Unit 2

Graphs and equations

Introduction

Mathematics is often used to represent and analyse the relationship between two quantities. For example, consider a car travelling along a road from some starting point. There is a relationship between the time that has elapsed since the start of the journey and the distance the car has travelled. This relationship can be investigated mathematically.

The use of mathematics to represent and study real-life situations is known as **mathematical modelling**. When creating a **mathematical model**, we usually simplify the real-life situation in order to concentrate on the aspects that we think are the most important. This often allows the relationship between the quantities of interest to be expressed as an equation.

In the example of the travelling car, if the variable t is used to represent the time in hours since the car left its starting point, and the variable s is used to represent the distance in kilometres that the car has travelled since the start, then there is a relationship between the variables t and s. In this unit you will consider some cases where the relationship between the variables s and t is modelled by using a relatively simple equation. Note that mathematicians often use s to represent distance.

Every equation in two variables, whether it models a practical situation or not, represents a relationship between those variables. For example, the equation

$$y = 2x - 1 \tag{1}$$

represents a relationship between the variables x and y. Each particular value of x corresponds to a particular value of y.

A useful way to visualise a relationship between two variables is to draw its *graph*. If the relationship is specified by an equation, then this graph is called the graph of the equation. Figure 1 shows the graph of equation (1).

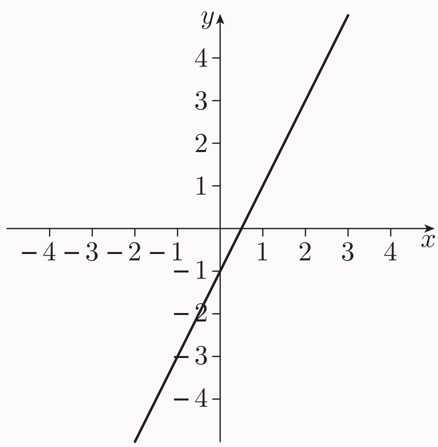


Figure 1 The graph of $y = 2x - 1$

Each point on the graph corresponds to a pair of values of x and y that are related by the equation. The graph of an equation in two variables can give new insights into the relationship between the variables. These complement those that you can obtain by using an algebraic approach.

In this unit you will revise two particular types of equation and their graphs. You have probably met these in your previous study of mathematics. They are covered here because they occur frequently in the later units of this module and you need to be confident in working with them, both graphically and algebraically. If you are familiar with them, then you may not wish to spend long reading the sections about them, but you should try the activities and ensure that you are not missing anything new to you, particularly in Subsection 2.4, which is important for the calculus that you'll study later. You will also begin to learn how to use the module computer algebra system to help you investigate mathematical problems.

In Section 1 you will revise the concepts of coordinates and graphs, and then in Sections 2 and 3 you will consider equations of the form

$$y = mx + c,$$

where x and y are variables, and m and c are constants. Equation (1) is of this form, with $m = 2$ and $c = -1$. The graph of every equation of this form is a straight line. You will see how such equations can be used to model some real-life situations, and revise how to solve pairs of such equations *simultaneously*.

In Section 4, you will consider equations of the form

$$y = ax^2 + bx + c,$$

where again x and y are variables, and a, b and c are constants, with $a \neq 0$. For example, the equation

$$y = x^2 - 2 \qquad (2)$$

is of this form, with $a = 1$, $b = 0$ and $c = -2$. The graph of every equation of this form has a particular curved shape called a *parabola*. Figure 2 shows the parabola that is the graph of equation (2).

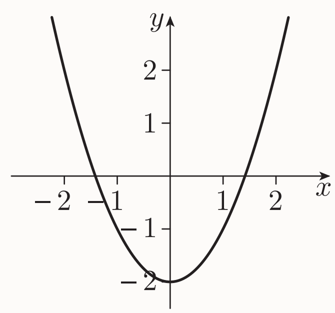


Figure 2 The graph of $y = x^2 - 2$

Equations of this form can be used to model some types of real-life situations, such as the motion of an object falling under the influence of gravity. To find the values of x where a graph such as that in Figure 2 crosses the x-axis you have to solve a *quadratic equation*, and you will revise various techniques for solving equations of this type.

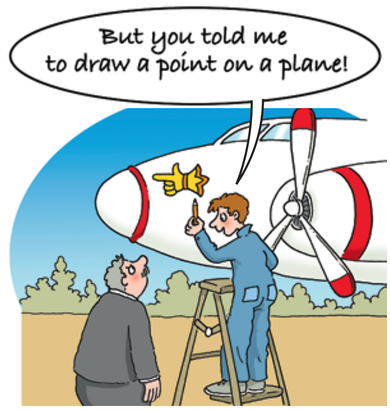
Finally, in Section 5 you will learn how to use the module computer algebra system to plot the graphs of various equations in two variables, manipulate expressions and solve equations.

1 Plotting graphs

In this section you'll revise the idea of the graph of an equation, and practise plotting graphs using tables of values. We begin with a brief reminder about *coordinates*.

1.1 Coordinates

The location of each point in a plane (that is, on a flat surface) can be specified using a pair of **coordinates** that give the position of the point relative to two *axes* at right angles to each other, as illustrated in Figure 3.



The horizontal and vertical axes are usually labelled x and y, respectively, and are referred to as the ***x*-axis** and the ***y*-axis**. The point where the axes intersect is called the **origin**, and is sometimes labelled O. Each axis is drawn with an arrowhead to indicate the positive direction (the direction in which the numbers increase), and is usually marked with numbers to show the scale.

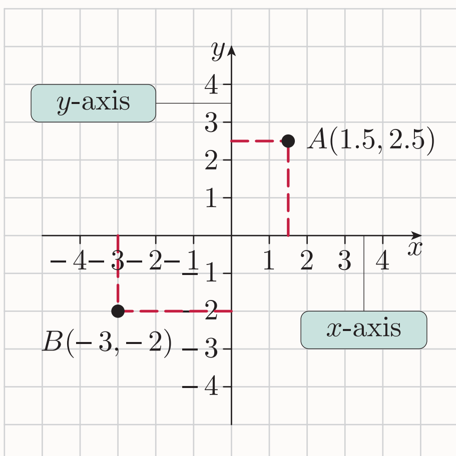


Figure 3 Coordinates

The coordinates of a point are written in brackets, separated by a comma, like this: $(1, 2)$. The first number is the number on the x-axis directly below or above the point, and is called the **x-coordinate**. The second number is the number on the y-axis directly to the left or right of the point, and is called the **y-coordinate**. For example, Figure 3 shows two points, A and B, with coordinates $(1.5, 2.5)$ and $(-3, -2)$, respectively.

This method of specifying the position of points is known as a *rectangular* or *Cartesian* coordinate system. The plane in which the axes and the points lie is sometimes called the **Cartesian plane**, the **x,y-plane**, the **coordinate plane** or, when there is no ambiguity, simply the **plane**. The axes are sometimes referred to as the **coordinate axes**.

The adjective *Cartesian* comes from the surname of the French mathematician and philosopher René Descartes (1596–1650). He is credited with being the first person to realise that the shapes of curves and surfaces can be studied using algebra. Poor health at school led to his life-long habit of never rising from bed until 11 o'clock in the morning. He later suggested this was an essential requirement for doing good mathematics!

Although x and y are the standard labels for graph axes, other labels can be used. For example, if a graph represents the relationship between the variables s and t, then these letters are used to label the axes. You can also refer to the horizontal and vertical axes, and the horizontal and vertical coordinates, and adjust the way you describe other quantities accordingly. Whatever the axis labels are, the first number in a pair of coordinates is always the position along the horizontal axis, and the second number is always the position along the vertical axis.

The position of a point in a plane can be marked with a dot, as in Figure 3, or with a small cross. It can be left unlabelled, or it can be labelled in any of various ways: with its coordinates, or with a letter, or with both, as in Figure 3.

1.2 Graphs of equations

Suppose that you have an equation in x and y, such as

$$y = 2x - 1, \quad y = x^2 + x + 3 \quad \text{or} \quad x^2 + y^2 = 1.$$

A point (x, y) is said to **satisfy** the equation if the equation is true for the point's values of x and y.

Example 1 *Checking whether a point satisfies an equation*

Show that the point $(3, 2)$ satisfies the equation $y = x^2 - 2x - 1$.

Solution

Substitute the x- and y-coordinates of the point into the left-hand side and right-hand side of the equation, and check that the two sides are equal.

When $x = 3$ and $y = 2$,

$$\text{LHS} = y = 2$$

and

$$\text{RHS} = x^2 - 2x - 1 = 3^2 - 2 \times 3 - 1 = 9 - 6 - 1 = 2.$$

The LHS and RHS are equal, so the point $(3, 2)$ satisfies the equation.

The method in Example 1 is the same as that in Example 22 in Unit 1.

Activity 1 *Checking whether points satisfy an equation*

Determine whether the following points satisfy the equation $y - 2 = 3x$.

(a) $(6, 20)$ (b) $(-2, 8)$

You know from Unit 1 that if a pair of values of x and y satisfy an equation in x and y, then they also satisfy any rearranged version of the equation. This means that the collection of points that satisfy an equation does not change when the equation is rearranged. For example, the points that satisfy the equation $y - 2x = -1$ are the same as the points that satisfy the equation $y = 2x - 1$.

Usually, for any particular equation in x and y, the points that satisfy the equation are the points that lie on a particular line or curve. If the equation has the property that it can be rearranged to express y as a formula in terms of x (so each value of x determines just one value of y), then the line or curve is called the **graph** of the equation. The word 'graph' is also used to refer to the whole diagram, including the line or curve and the coordinate axes.

A simple way to get an idea of the shape of the graph of an equation is to choose a few values for x, substitute them into the equation to find the corresponding values of y, plot the resulting points, and draw a smooth line or curve through them. This is illustrated in the next example. It is convenient to use a *table of values* to record the values of x and y.

Example 2 *Plotting the graph of an equation*

Plot the graph of the equation $y = \dfrac{x}{2} + 2$.

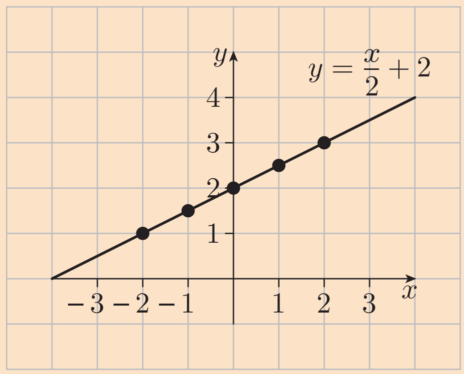
Solution

Construct a table of values. Choose some equally-spaced values of x, and work out the corresponding values of y by substituting into the equation. For example, substituting $x = -2$ into the equation gives $y = (-2)/2 + 2 = -1 + 2 = 1$.

A table of values for the equation $y = x/2 + 2$ is as follows.

x	-2	-1	0	1	2
y	1	1.5	2	2.5	3

Draw the axes and plot the points. They seem to lie in a straight line, so draw the straight line through them. Label the line with its equation, either on the graph or in a title.



We will consider why the points in Example 2 lie on a straight line in Subsection 2.2.

Notice that the graph drawn in Example 2 has been extended beyond the points that were calculated. This is because every straight line continues infinitely far in each direction. A graph can show only a small, finite part of a line. A finite part of a line, such as the part between two particular points, is called a **line segment**. Also notice that the graph was plotted with x on the horizontal axis and y on the vertical axis. When you plot the graph of an equation in which one variable y is expressed in terms of another variable x, the variable x is always on the horizontal axis and the variable y on the vertical axis.

You can practise using a table of values to plot a graph in the next activity. Remember that whenever you draw a graph, you should label it with its equation, either on the graph or in a title.

Activity 2 *Plotting the graph of an equation*

Complete the table of values below for the equation

$$y = x^2 + 2x + 2,$$

and hence plot the graph of this equation.

x	-2	-1	0	1	2
y					

When you plot the graph of an equation using a table of values alone, you cannot be absolutely sure that the graph is correct. This is because you do not know what happens between the points that you plotted and to each side of them. For example, consider the equation $y = 5x^3 - x^5$. A table of values for this equation, using integer values of x from -2 to 2, is given in Table 1.

Table 1 A table of values for $y = 5x^3 - x^5$

x	-2	-1	0	1	2
y	-8	-4	0	4	8

These points lie on a straight line, as shown in Figure 4(a). However this line is *not* the graph of the equation $y = 5x^3 - x^5$. A correct graph is shown in Figure 4(b).

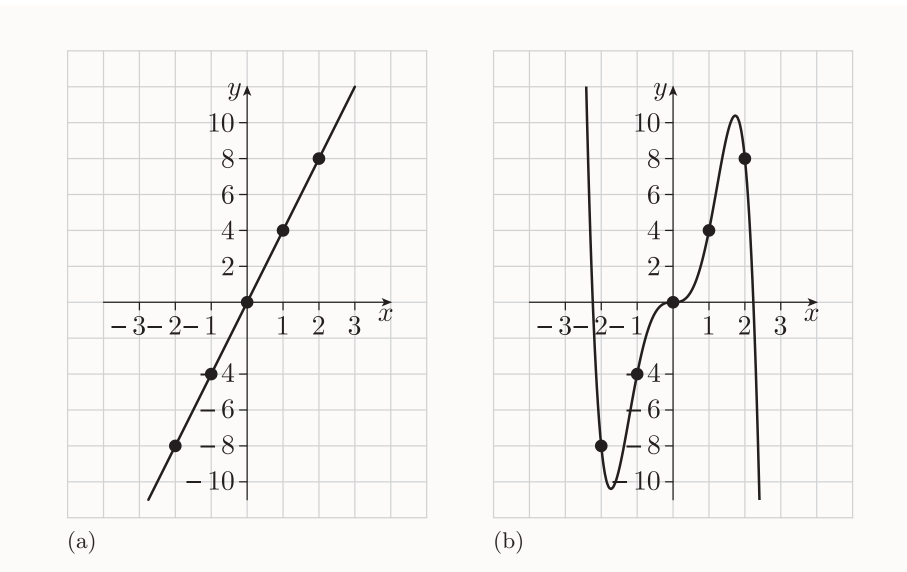


Figure 4 (a) The straight line through the points in Table 1 (b) the graph of $y = 5x^3 - x^5$, which goes through the same points

Because you can get the wrong idea about the shape of a graph from an unfortunate choice of plotted points, it's helpful to get to know the general shapes of the graphs of some standard types of equation, and learn how to sketch such graphs without using tables of values. You'll do this for various types of equation throughout this module, starting in Subsection 2.2 with equations whose graphs are straight lines.

