that will drive student success.

The latest teaching trends and topics will include:

- **Access and Equity:** Teaching Mathematics with an Equity Stance
- **Assessment:** A Tool for Purposeful Planning and Instruction
- **Building Conceptual and Procedural Understanding**
- **Professionalism:** Learning Together as Teachers
- **Teaching, Learning, and Curriculum:** Best Practices for Engaging Students in Productive Struggle
- **The "M" in STEM/STEAM**
- **Tools and Technology:** Using Technology to Effectively Teach and Learn Mathematics

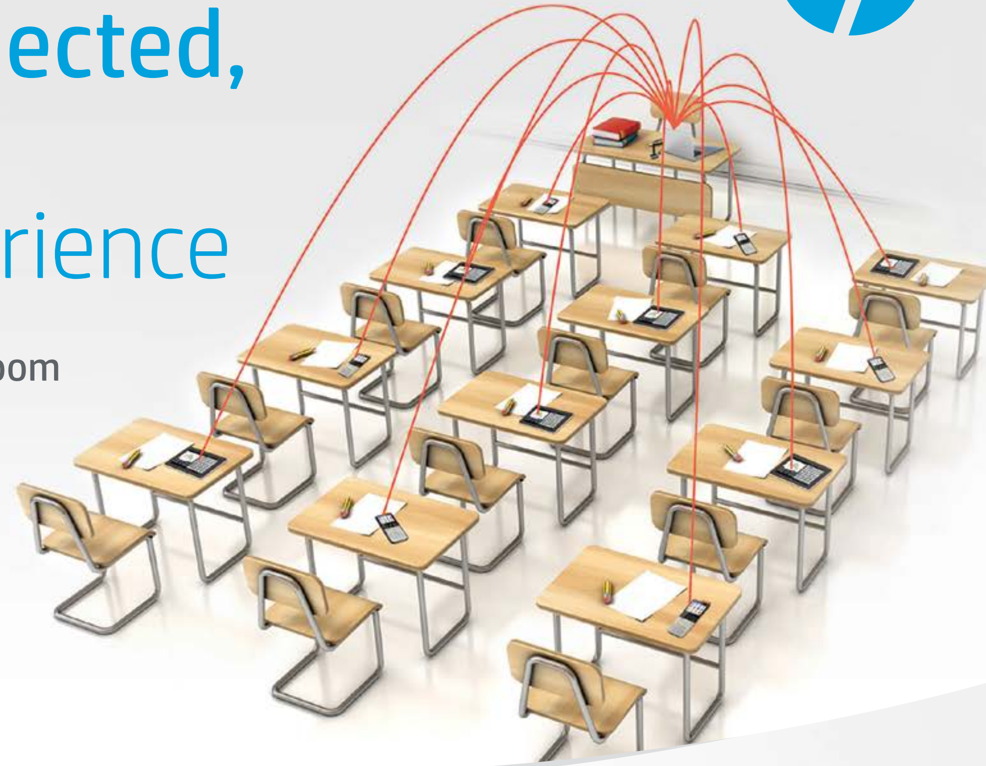
Registration Now Open

The NCTM Annual Meeting & Exposition is ideal for:

PRE-K–12 TEACHERS
MATH TEACHER EDUCATORS
NEW AND PROSPECTIVE TEACHERS
MATH COACHES AND SPECIALISTS
MATH RESEARCHERS
SCHOOL AND DISTRICT ADMINISTRATORS

Learn more at **nctm.org/annual** and follow us on #NCTMannual

Create a **connected, collaborative** learning experience



HP Prime Wireless Math Classroom

Meet the needs of your students with HP technology. Combine the HP Prime Graphing Calculator, HP Prime App[1], and HP Prime Wireless Kit.[2]

Discover more | **HP Prime Webinar**

A Breakthrough in Mathematics Education Technology!

November 29, 2016

Select one of two convenient times: 4:00 p.m. MT or 6:30 p.m. MT

Register at
hp.com/go/eduwebinars

Experience handheld calculating in the age of touch with the gesture-based, full-color HP Prime Graphing Calculator, which offers image backgrounds, function sketching, and wireless connectivity.[2]

Webinar attendees will receive the HP Prime Mobile App for Android, iOS or Windows for FREE! A $24.99 value

Mix and Match to create your ideal set-up

HP PRIME GRAPHING CALCULATOR

- College Board approved for AP, SAT and more*
- Advanced Graphing: plots what others cannot
- Precise Pinch-to-zoom

HP PRIME WIRELESS KIT

- Poll and quiz students
- Send and modify data sets
- Monitor and project calculator screens

HP PRIME MOBILE APP

- Same features and layout as the HP Prime Calculator
- Context-sensitive just-in-time help
- Windows, iOS and Android compatible

[1] Sold Separately.
[2] HP Prime Wireless Kit is sold separately and includes one PC antenna and 30 wireless modules, with bi-directional transmission range up to 15 meters.