2 Straight-line graphs

In this section you'll revise how to recognise equations that have straight-line graphs, and how to sketch such graphs from their equations. First, it's helpful to revise some basic properties of lines. Note that, in mathematics, the word 'line' is generally used to mean a straight line.

2.1 Gradients and intercepts of straight lines

Gradients

The *gradient* of a straight line is a measure of how steep it is. To understand what gradient means, imagine tracing your pen tip along a straight line. The **gradient** (or **slope**) of the line is the number of units that your pen tip moves *up* for every one unit that it moves *to the right*. For example, you can see that the line in Figure 5(a) has gradient 2.

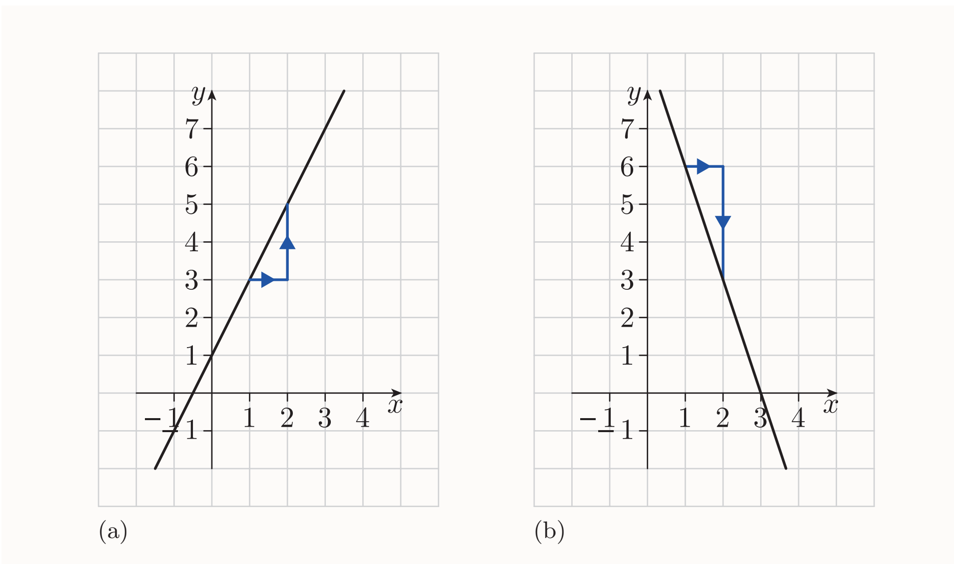


Figure 5 Straight lines with gradients (a) 2 (b) -3

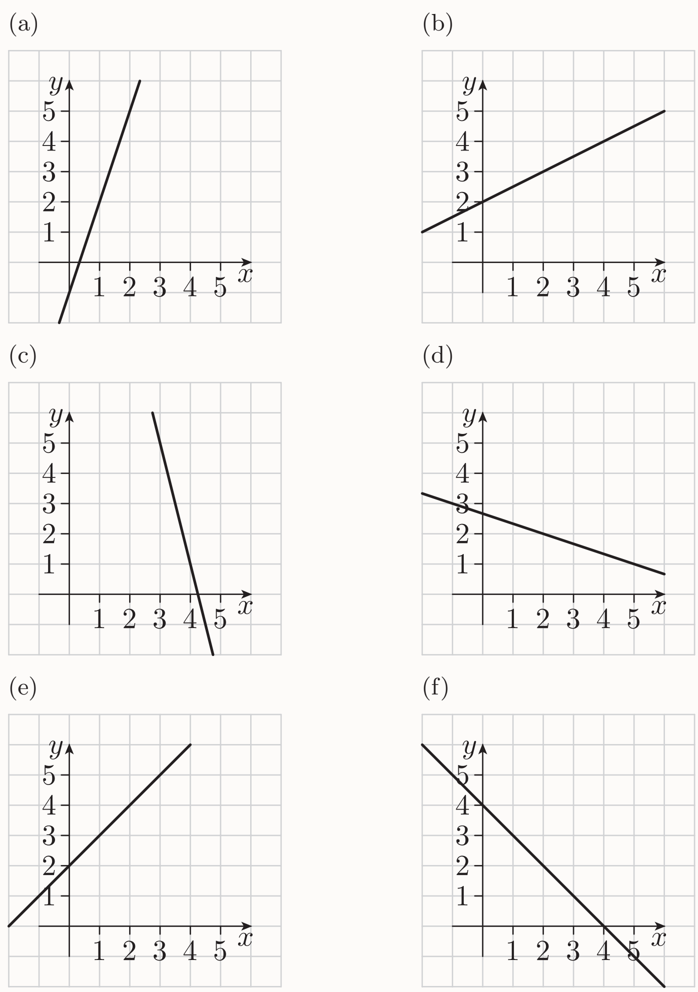
If you imagine tracing your pen tip along the line in Figure 5(b), you can see that it will move *down*, rather than up, as it moves to the right. It will move down by 3 units for every one unit that it moves to the right. A movement of 3 units down can be thought of as a movement of -3 units *up*. So your pen tip moves up by -3 units for every one unit that it moves to the right, and hence the gradient of this line is -3.

It's helpful to remember the following facts.

> A line that slopes *up* from left to right has a positive gradient.
>
> A line that slopes *down* from left to right has a negative gradient.

Activity 3 *Thinking about gradients*

By thinking about moving your pen tip along each of the lines below, and counting how many units it would move up or down for each unit it moves to the right, write down the gradients of the following lines.



Lines with large positive or negative gradients, such as $10, 50, -10$ or -50, are steeper than those with smaller positive or negative gradients, such as $1, 0.5, -0.5$ or -1. This fact can be expressed more neatly by using the idea of the **magnitude** of a number, which is its value without its minus sign, if it has one. For example, the magnitudes of 5 and -5 are both 5. The magnitude of a number is also called its **modulus** or its **absolute value**. The greater the magnitude of the gradient of a line, the steeper the line.

When the coordinate axes have *equal scales* (that is, when the distance representing one unit is the same for both the horizontal and vertical axes), a line with gradient 1 or -1 makes an angle of 45° with the horizontal axis, as shown in Figure 6. So a line whose gradient has magnitude greater than 1 makes an angle of more than 45° with the horizontal axis, and a line whose gradient has magnitude less than 1 makes an angle of less than 45° with the horizontal axis. Remember, though, that these facts are true *only if the coordinate axes have equal scales*.

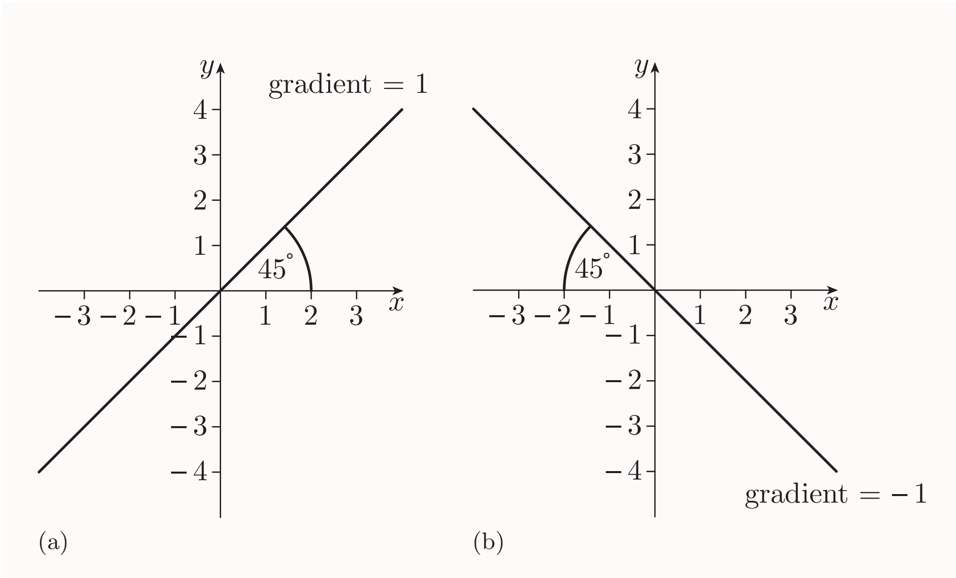


Figure 6 Lines with gradient (a) 1 (b) -1

Calculating gradients

You can calculate the gradient of a straight line by choosing any two points on the line, and proceeding as follows.

First you choose one of the two points (it doesn't matter which) to be the 'first point', and the other point to be the 'second point'. Then you find the number of units by which x increases as you trace your pen tip from the first point to the second point. This is known as the **run** from the first point to the second point. You also find the number of units by which y increases as you trace your pen tip from the first point to the second point. This is known as the **rise** from the first point to the second point. If x or y (or both) actually *decreases* as you trace your pen tip from the first point to the second point, then the run or rise (or both) is negative.

For example, in Figure 7(a) the run is 4 and the rise is 2, whereas in Figure 7(b) the run is −1 and the rise is 3.

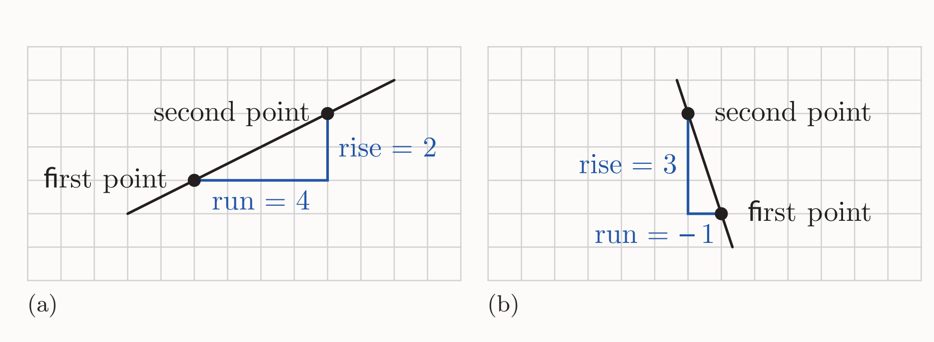


Figure 7 The run and rise from one point to another

Once you have found the run and rise from the first point to the second point, you can calculate the gradient as follows:

$$\text{gradient} = \frac{\text{rise}}{\text{run}}.$$

For example, the gradient of the line in Figure 7(a) is

$$\frac{\text{rise}}{\text{run}} = \frac{2}{4} = \frac{1}{2},$$

and the gradient of the line in Figure 7(b) is

$$\frac{\text{rise}}{\text{run}} = \frac{3}{-1} = -3.$$

This method for calculating the gradient of a line can be expressed as a formula in terms of the coordinates of the two points on the line. Let's denote the first and second points by (x_1, y_1) and (x_2, y_2), respectively.

Here x_1 and x_2 are particular values of x, and y_1 and y_2 are particular values of y. Mathematicians often use subscripts in this way to indicate particular values of variables. When you work with subscripts, be careful not to confuse x_2 with x^2, for example.

With this notation,

$$\text{run} = x_2 - x_1 \quad \text{and} \quad \text{rise} = y_2 - y_1.$$

So

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The gradient of a straight line

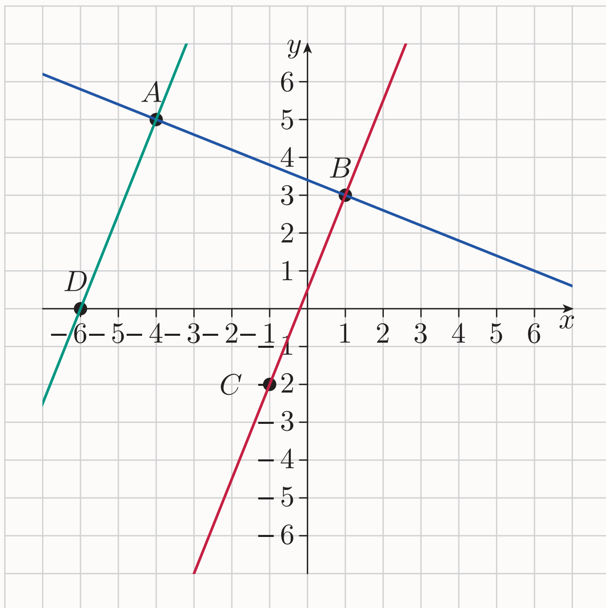
The gradient of the straight line through the points (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$, is given by

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Although it doesn't matter which point on the line you choose to be (x_1, y_1) and which to be (x_2, y_2) when you use the formula above, it is important to take them the same way round in both the numerator and the denominator.

Activity 4 *Using the formula for gradient*

Consider the following diagram.



(a) Write down the coordinates of the points A, B, C and D.

(b) Use the formula for gradient to calculate the gradients of the lines that pass through the following pairs of points.

(i) A and B (ii) A and D (iii) B and C

Gradients of horizontal and vertical lines

The gradient of a horizontal line is zero. This is because the gradient is the rise divided by the run, and the rise between any two points on a horizontal line is zero, as illustrated in Figure 8(a). On the other hand, the gradient of a vertical line is *undefined*. This is because, again, the gradient is the rise divided by the run, but the run between any two points on a vertical line is zero, as illustrated in Figure 8(b). Since it is not possible to divide by zero, the gradient of a vertical line does not exist.

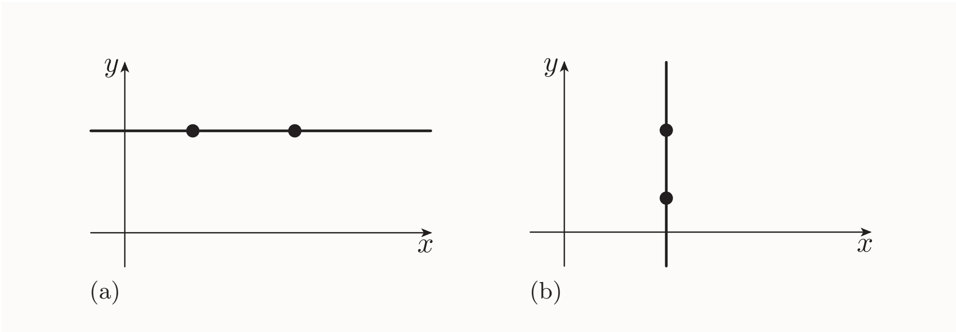


Figure 8 (a) The rise between two points on a horizontal line is zero (b) the run between two points on a vertical line is zero

Intercepts

The value of x where a line crosses the x-axis is called its $\boldsymbol{x}$**-intercept**, and the value of y where it crosses the y-axis is called its $\boldsymbol{y}$**-intercept**. For example, in Figure 9 the x-intercept is -3 and the y-intercept is 2.

Some mathematicians use the word 'intercept' to describe the *point* at which a line crosses an axis, rather than the value of the x- or y-coordinate there. They would say that the x- and y-intercepts in Figure 9 are $(-3, 0)$ and $(0, 2)$.

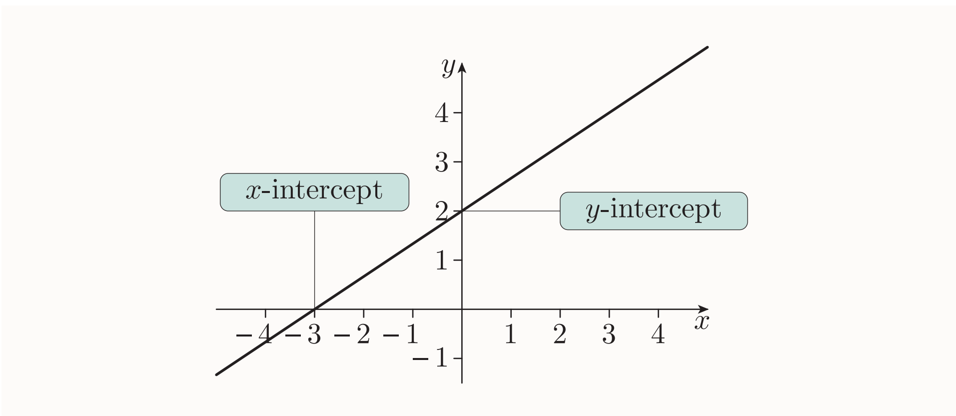


Figure 9 The x- and y-intercepts of a line

2.2 Straight lines and their equations

Let's now consider which equations in x and y are the equations of straight lines. First consider the simple equation $y = 2x$. The points that satisfy this equation are those whose y-coordinate is twice their x-coordinate. These are the points that lie on the straight line through the origin with gradient 2, as illustrated in Figure 10(a).

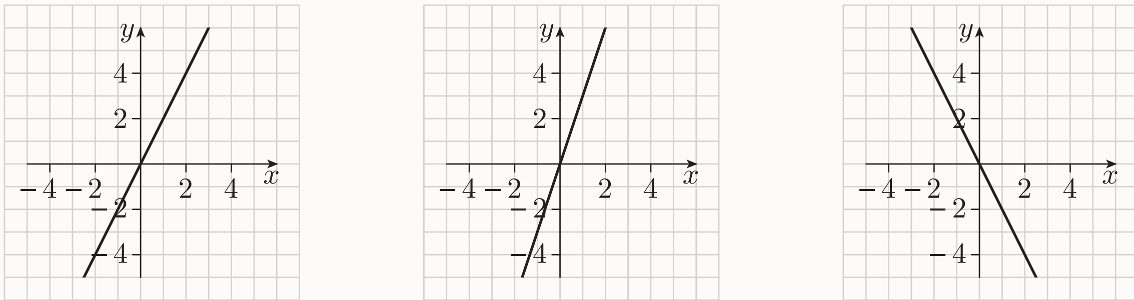


Figure 10 Lines through the origin with gradient (a) 2 (b) 3 (c) -2

Similarly, the points (x, y) that satisfy the equations $y = 3x$ and $y = -2x$ are the points that lie on the straight lines through the origin with gradients 3 and -2, respectively, as you can see in Figure 10(b) and (c).

In general, for any value of m, the points (x, y) that satisfy the equation $y = mx$ are the points that lie on the straight line through the origin with gradient m.

Vincenzo Riccati (1707–75)

It is traditional in the UK to use the letter m to represent gradient, though the reason is no longer known! Some countries traditionally use other letters, such as s or k. The earliest known use of the letter m for gradient is by the Italian mathematician Vincenzo Riccati in 1757. In addition to making important contributions to several areas of mathematics, Riccati taught Italian literature and Latin, and was responsible for developing flood control measures around Venice and Bologna.

Now consider the equation $y = 2x + 1$. Note that if a point satisfies $y = 2x$, then adding 1 to its y-coordinate gives a point that satisfies $y = 2x + 1$.

So the points that satisfy the equation $y = 2x + 1$ are all the points on the line that is obtained by moving the line in Figure 10(a) vertically up by

1 unit, as shown in Figure 11. Moving the line up by this amount changes its y-intercept from 0 to 1.

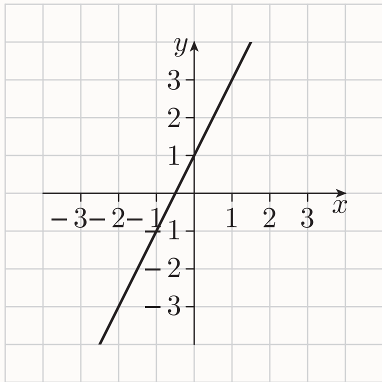


Figure 11 The line with equation $y = 2x + 1$

In general, for any constant c the graph of the equation $y = mx + c$ is obtained by moving the graph of the equation $y = mx$ vertically by c units. So we have the following fact.

Graphs of equations of the form $y = mx + c$

The graph of the equation $y = mx + c$ is the straight line with gradient m and y-intercept c.

It follows that the graph of any equation that can be rearranged into the form $y = mx + c$ is a straight line. For example, the graph of the equation $3x + 2y - 4 = 0$ is a straight line, since this equation can be rearranged as $y = -\frac{3}{2}x + 2$. Any equation that's of the form $y = mx + c$, or that can be rearranged into this form, is called a **linear equation** in the variables x and y. (You met the general definition of *linear equation* in Unit 1, where you revised how to solve linear equations in one unknown.)

When the equation of a line is written in the form $y = mx + c$, it's straightforward to 'read off' the gradient and the y-intercept. The gradient is the coefficient of x, and the y-intercept is the constant term. For example, the line $y = -\frac{3}{2}x + 2$ has gradient $-\frac{3}{2}$ and y-intercept 2.

To find the x-intercept of a line, you need to find the value of x for which $y = 0$. Here is an example.

Example 3 *Finding the x-intercept of a line from its equation*

Find the x-intercept of the line with equation $y = 4x - 3$.

Solution

The x-intercept is the value of x when $y = 0$.

Putting $y = 0$ gives

$$4x - 3 = 0.$$

Solving this equation gives

$$4x = 3$$
$$x = \tfrac{3}{4}.$$

Hence the x-intercept is $\frac{3}{4}$.

Leave the answer as a fraction.

(Check: substituting $x = \frac{3}{4}$ into $y = 4x - 3$ gives $y = 4 \times \frac{3}{4} - 3 = 0$, as expected.)

As illustrated in Example 3, if the x-intercept, y-intercept or gradient of a straight line is a fraction, then there is no need to express it as a decimal.

Activity 5 *Finding the gradients and intercepts from equations of lines*

Find the gradient, and x- and y-intercepts, of each of the following lines.

(a) $y = -4x + 3$ (b) $3y - x + 2 = 0$

Equations of horizontal and vertical lines

You saw earlier that the gradient of a horizontal line is zero. So the horizontal line with x-intercept c has equation $y = 0x + c$; that is, $y = c$. For example, the horizontal line in Figure 12(a) has y-intercept 3, so its equation is $y = 3$.

An alternative way to think of this fact is to notice that every point on the line in Figure 12(a) has y-coordinate 3, so the equation $y = 3$ describes each point on the line. It is therefore the equation of the line.

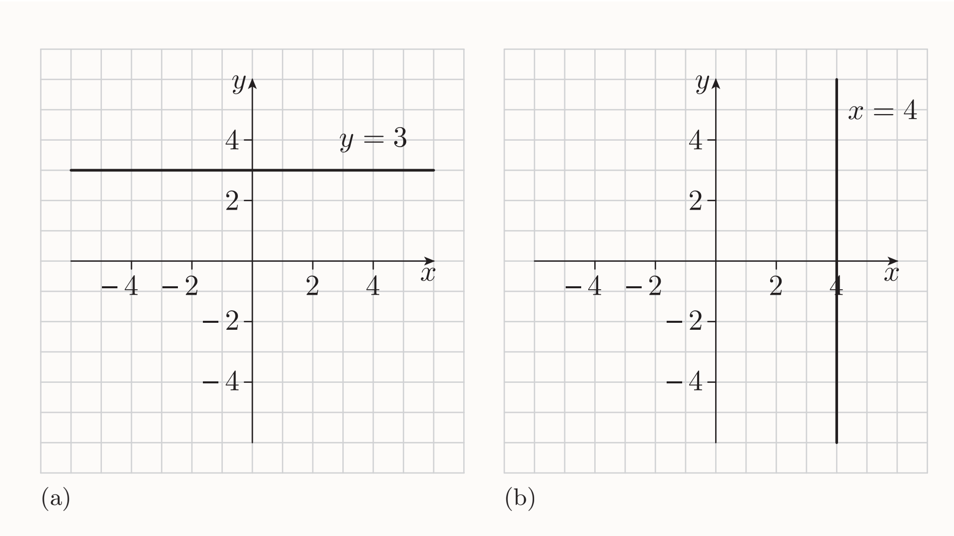


Figure 12 (a) A horizontal line (b) a vertical line

What about vertical lines? A vertical line has no gradient, so it doesn't have an equation of the form $y = mx + c$. Vertical lines are the only lines that do not have equations of this form.

However, every point on a vertical line has the same x-coordinate, so the line has an equation of the form $x = d$. The constant d is the x-intercept. For example, consider the vertical line shown in Figure 12(b). Every point on this line has x-coordinate 4, so the equation $x = 4$ describes each point on the line and is therefore the equation of the line.

Equations of horizontal and vertical lines

The horizontal line with y-intercept c has equation $y = c$.

The vertical line with x-intercept d has equation $x = d$.

In particular, the equation of the x-axis is $y = 0$, and the equation of the y-axis is $x = 0$.

Drawing a line from its equation

In Section 1 you saw how to plot the graph of an equation by finding several points that satisfy the equation, and drawing a smooth line or curve through them. If you can recognise an equation as the equation of a straight line, then to draw this line you just need to find *two* points that satisfy the equation, and draw the straight line through them.

One way to find two suitable points is to choose two values of x, and use the equation to find the corresponding values of y. You should try to choose values of x that are reasonably far apart and that lead to simple calculations. For example, if the equation is $y = \frac{1}{3}x + 2$ then you might choose $x = 0$ and $x = 3$, to avoid fractions. An alternative way to find two suitable points is to find the x- and y-intercepts.

Drawing a horizontal or vertical line from its equation is even more straightforward. To draw the line with equation $y = c$, you just mark the y-intercept c and draw the horizontal line through it. Similarly, to draw the line with equation $x = d$, you just mark the x-intercept d and draw the vertical line through it.

Activity 6 *Drawing lines from their equations*

Draw the straight lines with the following equations.

(a) $y = \frac{1}{3}x + 2$ (b) $y = -2x + 4$ (c) $y = \frac{7}{2}$ (d) $x = -3$

Finding the equation of a straight line

Every straight line that you can draw in a plane, with the exception of any vertical line, has a gradient and a y-intercept, and hence has an equation of the form $y = mx + c$.

If you know the gradient and the y-intercept of the line, then you can immediately write down the equation of the line. For example, the line with gradient 3 and y-intercept -5 has equation $y = 3x - 5$.

Sometimes, however, you might know different information about a line. The next example demonstrates a method for finding the equation of a line when you know its gradient and a point on it.

Example 4 *Finding the equation of a line from its gradient and a point on it: Method 1*

Find the equation of the line that has gradient -6 and passes through the point $(-1, 4)$.

Solution

A straight line has an equation of the form $y = mx + c$, where m is the gradient.

The equation is of the form $y = -6x + c$.

The point $(-1, 4)$ lies on the line, so this point must satisfy the equation.

Substituting $x = -1$ and $y = 4$ into the equation gives

$$4 = -6 \times (-1) + c$$
$$4 = 6 + c.$$

Solve this equation to find c.

$$-2 = c$$

So the equation of the line is $y = -6x - 2$.

(Check: substituting $x = -1$ into the equation $y = -6x - 2$ gives $y = -6 \times (-1) - 2 = 4$, so the point $(-1, 4)$ lies on the line, as expected.)

Here's an alternative way to find the equation of a straight line from its gradient and a point on it. Suppose that the gradient is m and the point is (x_1, y_1). If (x, y) is any other point on the line, then, by the formula for the gradient on page 127,

$$m = \frac{y - y_1}{x - x_1}. \tag{3}$$

Rearranging this equation gives

$$y - y_1 = m(x - x_1). \tag{4}$$

Equation (3) does not hold when $(x, y) = (x_1, y_1)$, since that would require division by zero, but it does hold for all other points (x, y) on the line. However, the rearranged equation, equation (4), holds for *all* points (x, y) on the line, including $(x, y) = (x_1, y_1)$, since for this point both sides are equal to zero. So equation (4) is the equation of the line. This fact is summarised below.

The equation of the straight line with gradient m that passes through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

Example 5 *Finding the equation of a line from its gradient and a point on it: Method 2.*

Find the equation of the line that has gradient -2 and passes through the point $(1, 4)$.

Solution

Substitute $m = -2$, $x_1 = 1$ and $y_1 = 4$ into the equation in the box above, and simplify it.

The equation of the line is

$$y - 4 = -2(x - 1).$$

It can be simplified as follows:

$$y - 4 = -2x + 2$$
$$y = -2x + 6.$$

So the equation of the line is $y = -2x + 6$.

(Check: substituting $x = 1$ into $y = -2x + 6$ gives $y = -2 \times 1 + 6 = 4$, so the point $(1, 4)$ lies on the line, as expected.)

Sometimes you might want to find the equation of a line from the coordinates of two points on it. If the two points have the same x-coordinate or the same y-coordinate, then you can immediately write down the equation of the horizontal or vertical line that they lie on. Otherwise, you can use the coordinates of the two points to calculate the gradient of the line, and then apply the method of Example 4 or Example 5.

Activity 7 *Finding the equations of lines*

Find the equations of the following lines.

(a) The line through the point $(2, 1)$ with gradient 3

(b) The line through the points $(2, 3)$ and $(4, 5)$

(c) The line with y-intercept 3 and gradient 2

(d) The line with x-intercept 2 and gradient -3

(e) The vertical line that passes through the point $(1, 0)$

(f) The line through the points $(-2, 3)$ and $(4, 3)$

2.3 Parallel and perpendicular lines

Two straight lines are **parallel** if they never cross, even when extended infinitely far in each direction, as illustrated in Figure 13.

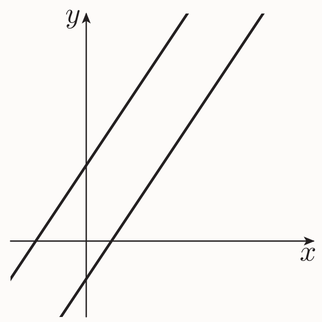


Figure 13 Two parallel lines

In Activity 4 on page 128, the line through the points A and D is parallel to the line through the points B and C. You may have noticed from your solution to this activity that these two lines have the same gradient. In general, saying that two non-vertical lines are parallel means the same as saying that they have the same gradient. Any two vertical lines are also parallel.

Two lines are **perpendicular** if they are at right angles to each other. You'd expect the gradients of two perpendicular lines to be related in some way – but how? To work this out, consider any two perpendicular lines that are not parallel to the axes, as illustrated in Figure 14.

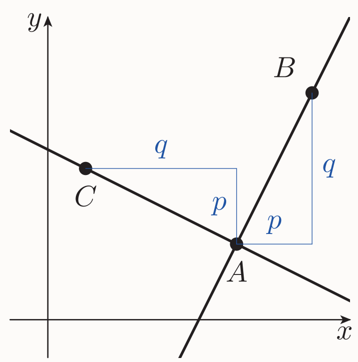


Figure 14 Two perpendicular lines

Let A be the point where the lines cross, let B be any point that lies on one of the lines and is above and to the right of A, and let C be the point on the other line that is obtained by rotating B anticlockwise through a quarter turn about A, as shown in Figure 14.

Let's denote the run from A to B by p, and the rise from A to B by q. Both p and q are positive numbers. If you rotate the right-angled triangle with hypotenuse AB in Figure 14 anticlockwise through a quarter turn about A, then it will lie exactly on top of the right-angled triangle with hypotenuse AC. It follows that the run from A to C is $-q$ (it's negative because C lies to the left of A), and the rise from A to C is p.

Hence

$$\text{gradient of the line through } A \text{ and } B = \frac{q}{p},$$

and

$$\text{gradient of the line through } A \text{ and } C = \frac{p}{-q} = -\frac{p}{q}.$$

Multiplying these two gradients together gives

$$\frac{q}{p} \times \left(-\frac{p}{q}\right) = -1.$$

So we have the following fact.

Gradients of perpendicular lines

The gradients of any two perpendicular lines (not parallel to the axes) have product -1.

Note that if two perpendicular lines are parallel to the axes, then one of them is parallel to the y-axis and hence has undefined gradient.

Activity 8 *Finding the equation of a line perpendicular to another line*

(a) Find the gradient of a line perpendicular to the line $y = 3x + 5$.

(b) Hence find the equation of the line that is perpendicular to $y = 3x + 5$ and passes through the point $(2, 1)$.

2.4 Applications of straight-line graphs

As mentioned in the introduction to this unit, the relationship between two real-life quantities can often be modelled by an equation in two variables. If the equation representing this relationship is of the form $y = mx + c$, and hence has a straight-line graph, then we say that the model is **linear**.

In this subsection, you'll look at some examples of linear models, and practise working with them. It's particularly important that you work carefully through this subsection, as it underpins *calculus*, which you'll start studying in Unit 6.

First, consider the graph in Figure 15, which represents the journey of a car along a road. It shows the relationship between the time that has elapsed since the car began its journey, and the distance that it has travelled since the start of the journey. The axes of the graph are labelled with 'time' and 'distance', as well as with t and s, the letters chosen to represent these quantities. The units in which each quantity is measured, kilometres (abbreviated to km) and hours (abbreviated to h), are also included. A graph like this, in which distance is plotted against time, is known as a **distance–time graph**.

The variables s and t are often used for distance and time, respectively, so you need to be careful to avoid possible confusion if the unit s (seconds) is used for time.

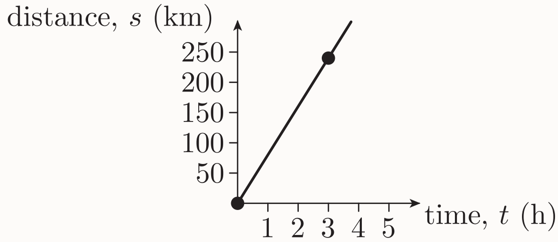


Figure 15 A distance–time graph for the journey of a car

This graph includes only non-negative values of t and s, as the times elapsed and the distances travelled since the start of the journey are all non-negative.

You can calculate the gradient of the graph in Figure 15 by choosing two points on it in the usual way. The two points marked on the graph have coordinates $(0, 0)$ and $(3, 240)$, so

$$\text{gradient} = \frac{(240 - 0)\,\text{km}}{(3 - 0)\,\text{h}} = 80\,\text{km/h}.$$

Notice that, because the numbers on the axes of the graph have units, the gradient also has units. Since the rise is measured in kilometres (km) and the run is measured in hours (h), the units of the gradient are kilometres divided by hours, that is, kilometres per hour (km/h). In general, the units of the gradient of a graph are the units on the vertical axis divided by the units on the horizontal axis.

The units are shown in the calculation above to demonstrate this fact, but in general it's not necessary to include units in the calculation of a gradient. You just need to state the units in the final answer.

Units such as km/h, which are obtained by combining simpler units (in this case km and h), are called **derived units**.

The fact that the gradient of the graph in Figure 15 is 80 km/h tells you that the distance travelled by the car changes by 80 kilometres for each hour of the journey. That is, it tells you that the car travels 80 kilometres in each hour, or, in other words, that the speed of the car is 80 kilometres per hour. The fact that the graph is a straight line tells you that the car is travelling at a *constant* speed.

In general, if a graph is a straight line, then it means that the quantity on the vertical axis is changing at a *constant rate* with respect to the quantity on the horizontal axis. The gradient of the graph tells you how many units the quantity on the vertical axis changes for every one unit that the quantity on the horizontal axis changes. In other words, the gradient of the graph is the **rate of change** of the quantity on the vertical axis **with respect to** the quantity on the horizontal axis. So, for example, the gradient of a distance–time graph is the rate of change of distance with respect to time, which is speed.

Here's another example of a linear model. Consider the graph in Figure 16. It represents the change in the concentration of a chemical as it undergoes a chemical reaction over time. The concentration is denoted by c and measured in a unit called the *molar* (M), and time is denoted by t and measured in seconds (s). A chemical reaction in which the concentration falls linearly, as in Figure 16, is called a *zero-order* reaction.

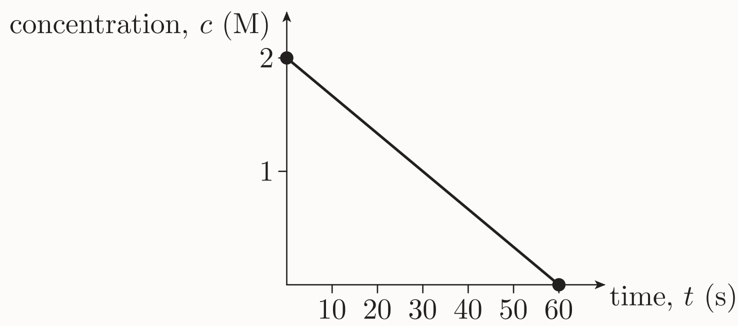


Figure 16 The relationship between the concentration of a reacting chemical and time

The gradient of the graph in Figure 16 is negative, because the concentration of the chemical *decreases* as time goes on. Notice that the graph includes values of t only between 0 and 60, and the corresponding values of c. This is because the times since the beginning of the reaction are all positive, and after 60 seconds the concentration of the chemical has decreased to 0, so the model does not apply for later times.

The gradient of the graph in Figure 16 can be calculated as follows. The two points marked in Figure 16 are $(0, 2)$ and $(60, 0)$, so

$$\text{gradient} = \frac{0-2}{60-0} = \frac{-2}{60} = -\frac{1}{30}\,\text{M/s}.$$

This tells you that the rate of change of the concentration with respect to time is approximately $-0.03\,\text{M/s}$. In other words, in each second that passes, the concentration of the chemical decreases by approximately $0.03\,\text{M}$.

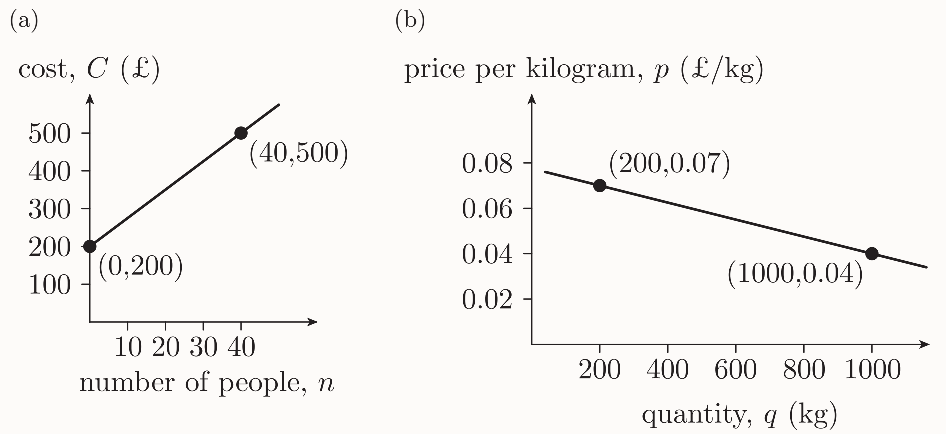
In the next activity, you're asked to find and interpret the gradients of two straight-line graphs.

Activity 9 *Calculating gradients of real-life graphs*

For each of the graphs below, find the gradient in appropriate units and explain what the gradient represents.

Graph (a) represents the relationship between the number n of people attending a meeting and the cost C (in £) of hiring the meeting room (including lunch and refreshments for the attendees).

Graph (b) represents the relationship between the price p (in £/kg) charged for building sand and the quantity q (in kg) purchased by a customer.



The intercepts of real-life graphs often have practical interpretations too. For example, look back at the graph in Figure 16, which represents the decrease in the concentration of a reacting chemical over time. The intercept on the c-axis is $2\,\mathrm{M}$, which is the concentration of the chemical at the start of the chemical reaction. The intercept on the t-axis is $60\,\mathrm{s}$, which is the time at which the concentration of the chemical falls to zero.

Activity 10 *Interpreting intercepts of real-life graphs*

Look back at the graph in Activity 9(a). State the vertical intercept, and explain what it means.

The intercepts of real-life graphs don't always have useful meanings. For example, look back at the graph in Activity 9(b). Notice that it has been drawn as a line segment that doesn't cross either of the axes. This is because the model isn't valid for the values of q and p that don't correspond to points on this line segment. In particular, it isn't valid when $q = 0$ and $p = 0$, so the intercepts have no meaning in this case.

When you're working with a linear model, it's usually helpful to use the equation of the associated straight-line graph. You can often find the equation using the methods that you practised earlier in this section – you use the variables that represent the real-life quantities in place of the standard variables x and y. For example, for the distance–time graph in Figure 15, the variables on the horizontal and vertical axes are t and s, respectively, so you use t in place of x and s in place of y. The gradient of this graph is 80 km/h and the y-intercept is 0 km, so the equation of the graph is

$$s = 80t. \tag{5}$$

An alternative way to obtain the equation of the graph in this case is to simply use the familiar relationship

distance travelled = constant speed × time elapsed.

When you use relationships like this, it's important to remember that they're valid only if the units in which the quantities are measured are *consistent*. For example, for the relationship above, the units in which time is measured must be the same as the units of time contained within the derived units in which speed is measured. Here are some examples of consistent sets of units:

- time in seconds, speed in metres per second, distance in metres
- time in minutes, speed in metres per minute, distance in metres
- time in hours, speed in kilometres per hour, distance in kilometres.

The units used for the quantities in equation (5) are consistent since the number 80 represents the speed of the car in km/h, the variable t represents time in hours, and the variable s represents distance in km.

As another example of consistent units, consider the equation representing the line in Figure 16 on page 140, which is

$$c = -\frac{1}{30}t + 2. \tag{6}$$

The units used for the quantities in this equation are consistent since all three terms, c, $-\frac{1}{30}t$ and 2, have the same units, namely molars (M). Note that the gradient $-\frac{1}{30}$ is measured in M/s, and the time t in seconds.

Activity 11 *Finding equations of real-life graphs*

(a) Using your answers to Activity 9, find the equations of the graphs in Activity 9.

(b) Use these equations to find the following.

  (i) The maximum number of people that can be accommodated in the meeting room, if the maximum budget for the meeting is £560.

  (ii) The price per kilogram of building sand that corresponds to a quantity of 500 kg.

Displacement and velocity

When you're considering a moving object, such as a car travelling along a road, it's often helpful to consider not the *distance* that the object has travelled, but its *displacement* from a particular point.

To see what this means, consider any object that's moving along a straight line – we'll consider only motion along a straight line here, for simplicity. We choose any point on the straight line to be a reference point and we choose one of the two directions along the line to be the positive direction. Then the object's **displacement** from the reference point is its distance from that point, with a positive or negative sign to indicate the direction from that point.

For example, consider the straight line in Figure 17. The reference point has been chosen to be the point marked R, and the positive direction has been chosen to be rightwards. An object at position A has a displacement of 3 cm, while an object at position B has a displacement of -5 cm.

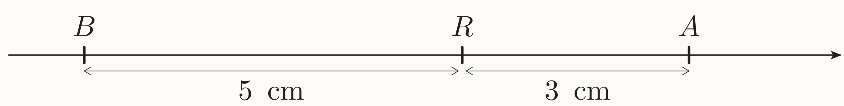


Figure 17 Positions along a straight line

Imagine placing your pen tip at R, moving it to A, and then to B. The *distance moved* by the pen tip is the distance from R to A, plus the distance from A to B, which is $3\,\text{cm} + 8\,\text{cm} = 11\,\text{cm}$. However, the final *displacement* of the pen tip is -5 cm, since it is at B.

To see how the idea of displacement works in practice, suppose that a car is travelling along a straight road, and that a particular reference point R

on the road and direction along it have been chosen, as illustrated in Figure 18.

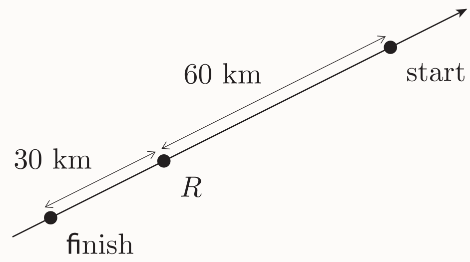


Figure 18 A reference point and a direction on a straight road

The graph in Figure 19 shows the relationship between the time that has elapsed since the start of the car's journey, and the car's displacement from the reference point. You can see that the car has displacements of 60 km and −30 km at the start and end of its journey, respectively, and that it drives past the reference point 1 hour after the start of its journey. A graph like the one in Figure 19, in which displacement is plotted against time, is known as a **displacement–time graph**.

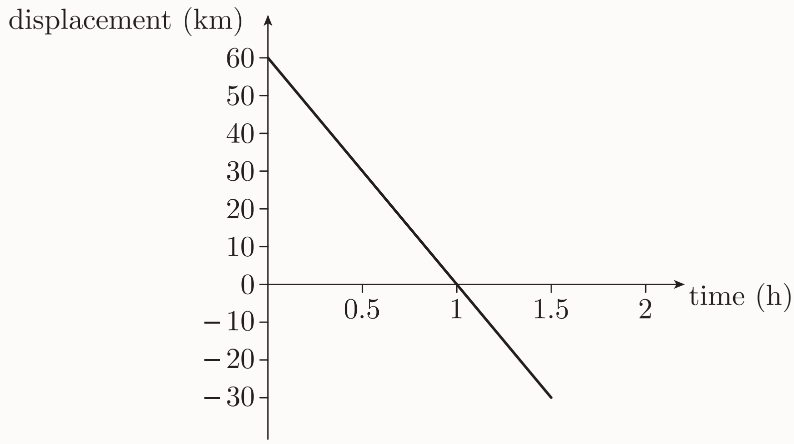


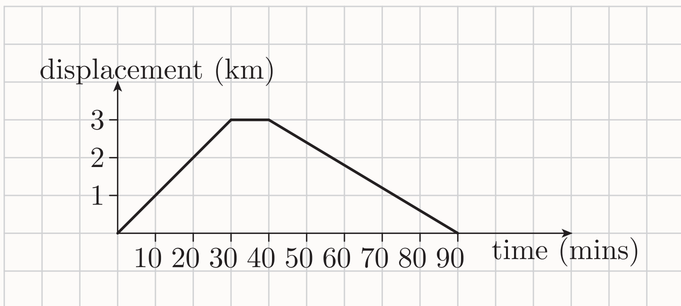
Figure 19 A displacement–time graph for a car's journey

The gradient of the graph in Figure 19 is −60 km/h. This tells you that in each hour the displacement of the car changes by −60 km. In other words, the car is moving at a speed of 60 km/h in the negative direction, that is, in the direction opposite to the direction chosen as the positive direction in Figure 18.

The gradient of the displacement–time graph of an object moving in a straight line is called the *velocity* of the object. In other words, the **velocity** of an object is its *rate of change of displacement with respect to time*. So the velocity of an object that's moving along a straight line is the same as its speed, except that it has a positive or negative sign to indicate the direction in which the object is moving along the line.

Activity 12 *Working with a displacement–time graph*

The displacement–time graph below, which consists of three line segments, represents a woman's walk along a straight path. The woman walks at a constant speed to a bench, sits there for some minutes, and then returns, again at a constant speed, to her starting point. The reference point has been chosen to be the point where she begins her walk, the positive direction has been chosen to be the direction in which she first walks, and time is measured from the time when she begins her walk.



(a) What is the displacement of the bench from the woman's starting point?

(b) How long does the woman remain at the bench?

(c) Calculate the woman's velocity as she walks to the bench.

(d) Calculate her velocity as she walks back to her starting point.

(e) What is the woman's speed as she walks to the bench, and what is her speed as she walks back?

(f) Find the equation of the line segment that represents the first part of the woman's walk.

(g) Use the equation that you found in part (f) to find what the woman's displacement would be after 50 minutes if she hadn't stopped at the bench but had instead carried on walking at the same speed and in the same direction.

Derived units, such as km/h, are often written using *index notation*. For example, the unit km/h can be written as $\mathrm{km\,h^{-1}}$, since $1/\mathrm{h} = \mathrm{h^{-1}}$. This way of writing derived units makes no difference to the way that you read them: for example, $\mathrm{km\,h^{-1}}$ is read as 'kilometres per hour'. Derived units are often written in index notation throughout the rest of this module.

3 Intersection of lines

In this section you'll revise how to find the point at which two lines cross. This point is called the **point of intersection** of the lines.

3.1 Simultaneous equations

Here is an example where we want to find the point of intersection of two lines. Consider Figure 20, which shows the displacement–time graphs for two cars travelling on the same road, drawn on the same coordinate axes. Both graphs are straight lines. Notice that, as time goes on, the displacement of one car increases and the displacement of the other car decreases, which tells you that they're travelling in opposite directions. At the point of intersection of the lines, the displacements of the two cars are equal at the same time.

So this point corresponds to the time and displacement when the two cars pass each other. How can we find this point of intersection?

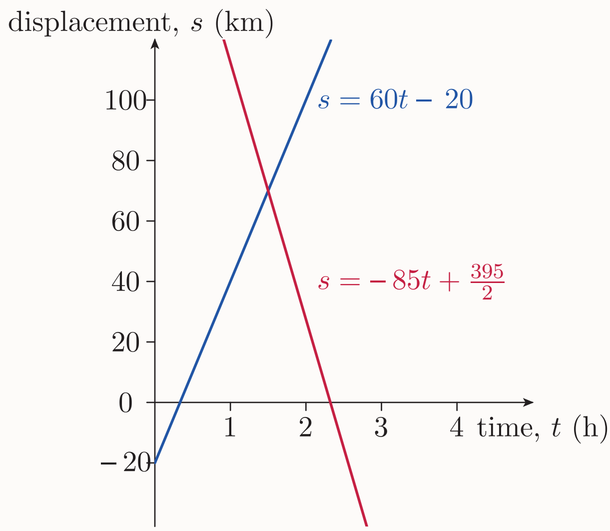


Figure 20 Displacement–time graphs for two cars travelling in opposite directions on the same road

You could find approximate values for the time at which the two cars pass each other, and the displacement at which this happens, by reading off the point of intersection from the graph. Alternatively, you can find accurate values by working with the equations of the lines and using algebra.

The equation of the displacement–time graph of the first car is

$$s = 60t - 20, \tag{7}$$

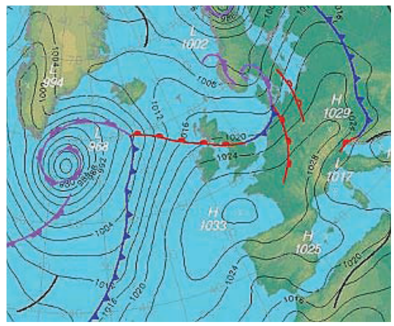
where s is the displacement in km and t is the time in hours. The corresponding equation for the second car is

$$s = -85t + \frac{395}{2}. \tag{8}$$

The coordinates (t, s) of the point of intersection must satisfy equation (7), since the point lies on the graph of that equation, and must also satisfy equation (8), since it also lies on the graph of this equation.

So to find the point of intersection you have to find a pair of values of t and s that satisfy *both* equations. The process of finding these values is known as solving the equations **simultaneously**, and in this context the two equations are called **simultaneous equations**. Since the equations are *linear* equations, they're **simultaneous linear equations**. The variables t and s are unknowns, and any pair of values of t and s that satisfy both equations is called a **solution** of the simultaneous equations.

Simultaneous equations are not restricted to two equations and two unknowns; systems containing larger numbers of equations and unknowns frequently arise in practical situations.



The calculation of weather forecasts requires the solution of millions of simultaneous equations.

In the next subsection you'll revise two methods for solving simultaneous linear equations in two unknowns. You'll meet a third method in Unit 9.

3.2 Solving simultaneous equations

In this subsection we'll assume that the simultaneous equations that you want to solve are linear and have been rearranged so that the terms in the unknowns are on the left, and the constant terms are on the right. This is a standard way to present simultaneous equations.

For example, rearranging equations (7) and (8) in this way, and also clearing the fractions in equation (8), gives the pair of equations

$$s - 60t = -20,$$
$$2s + 170t = 395.$$

Here's the first method for solving simultaneous linear equations.

Substitution method

This is described in the following strategy and illustrated in Example 6.

Strategy:
To solve simultaneous equations: substitution method

1. Rearrange one of the equations, if necessary, to obtain a formula for one unknown in terms of the other.
2. Use this formula to substitute for this unknown in the other equation.
3. You now have an equation in one unknown. Solve it to find the value of that unknown.
4. Substitute this value into an equation involving both unknowns to find the value of the other unknown.

(Check: confirm that the two values satisfy the original equations.)

Example 6 *Solving simultaneous equations by substitution*

Use the substitution method to solve the following simultaneous equations.

$$s - 60t = -20$$
$$2s + 170t = 395$$

Solution

Label the equations, so you can refer to them easily.

The equations are

$$s - 60t = -20, \qquad (9)$$
$$2s + 170t = 395. \qquad (10)$$

Rearrange one of the equations to express one unknown in terms of the other. The simplest choice here is to rearrange equation (9) to express s in terms of t. Any other choice of equation or unknown leads to fractions.

Rearranging equation (9) gives

$$s = 60t - 20. \qquad (11)$$

Use this formula to substitute for s in the other equation.

Substituting in equation (10) gives

$$2(60t - 20) + 170t = 395.$$

Solve this equation to find t.

$$120t - 40 + 170t = 395$$
$$290t = 435$$
$$t = \frac{435}{290} = \frac{3}{2}$$

Substitute this value of t into an equation containing s, and solve it to find s.

Substituting into equation (11) gives

$$\begin{aligned} s &= 60t - 20 \\ &= 60 \times \tfrac{3}{2} - 20 \\ &= 90 - 20 \\ &= 70. \end{aligned}$$

So the solution is $s = 70$, $t = \frac{3}{2}$.

It's a good idea to check that these values are correct by substituting them into the original equations.

When you solve simultaneous equations, it's fine to label them (1) and (2), every time. If you want to refer to another equation in your working, then you can label it (3), and so on. (The simultaneous equations in Example 6 are labelled (9) and (10) just so that they're in sequence with the other labelled equations in this unit.)

Activity 13 *Solving simultaneous equations by substitution*

Use the substitution method to solve the following pairs of simultaneous equations.

(a) $s - 5t = -3$
$s + 3t = 13$

(b) $x + 4y = 2$
$2x + 5y = 3$

Here's a variation of the substitution method that's sometimes useful. You've seen that the first step of the method is to rearrange one of the equations to express one unknown in terms of the other. If instead you start by rearranging *both* equations to obtain *two* formulas, each of which expresses one unknown (the same in both cases) in terms of the other unknown, then you can obtain an equation in one unknown by equating the right-hand sides of both formulas. You can then proceed in the usual way.

For example, notice that both the equations of the lines in Figure 20 were obtained originally in equations (7) and (8) as formulas for s in terms of t:

$$s = 60t - 20,$$
$$s = -85t + \frac{395}{2}.$$

You can equate the two right-hand sides to give the following equation in t:

$$60t - 20 = -85t + \frac{395}{2}.$$

You can then solve this equation to find t, and substitute this value into one of the original equations to find s.

Elimination method

Often when you want to solve a pair of simultaneous equations, it's difficult to use the substitution method without introducing fractions. In most cases it's better to use the method described in the following strategy, which is illustrated in Example 7.

Strategy:
To solve simultaneous equations: elimination method

1. Multiply one or both of the equations by suitable numbers, if necessary, to obtain two equations that can be added or subtracted to eliminate one of the unknowns.
2. Add or subtract the equations to eliminate this unknown.
3. You now have an equation in one unknown. Solve it to find the value of that unknown.
4. Substitute this value into an equation involving both unknowns to find the value of the other unknown.

(Check: confirm that the two values satisfy the original equations.)

Example 7 *Solving simultaneous equations by elimination*

Use the elimination method to solve the following simultaneous equations.

$$5u - 40v = 155$$
$$2u + 9v = 12$$

Solution

Label the equations.

The equations are

$$5u - 40v = 155, \quad (12)$$
$$2u + 9v = 12. \quad (13)$$

Multiply the first equation by 2, and multiply the second equation by 5, to obtain two equations in which u has the same coefficient. (Alternatively, you could multiply the first equation by 9 and the second equation by 40, to obtain two equations in which v can be eliminated, but that involves harder arithmetic.)

Multiplying equation (12) by 2 and equation (13) by 5 gives

$$10u - 80v = 310, \quad (14)$$
$$10u + 45v = 60. \quad (15)$$

Subtract equation (14) from equation (15) to eliminate u.

Subtracting equation (14) from equation (15) gives

$$10u - 10u + 45v + 80v = 60 - 310$$

Solve this equation to find v.

$$125v = -250$$
$$v = -2.$$

Substitute this value of v into an equation containing u, say equation (12), and solve it to find u.

Hence

$$\begin{aligned} 5u &= 40v + 155 \\ &= 40 \times (-2) + 155 \\ &= -80 + 155 \\ &= 75 \\ u &= \tfrac{1}{5} \times 75 = 15. \end{aligned}$$

So the solution is $u = 15$, $v = -2$.

The 'check' referred to in the strategy has not been included in the solution to Example 7. You should check your answers wherever possible, but you don't have to write the check down as part of your solution.

The aim of the first step in the solution to Example 7 was to find two equations in which u has the *same* coefficient. This was so that *subtracting* the equations will eliminate u. An alternative strategy in the first step is to find two equations in which the coefficients of u are *negatives of each other*, such as

$$-10u + 80v = -310,$$
$$10u + 45v = 60.$$

(These equations are obtained by multiplying the first of the original equations by -2 and the second by 5.) Then *adding* the equations will eliminate u.

Activity 14 *Solving simultaneous equations by elimination*

Use the elimination method to solve the following pairs of simultaneous equations.

(a) $s + 6t = 20$
$2s + 7t = 35$

(b) $2x + 3y = -5$
$3x - 2y = 12$

(c) $3u - v = -\frac{5}{2}$
$2u + 5v = 21$

3.3 The number of solutions of simultaneous equations

Each pair of simultaneous linear equations in the last subsection had exactly one solution, but it's also possible for a pair of simultaneous linear equations to have no solutions, or infinitely many solutions.

To see this, consider the lines that are the graphs of the two equations. They might intersect in a single point, or they might be parallel, or they might be the same line. The three possibilities are illustrated in Figure 21.

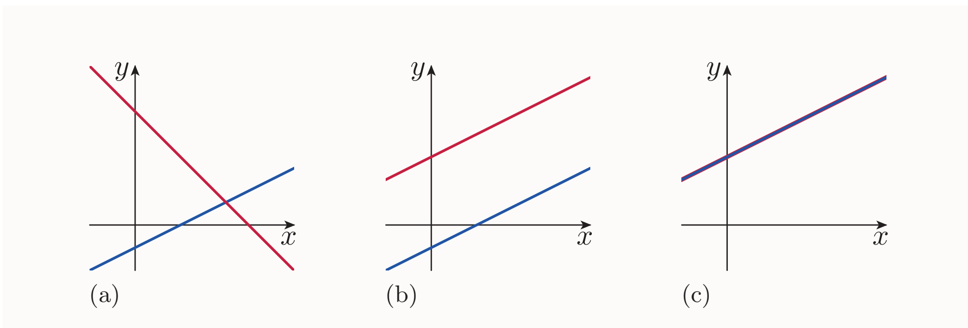


Figure 21 Three possible configurations of two lines

If the two lines have different gradients, as illustrated in Figure 21(a), then they intersect at a single point. So there is exactly one solution.

If the two lines have the same gradient but different y-intercepts, as illustrated in Figure 21(b), then they are parallel and hence do not intersect. In this case, you will not be able to solve the simultaneous equations. For example, the simultaneous equations

$$y - 2x = 3,$$
$$y - 2x = 4$$

have no solution; no matter what values of x and y are chosen, it is impossible for $y - 2x$ to be equal to both 3 and 4.

If the two lines have the same gradient and the same y-intercept, as illustrated in Figure 21(c), then the graphs of the two equations are identical. So every point that satisfies the first equation also satisfies the second equation, and hence there are infinitely many solutions. This situation occurs if one equation can be obtained by rearranging the other. For example, consider the simultaneous equations

$$y - 2x = 3,$$
$$2y - 4x = 6.$$

The second equation is obtained by multiplying the first equation by 2, so they represent the same line, and hence they have infinitely many solutions.

Activity 15 *Finding the number of solutions of simultaneous equations*

Determine whether each of the following pairs of simultaneous equations has one, infinitely many or no solutions. If there is one solution, find it.

(a) $y - 3x = -2$
$2y + x = 10$

(b) $y = 4x - 5$
$2y - 8x = 10$

(c) $4y = 2x + 6$
$2y - x = 3$

In practice, when you're solving simultaneous equations there's no need to start by investigating the number of solutions. You can just try to solve the equations, and see what happens!

4 Quadratics

In this section we turn our attention to equations of the form

$$y = ax^2 + bx + c, \tag{16}$$

where a, b and c are constants and $a \neq 0$. An expression of the form of the right-hand side of this equation is called a **quadratic expression**, or simply a **quadratic**. An equation of the form

$$ax^2 + bx + c = 0, \tag{17}$$

where a, b and c are constants and $a \neq 0$, is called a **quadratic equation**. Since equation (17) is obtained by putting $y = 0$ in equation (16), its solutions are the x-intercepts of the graph of equation (16).

In the first subsection of this section, you'll revise the shapes of the graphs of equations of form (16). Then in the next few subsections you'll revise two useful techniques for rearranging quadratic expressions, namely *factorisation* and *completing the square*. You'll see how each of these techniques leads to a method for solving quadratic equations. You'll also revise a third method for solving quadratic equations, the *quadratic formula*.

It's important to make sure that you're familiar with all three of these methods for solving quadratic equations, as they're all used later in the module. You'll also need to use the techniques of factorisation of quadratics and completing the square for purposes other than solving quadratic equations.

Finally in this section you'll learn how to sketch the graphs of equations of form (16), and you'll see a few applications of quadratics.

Note that the reason for the condition '$a \neq 0$' in the definitions above is that, if $a = 0$, then the expression $ax^2 + bx + c$ reduces to $bx + c$, which is a *linear* expression.

4.1 Quadratic graphs

You have already seen one quadratic graph in this unit. In Activity 2 on page 123 you were asked to draw the graph of $y = x^2 + 2x + 2$ by first plotting points. In the next activity you can explore the shape of the graph of the equation $y = ax^2 + bx + c$ for various values of a, b and c.

Activity 16 *Investigating quadratic graphs*

Use the *Quadratic graphs* applet to investigate how the shape of the graph of the equation $y = ax^2 + bx + c$ changes as you vary the values of a, b and c.

In Activity 16, you should have seen that no matter what the values of a, b and c are (as long as $a \neq 0$), the graph of the equation $y = ax^2 + bx + c$ always has the same type of shape, which is called a **parabola**. If a is positive, then the parabola is **u-shaped**, as shown in Figure 22(a). If a is negative, then it is **n-shaped**, as shown in Figure 22(b). In both cases, the parabola has a vertical *axis of symmetry*.

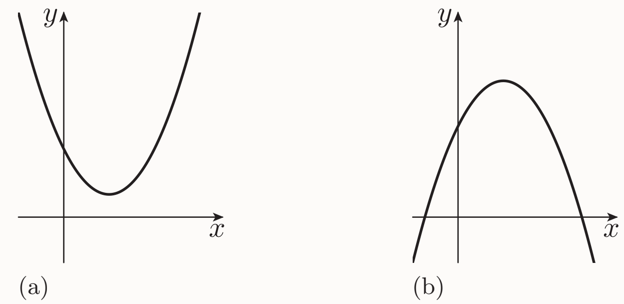


Figure 22 Typical graphs of $y = ax^2 + bx + c$ when (a) a is positive (b) a is negative

> The word 'parabola' was first used for curves like these by the Greek geometer and astronomer Apollonius of Perga (262BC–190BC), though the shape itself was discovered even earlier.

You should have seen that changing the values of a, b and c can change the position of the parabola, and can stretch or squash it in a direction parallel to one of the axes. In particular, as you might have expected from your study of straight lines, changing the value of c on its own moves the parabola vertically, but keeps the shape the same.

The lowest point on a u-shaped parabola, or the highest point on an n-shaped parabola, is called the **vertex** of the parabola. The parabola gets steeper and steeper on each side of the vertex, but never becomes vertical.

A u-shaped or n-shaped parabola can have two, one or no x-intercepts, as illustrated for u-shaped parabolas in Figure 23. It always has exactly one y-intercept, because there is exactly one value of y for each value of x, including $x = 0$.

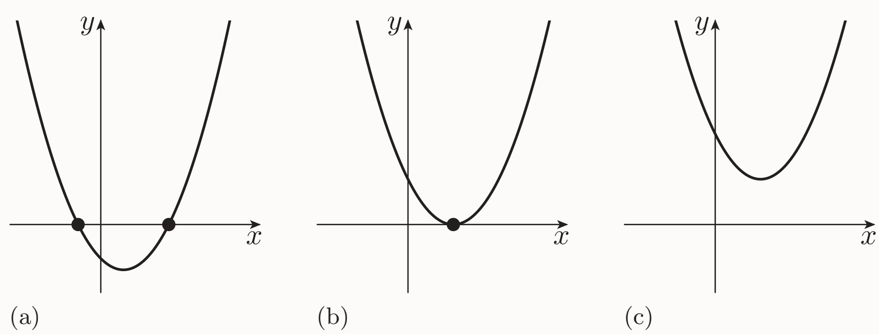
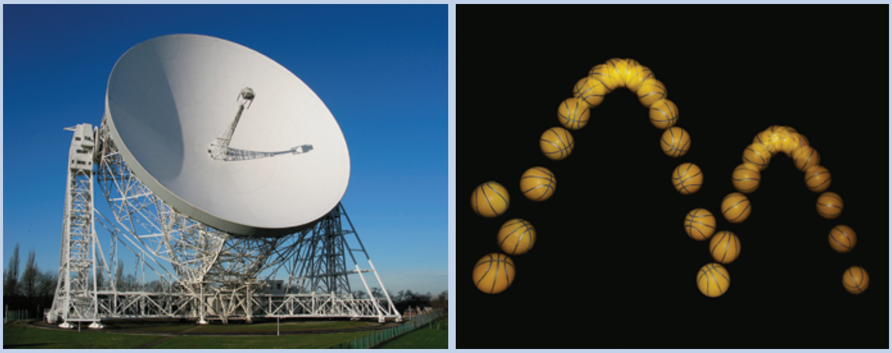


Figure 23 A quadratic graph can have two, one or no x-intercepts

Parabolas have many practical uses, mainly due to the reflecting properties of surfaces with parabolic cross-sections. If such a surface is made of a suitable reflecting material, then all light waves or radio waves that arrive at it by travelling along a direction parallel to its axis of symmetry are reflected to pass through a single point, known as the *focus*. Similarly, any wave that travels from the focus to the surface is reflected to travel along a line parallel to the axis of symmetry. Parabolic reflecting surfaces can be found in car headlights, reflecting and radio telescopes, and satellite dishes. The path of a ball thrown through the air is also parabolic (as long as the effects of air resistance are negligible).



The parabolic dish of a radio telescope and the parabolic path of a ball

4.2 Factorising quadratic expressions

In this subsection you'll revise how to *factorise* a quadratic expression. This usually involves more than simply taking out a common factor, which you revised in Unit 1.

If you take any two linear expressions and multiply them together, then you get a quadratic expression. For example,

$$(2x + 3)(x - 2) = 2x^2 - x - 6.$$

Factorisation of a quadratic is the reverse of this process – it means writing the quadratic as the product of two linear expressions. For example, factorising the quadratic $2x^2 - x - 6$ means writing it as $(2x + 3)(x - 2)$ or, equivalently, as $(x - 2)(2x + 3)$.

Here, you'll see how to factorise certain quadratics *with integer coefficients*. For example, the quadratic $2x^2 - x - 6$ is of this type, as its coefficients 2, -1 and -6 are integers.

You'll see how to factorise such a quadratic into a product of two linear expressions *whose coefficients are also integers*, where this is possible. We'll refer to factorisation of this sort as factorisation *using integers*. For example, you've just seen that the quadratic $2x^2 - x - 6$ can be factorised using integers, since all the coefficients in the expression $(2x+3)(x-2)$ are integers. However, many quadratics with integer coefficients can't be factorised using integers.

Factorising quadratics in which x^2 has coefficient 1

We'll begin by looking at quadratics in which x^2 has coefficient 1, such as

$$x^2 - 2x - 15 \quad \text{and} \quad x^2 - 8x + 12.$$

These are usually easier to factorise than other quadratics.

If it's possible to factorise a quadratic of this type using integers, then the factorisation must be of the form

$$(x \qquad)(x \qquad), \tag{18}$$

where the two gaps are filled by constant terms. Such an arrangement of brackets with gaps for the constants is called a 'framework' in this unit.

To factorise the quadratic, you have to find two numbers (each a positive or negative integer, or zero) to fill the gaps in the framework (18), such that when you multiply out the brackets you get the quadratic that you started with. For example, if your quadratic is $x^2 - 2x - 15$, then the two numbers that you're looking for are $+3$ and -5, because

$$(x+3)(x-5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15.$$

You can use two facts to work out what the two numbers must be.

First, the two numbers must multiply together to give the constant term in the quadratic. To see why, consider what happens when you multiply out an expression of form (18). For example,

$$(x+3)(x-5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15.$$

The two numbers are multiplied to give the constant term in the quadratic.

Second, the two numbers must add together to give the coefficient of x in the quadratic. To see why, again consider what happens when you multiply out an expression of form (18).

For example,

$$(x+3)(x-5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15.$$

The two numbers become coefficients of x.

They are added to give the overall coefficient of x.

The next example demonstrates an efficient strategy for using these facts to find the two numbers.

This strategy is quick to use once you've had some practice with it, and it can be adapted to deal with quadratics in which x^2 does *not* have coefficient 1, as you'll see shortly. However, there are other strategies – there's a different one in the document *An alternative strategy for factorising quadratics* on the website that you might like to look at, and you may have learned another one elsewhere. It's fine to use any strategy that works for you.

You might find it easier to understand the strategy if you hear someone explain it to you, so it's a good idea to watch the tutorial clip for Example 8. The strategy is summarised after the example.

Example 8 *Factorising quadratics in which x^2 has coefficient 1*

Factorise the following quadratics.

(a) $x^2 - 8x + 12$ (b) $x^2 + 2x - 15$

Solution

(a) Start by writing out the framework.

$$x^2 - 8x + 12 = (x \quad)(x \quad)$$

The numbers to go in the gaps must have product 12 (the constant term) and sum −8 (the coefficient of x). List the factor pairs of 12:

$$1, 12; \quad -1, -12; \quad 2, 6; \quad -2, -6; \quad 3, 4; \quad -3, -4.$$

Find a pair in this list with sum −8. The only such pair is −2, −6.

$$x^2 - 8x + 12 = (x - 2)(x - 6)$$

You can check the answer by multiplying out the brackets.

(b) Write out the framework.

$$x^2 + 2x - 15 = (x \quad)(x \quad)$$

The numbers to go in the gaps must have product −15 and sum 2.
List the factor pairs of −15:

$$1, -15; \quad -1, 15; \quad 3, -5; \quad -3, 5.$$

The only pair with sum 2 is −3, 5.

$$x^2 + 2x - 15 = (x - 3)(x + 5)$$

The strategy demonstrated in Example 8 is summarised below.

Strategy:
To factorise a quadratic of the form $x^2 + bx + c$

1. Start by writing

   $$x^2 + bx + c = (x \qquad)(x \qquad).$$

2. Find the factor pairs of c (including both positive and negative factors).
3. Choose a factor pair with sum b.
4. Write your factor pair p, q in position:

   $$x^2 + bx + c = (x + p)(x + q).$$

For some quadratics, this strategy doesn't give a factorisation. When this happens, it means that the quadratic can't be factorised using integers. For example, if you try to apply the strategy to the quadratic $x^2 + 2x + 3$, then you find that neither of the factor pairs of 3, namely $1, 3$ and $-1, -3$, have sum 2. So this quadratic can't be factorised using integers.

As you become more familiar with factorising quadratics, you'll find ways to apply the strategy above more efficiently. For example, rather than writing down *all* the factor pairs before considering their sums, you can consider the sum of each pair as you go along, and stop once you've found a pair with the required sum. Similarly, if you're trying to factorise a quadratic $x^2 + bx + c$ where both b and c are *positive*, then you need consider only *positive* factor pairs of c, since the sum of two negative numbers can't be the positive number b.

Activity 17 *Factorising quadratics of the form $x^2 + bx + c$*

Factorise the following quadratics. (They *can* all be factorised.)

(a) $x^2 + 5x + 6$ (b) $x^2 - 8x + 15$ (c) $x^2 + 4x - 5$

(d) $x^2 - 2x - 35$ (e) $x^2 - 6x + 9$ (f) $x^2 - 6x + 8$

(g) $y^2 - 7y - 18$ (h) $u^2 + 4u + 4$ (i) $p^2 - 4p - 12$

(j) $s^2 + s - 30$ (k) $v^2 + 5v - 50$ (l) $r^2 - 10r + 16$

Factorising quadratics in which x^2 doesn't have coefficient 1

We'll now look at how to factorise quadratics in which x^2 doesn't have coefficient 1, such as $6x^2 - 11x - 35$.

The first thing to do when you want to factorise a quadratic like this is to check whether there are any numerical common factors. If there are, then take them out. For example, if the quadratic is $4x^2 + 2x - 2$, then write it as

$$4x^2 + 2x - 2 = 2(2x^2 + x - 1).$$

Also, if the coefficient of x^2 is negative, then take out a factor of -1. For example, if the quadratic is $-3x^2 + x + 2$, then write it as

$$-3x^2 + x + 2 = -(3x^2 - x - 2).$$

Once you've done these things, you can focus on factorising the simpler quadratic inside the brackets.

Because you can take out a factor of -1 if necessary, you only ever need to factorise quadratics in which the coefficient of x^2 is positive. You can do that by using the strategy that you saw earlier, with two adaptations when the coefficient of x^2 is not 1.

The first adaptation is needed to deal with the fact that there can be more than one possibility for the initial framework that you write out. Each pair of positive factors of the coefficient of x^2 gives you a possible framework. For example, if the quadratic that you want to factorise is

$$6x^2 + 11x - 35,$$

then there are two possible frameworks, namely

$$(6x \qquad)(x \qquad) \quad \text{and} \quad (2x \qquad)(3x \qquad).$$

To deal with this, you apply the factorisation strategy to each of the possible frameworks in turn, until you find one that gives you a factorisation. If none of the possibilities gives you a factorisation, then it means that the quadratic can't be factorised using integers.

The second adaptation is a little more complicated. In the earlier strategy, once you've written out the framework, you consider the factor pairs of the constant term, and for each factor pair you consider whether its sum is equal to the coefficient of x. In the adapted strategy, you still consider the factor pairs of the constant term, but you don't consider the sum of each factor pair, as this doesn't tell you anything useful when the coefficient of x^2 isn't 1. Instead, you consider directly whether each factor pair leads to the correct term in x when you multiply out. This process is explained more fully in the following example.

The adapted strategy might seem quite complicated when you first meet it, but it should seem more straightforward after you've practised it a few times. Again, you might find it easier to understand if you hear someone explain it to you, so it's a good idea to watch the tutorial clip for Example 9.

Example 9 *Factorising a quadratic in which x^2 doesn't have coefficient 1*

Factorise the quadratic $2x^2 - 7x - 15$.

Solution

There are no numerical common factors that can be taken out. The coefficient of x^2 is a prime number, so there's only one possible framework.

$$2x^2 - 7x - 15 = (2x \qquad)(x \qquad)$$

Consider the factor pairs of the constant term -15:

$$1, -15; \quad -1, 15; \quad 3, -5; \quad -3, 5.$$

Each factor pair can go in the brackets in two different ways, giving eight possible cases, as follows.

$$\begin{array}{ll} (2x+1)(x-15) & (2x-15)(x+1) \\ (2x-1)(x+15) & (2x+15)(x-1) \\ (2x+3)(x-5) & (2x-5)(x+3) \\ (2x-3)(x+5) & (2x+5)(x-3) \end{array}$$

For each case, calculate the term in x that you obtain when you multiply out the brackets.

$$\begin{array}{llll} (2x+1)(x-15) & -29x & (2x-15)(x+1) & -13x \\ (2x-1)(x+15) & 29x & (2x+15)(x-1) & 13x \\ (2x+3)(x-5) & -7x & (2x-5)(x+3) & x \\ (2x-3)(x+5) & 7x & (2x+5)(x-3) & -x \end{array}$$

Identify the case that gives $-7x$.

$$2x^2 - 7x - 15 = (2x+3)(x-5)$$

You can check the answer to Example 9 by multiplying out the brackets.

The solution to Example 9 includes some working to help with finding the factorisation. You might find it helpful to write down working like this while you're getting used to factorising quadratics, but after a while you'll probably find that you can usually do it in your head. So, when you factorise a quadratic, you don't need to write down any working – it's fine to just write down the quadratic and its factorisation, like this:

$$2x^2 - 7x - 15 = (2x+3)(x-5).$$

Here's a summary of the method demonstrated in the example above.

Strategy:
To factorise a quadratic of the form $ax^2 + bx + c$

1. Take out any numerical common factors. If the coefficient of x^2 is negative, also take out the factor -1. Then apply the steps below to the quadratic inside the brackets.
2. Find the positive factor pairs of a, the coefficient of x^2. For each such factor pair d, e write down a framework $(dx \quad)(ex \quad)$.
3. Find all the factor pairs of c, the constant term (including both positive and negative factors).
4. For each framework and each factor pair of c, write the factor pair in the gaps in the framework in both possible ways.
5. For each of the resulting cases, calculate the term in x that you obtain when you multiply out the brackets.
6. Identify the case where this term is bx, if there is such a case. This is the required factorisation.

As with the earlier strategy, if this strategy doesn't lead to a factorisation, then the quadratic can't be factorised using integers.

Activity 18 *Factorising quadratics of the form $ax^2 + bx + c$*

Factorise the following quadratics. (They *can* all be factorised.)

(a) $5x^2 + 13x - 6$	(b) $3x^2 + 16x + 5$	(c) $6x^2 - 11x + 3$
(d) $5x^2 - 8x - 21$	(e) $18x^2 + 9x - 2$	(f) $4x^2 - 8x + 3$
(g) $4p^2 - 19p - 5$	(h) $6u^2 + 11u - 35$	(i) $4t^2 + 4t + 1$
(j) $9v^2 - 12v + 4$	(k) $-4s^2 + 4s + 3$	(l) $12y^2 - 10y - 2$

There are two special types of quadratic that can be factorised more easily than those that you have seen so far in this subsection. You should always check whether your quadratic is one of these before you embark on either of the two strategies above.

Quadratics with no constant term

These can be factorised by taking x out as a common factor. For example,

$$x^2 + 4x = x(x + 4),$$
$$3x^2 - 6x = 3(x^2 - 2x) = 3x(x - 2).$$

Differences of two squares

As you saw in Unit 1, a **difference of two squares** is any expression of the form

$$A^2 - B^2,$$

where A and B are subexpressions. You can check, by multiplying out the brackets, that

$$A^2 - B^2 = (A + B)(A - B). \tag{19}$$

If you can recognise a quadratic as a difference of two squares, then you can use equation (19) to factorise it immediately. For example,

$$x^2 - 9 = x^2 - 3^2 = (x + 3)(x - 3),$$
$$x^2 - 1 = x^2 - 1^2 = (x + 1)(x - 1),$$
$$4x^2 - 1 = (2x)^2 - 1^2 = (2x + 1)(2x - 1).$$

Activity 19 *Factorising special quadratics*

Factorise the following quadratics.

(a) $x^2 - 4$ (b) $x^2 - 4x$ (c) $2x^2 + 5x$ (d) $x^2 - 25$

(e) $9y^2 - 4$ (f) $25t^2 - 1$ (g) $9u^2 - u$ (h) $p^2 + p$

4.3 Solving quadratic equations by factorisation

In this subsection you'll see how you can use the method of factorisation to solve quadratic equations. Remember that a quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where a, b and c are constants with $a \neq 0$.

Whenever you have a quadratic equation to solve, the first thing to do is to check whether it can be simplified, as this will make it easier to deal with. Here are some things that you might be able to do.

Simplifying a quadratic equation

- If necessary, rearrange the equation so that all the non-zero terms are on the same side.
- If the coefficient of x^2 is negative, then multiply the equation through by -1 to make this coefficient positive.
- If the coefficients have a common factor, then divide the equation through by this factor.
- If any of the coefficients are fractions, then multiply the equation through by a suitable number to clear them.

Once you've simplified the quadratic equation as much as you can, you can go on to solve it, using one of the three methods covered in this section.

The method of solving quadratic equations by *factorisation* depends on the techniques that you practised in the last subsection, together with the following crucial fact.

If the product of two or more numbers is 0, then at least one of the numbers must be 0.

You can see how this fact is used in the example below.

Example 10 *Solving a quadratic equation by factorisation*

Solve the equation

$$6x^2 + 14x - 12 = 0.$$

Solution

Simplify the equation if possible. Here you can divide through by 2.

$$3x^2 + 7x - 6 = 0$$

Factorise the quadratic expression.

$$(3x - 2)(x + 3) = 0$$

Use the fact in the box above.

$$3x - 2 = 0 \quad \text{or} \quad x + 3 = 0$$

Solve these linear equations for x.

$$x = \tfrac{2}{3} \quad \text{or} \quad x = -3$$

So the solutions are $x = \frac{2}{3}$ and $x = -3$.

You can check the answers to Example 10 by substituting each of the solutions into the original equation.

Some quadratic equations can't be solved using the method of Example 10, because it's not possible to factorise their associated quadratic expressions using integers. You'll see another method for solving quadratic equations later in this section.

Activity 20 *Solving quadratic equations by factorisation*

Solve the following quadratic equations by factorisation. (They *can* all be solved in this way.)

(a) $x^2 + 4x - 21 = 0$ (b) $3x^2 - 18x + 24 = 0$ (c) $-t^2 - 6t - 9 = 0$

(d) $8x + 4x^2 - 5 = 0$ (e) $3x^2 - x = 0$ (f) $x^2 - 16 = 0$

(g) $\frac{3}{2}u^2 + u - \frac{1}{2} = 0$ (h) $x^2 + \frac{9}{4}x + \frac{1}{2} = 0$ (i) $10(x^2 + 1) = 29x$

In part (c) of the activity above, you'll have noticed that when you factorised the quadratic, the two linear factors were the same, which led to only one solution. A quadratic equation with only one solution is said to have a **repeated solution**.

4.4 Expressing a quadratic in completed-square form

In this subsection you'll revise how to *complete the square* in a quadratic expression, and in the next subsection you'll see how you can use this technique to solve quadratic equations. Completing the square is also useful for other purposes, as you'll see later in the module.

Completing the square in a quadratic expression means rearranging it in a particular way. For example, the completed-square form of the quadratic $2x^2 - 12x + 25$ is

$$2(x - 3)^2 + 7.$$

You can check that this expression is equivalent to the original quadratic by multiplying out the brackets and simplifying:

$$\begin{aligned} 2(x - 3)^2 + 7 &= 2(x^2 - 6x + 9) + 7 \\ &= 2x^2 - 12x + 18 + 7 \\ &= 2x^2 - 12x + 25. \end{aligned}$$

In general, to **complete the square** in a quadratic $ax^2 + bx + c$, you have to rearrange it into the form

$$a(x + r)^2 + s,$$

where a, r and s are constants. The constant a is the same as the constant a in the original expression, and r and s are new constants, each of which can be positive, negative or zero. The rearranged form of the quadratic is called its **completed-square form**.

The key to completing the square in any quadratic is to first learn how to do it for quadratics of the form $x^2 + bx$. Some examples of quadratics of this form are $x^2 + 6x$, $x^2 - 10x$ and $x^2 + x$.

Let's start by considering the quadratic $x^2 + 6x$. To see how to write it in completed-square form, consider what happens when you multiply out the expression $(x+3)^2$. The '+3' here is obtained by halving the coefficient of x in the original quadratic, which is +6. You obtain

$$\begin{aligned}(x+3)^2 &= x^2 + 3x + 3x + 9\\ &= x^2 + 6x + 9.\end{aligned}$$

So expanding $(x+3)^2$ gives the original quadratic $x^2 + 6x$, with an extra term, namely 9, added on. So if you subtract 9 from $(x+3)^2$, then you'll have an expression that's equivalent to $x^2 + 6x$. That is,

$$x^2 + 6x = (x+3)^2 - 9.$$

The expression on the right of this equation is the completed-square form of $x^2 + 6x$.

The method can be summarised as follows.

Strategy:
To complete the square in a quadratic of the form $x^2 + bx$

1. Write down $(x \quad)^2$, filling the gap with the number that's half of b, the coefficient of x (including its + or − sign, of course).
2. Subtract the square of the number that you wrote in the gap.

The first step of the strategy ensures that you have squared brackets that, when expanded, give the quadratic $x^2 + bx$ together with an extra term, which is the square of the number written in the gap. In the second step of the strategy you subtract this extra term to obtain a final completed-square form that's equivalent to $x^2 + bx$. Here's an example.

Example 11 *Completing the square in a quadratic of the form $x^2 + bx$*

Write the quadratic expression $x^2 - 10x$ in completed-square form.

Solution

$$x^2 - 10x = (x - 5)^2 - 25$$

Halve this coefficient and write it here.

Square the constant term in brackets and subtract it.

Remember that once you've found the completed-square form of a quadratic, you can always check that it's correct by multiplying it out.

Here are some examples of completing the square for you to try. In this activity, try checking your answers by multiplying them out – this will help you understand how the method works.

Activity 21 *Completing the square in quadratics of the form $x^2 + bx$*

Write the following quadratics in completed-square form.

(a) $x^2 + 8x$ (b) $x^2 - 4x$ (c) $x^2 + x$ (d) $t^2 - 3t$

Once you know how to complete the square in any quadratic of the form $x^2 + bx$, you can also complete the square in any quadratic of the form $x^2 + bx + c$.

Strategy:
To complete the square in a quadratic of the form $x^2 + bx + c$

1. Use the earlier strategy to complete the square in the subexpression $x^2 + bx$.
2. Collect the constant terms.

Example 12 *Completing the square in a quadratic of the form $x^2 + bx + c$*

Write the quadratic $x^2 - 8x + 2$ in completed-square form.

Solution

Complete the square in the subexpression $x^2 - 8x$, leaving the $+2$ unchanged.

$$x^2 - 8x + 2 = (x - 4)^2 - 16 + 2$$

Collect the constant terms.

$$= (x - 4)^2 - 14.$$

Here are some examples for you to try.

Activity 22 *Completing the square in quadratics of the form* $x^2 + bx + c$

Write the following quadratics in completed-square form.

(a) $x^2 + 2x + 2$ (b) $x^2 + 3x - 1$ (c) $y^2 + \frac{1}{2}y - \frac{1}{4}$

In fact you can use the method that you've seen for completing the square in quadratics of the form $x^2 + bx$ to allow you to complete the square in any quadratic at all. To complete the square in a quadratic of the form $ax^2 + bx + c$, where $a \neq 1$, you begin by factorising the coefficient a out of the subexpression formed by the terms in x^2 and x. For example, if the quadratic is $2x^2 - 12x + 20$, then you begin by writing

$$2x^2 - 12x + 20 = 2(x^2 - 6x) + 20.$$

It doesn't matter if the coefficient of x^2 in the quadratic isn't a factor of the coefficient of x. For example, for the quadratic $-2x^2 + x + 1$, you begin by writing

$$-2x^2 + x + 1 = -2(x^2 - \tfrac{1}{2}x) + 1.$$

Once you've carried out this initial step, the quadratic in the brackets will be of the form $x^2 + bx$. You can then complete the square in this quadratic, and finally simplify the results to obtain the completed-square form of the original quadratic. Here's an example.

Example 13 *Completing the square in a quadratic of the form* $ax^2 + bx + c$

Express $2x^2 - 12x + 20$ in completed-square form.

Solution

Factorise the coefficient of x^2 out of the subexpression formed by the terms in x^2 and x.

$$2x^2 - 12x + 20 = 2\left(x^2 - 6x\right) + 20$$

Now the brackets contain a quadratic of the form $x^2 + bx$. Complete the square in it, keeping it enclosed within its brackets.

$$= 2\left((x-3)^2 - 9\right) + 20$$

Multiply out the *outer* brackets. Don't multiply out the inner brackets, because you want the square $(x-3)^2$ to appear in the final expression.

$$= 2(x-3)^2 - 18 + 20$$

Collect the constant terms.

$$= 2(x-3)^2 + 2$$

Here's a summary of the method demonstrated in Example 13.

Strategy:
To complete the square in a quadratic of the form $ax^2 + bx + c$, where $a \neq 1$

1. Rewrite the quadratic with the coefficient a taken out of the expression $ax^2 + bx$ as a factor. This generates a pair of brackets.
2. Use the earlier strategy to complete the square in the simple quadratic inside the brackets. This generates a second pair of brackets, inside the first pair.
3. Multiply out the *outer* brackets.
4. Collect the constant terms.

Activity 23 *Completing the square in quadratics of the form $ax^2 + bx + c$*

Write the following quadratics in completed-square form.

(a) $3x^2 + 6x + 5$ (b) $2y^2 - 5y + 4$ (c) $-x^2 + x - \frac{1}{2}$

4.5 Solving quadratic equations by completing the square

The example below demonstrates how you can use the technique of completing the square to solve a quadratic equation. You can use this method to solve any quadratic equation that has solutions, including those that can't be solved by factorisation using integers.

Example 14 *Solving a quadratic equation by completing the square*

Solve the quadratic equation $2x^2 - 8x + 5 = 0$ by completing the square.

Solution

Check whether the equation can be simplified (see the box on page 163). Here there's no simplification to be done. Next, complete the square on the left-hand side.

$$2(x^2 - 4x) + 5 = 0$$
$$2\left((x-2)^2 - 4\right) + 5 = 0$$
$$2(x-2)^2 - 8 + 5 = 0$$
$$2(x-2)^2 - 3 = 0$$

Rearrange the equation so that the left-hand side is of the form $(x \quad)^2$.

$$(x-2)^2 = \frac{3}{2}$$

Take square roots of both sides, remembering that a positive number has both a positive and a negative square root.

$$x - 2 = \sqrt{\frac{3}{2}} \quad \text{or} \quad x - 2 = -\sqrt{\frac{3}{2}}$$

Get x by itself on the left of each equation.

$$x = 2 + \sqrt{\frac{3}{2}} \quad \text{or} \quad x = 2 - \sqrt{\frac{3}{2}}$$

So the solutions are $x = 2 + \sqrt{3/2}$ and $x = 2 - \sqrt{3/2}$.

Here are some examples for you to try.

Activity 24 *Solving quadratic equations by completing the square*

Solve the following quadratic equations by completing the square.

(a) $x^2 + 4x + 1 = 0$ (b) $3t^2 - 12t + 11 = 0$ (c) $2x^2 + 3x - 3 = 0$

In practice, if you can't solve a particular quadratic equation by factorisation, then instead of solving it by completing the square, you may prefer to use the *quadratic formula*, which is given in the next subsection. This formula is obtained by completing the square in the general quadratic expression $ax^2 + bx + c$.

4.6 The quadratic formula

The formula below expresses the solutions of a general quadratic equation in terms of its coefficients.

The quadratic formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Recall from Unit 1 that the symbol $\pm$ means 'plus or minus'.

Later in this subsection you'll see how the quadratic formula is obtained by completing the square – for now you should concentrate on how to use it.

Remember that before you solve a quadratic equation you should always simplify it as much as possible, following the guidelines on page 163.

Example 15 *Using the quadratic formula*

Use the quadratic formula to solve the equation $2x^2 - 6x - 5 = 0$.

Solution

Check that the equation is in the form $ax^2 + bx + c = 0$, and find the values of a, b and c.

Here $a = 2$, $b = -6$ and $c = -5$.

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 2 \times (-5)}}{2 \times 2} \\ &= \frac{6 \pm \sqrt{36 + 40}}{4} \\ &= \frac{6 \pm \sqrt{76}}{4} \end{aligned}$$

Simplify this pair of surds.

$$\begin{aligned} &= \frac{6 \pm \sqrt{4 \times 19}}{4} \\ &= \frac{6 \pm 2\sqrt{19}}{4} \\ &= \frac{3 \pm \sqrt{19}}{2} \\ &= \tfrac{1}{2}(3 \pm \sqrt{19}) \end{aligned}$$

So the solutions are $x = \frac{1}{2}(3 + \sqrt{19})$ and $x = \frac{1}{2}(3 - \sqrt{19})$.

As you can see from Example 15, the calculations that you have to do when you use the quadratic formula can be quite complicated, and it's easy to make mistakes. Before you solve a quadratic equation using the quadratic formula, it's always worth checking whether you can solve it more easily by factorisation instead.

When you use the quadratic formula, the solutions that you obtain are often surds. In fact, if you don't obtain surds, then it means that the quadratic equation could have been solved by factorisation instead. If you obtain surds, then you should leave them as surds, expressing them in their simplest form, unless you've been asked for a decimal approximation, or the solutions to the quadratic equation are the answers to a practical problem.

You can practise using the quadratic formula in the next activity. Remember to simplify the equations before you apply the formula, where possible.

Activity 25 *Using the quadratic formula*

Solve the following quadratic equations by using the quadratic formula. In each case where the solutions are not surds, try solving the quadratic equation by factorisation as well.

(a) $x^2 - 6x - 1 = 0$ (b) $9x^2 + 15x - 6 = 0$ (c) $9x^2 + 6x = 11$

(d) $t^2 + \frac{5}{2}t + \frac{3}{2} = 0$ (e) $u^2 = 4u - 4$

To end this subsection, here's an explanation of where the quadratic formula comes from. Given the general quadratic equation,

$$ax^2 + bx + c = 0,$$

where $a \neq 0$, the first step is to complete the square:

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0.$$

Next rearrange the equation so that you get the constant terms on the right, and combine them into a single fraction:

$$\begin{aligned}\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a}\\ &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2}\\ &= \frac{b^2 - 4ac}{4a^2}.\end{aligned}$$

Now take square roots of both sides:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

The last step is to get x by itself on the left-hand side:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is the quadratic formula!

4.7 The number of solutions of a quadratic equation

Earlier in this unit, you saw that the graph of an equation of the form $y = ax^2 + bx + c$, where a, b and c are constants with $a \neq 0$, can have two, one or no x-intercepts. Since the x-intercepts of the graph of $y = ax^2 + bx + c$ are the solutions of the quadratic equation $ax^2 + bx + c = 0$, this means that a quadratic equation can have two, one or zero solutions that are real numbers.

In fact every quadratic equation has at least one solution if we allow solutions that are *complex numbers* – these numbers were mentioned in Unit 1, and they include all the real numbers and also many 'imaginary' numbers, such as the square root of -1. You'll see more about this in Unit 12, but until then you only need to consider the solutions of quadratic equations that are real numbers, which are known as their *real solutions*.

When you use the quadratic formula to solve a quadratic equation, the number of real solutions of the equation quickly becomes clear. For example, consider the equation

$$x^2 + 2x + 3 = 0.$$

Here $a = 1$, $b = 2$ and $c = 3$. Substituting these values into the quadratic formula, we obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 3}}{2 \times 1}$$
$$= \frac{-2 \pm \sqrt{4 - 12}}{2}$$
$$= \frac{-2 \pm \sqrt{-8}}{2}.$$

This expression involves $\sqrt{-8}$, but there is no such number in the set of real numbers. So this equation has no real solutions. This is confirmed by the graph of $y = x^2 + 2x + 3$ in Figure 24, which does not cross or touch the x-axis and so has no x-intercepts.

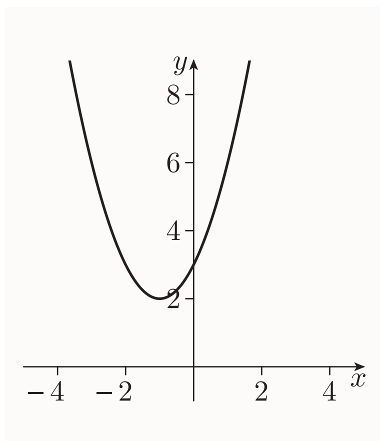


Figure 24 The graph of $y = x^2 + 2x + 3$

Similarly, consider the equation

$$x^2 + 2x + 1 = 0.$$

Here $a = 1$, $b = 2$ and $c = 1$. Substituting these values into the quadratic formula, we obtain

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{-2 \pm \sqrt{4 - 4}}{2} \\ &= \frac{-2 \pm \sqrt{0}}{2} \\ &= -1. \end{aligned}$$

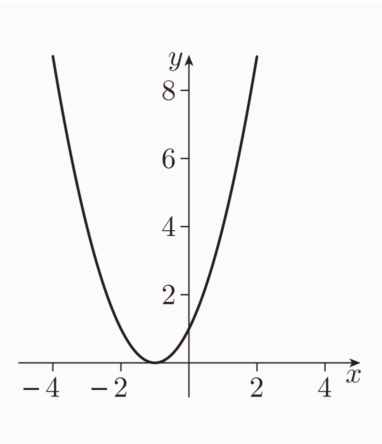


Figure 25 The graph of $y = x^2 + 2x + 1$

In this calculation the number under the square root sign turns out to be zero, and this leads to just one solution of the equation. This is confirmed by the graph of $y = x^2 + 2x + 1$ in Figure 25, which touches the x-axis but does not cross it. From these examples you can see that it's the value of the expression $b^2 - 4ac$, which appears under the square root sign in the quadratic formula, that determines how many solutions a quadratic equation has. The different possibilities are set out in the box below. The value $b^2 - 4ac$ is called the **discriminant** of the quadratic expression $ax^2 + bx + c$.

The number of real solutions of a quadratic equation

The quadratic equation $ax^2 + bx + c = 0$ has:

- two real solutions if $b^2 - 4ac > 0$ (the discriminant is positive)
- one real solution if $b^2 - 4ac = 0$ (the discriminant is zero)
- no real solutions if $b^2 - 4ac < 0$ (the discriminant is negative).

Activity 26 *Predicting the number of real solutions of a quadratic equation*

Use the discriminant to determine whether each of the following quadratic equations has two, one or no real solutions. Find any real solutions.

(a) $4x^2 - 20x + 25 = 0$ (b) $2x^2 + 6x + 5 = 0$ (c) $4x^2 = 4x + 5$

The discriminant of a quadratic also tells you whether the quadratic can be factorised using integers. You can see that if the discriminant of a quadratic with integer coefficients is a *perfect square*, then you won't get surds when you use the quadratic formula to solve the corresponding quadratic equation. This tells you that the expression can be factorised using integers.

4.8 Solving equations related to quadratic equations

In this section you'll see how to use the techniques for solving quadratic equations to solve some equations that don't look like quadratic equations at first sight, and to solve some equations that are not quadratic.

First, remember that in Unit 1 you saw how to solve some equations containing algebraic fractions. Usually the first step is to clear the fractions, by multiplying through by a suitable expression. All the equations containing algebraic fractions in Unit 1 turned out to be linear equations, but it's also possible for an equation containing algebraic fractions to turn out to be a quadratic equation. Here's an example.

Example 16 *Solving an equation involving algebraic fractions*

Solve the equation

$$\frac{4}{x+2} + x = 3.$$

Solution

There is a fraction with denominator $x+2$, so multiply through by $x+2$ to clear it. For this to be guaranteed to give an equivalent equation, you have to assume that $x+2 \neq 0$; that is, $x \neq -2$.

Assume that $x \neq -2$.

$$4 + x(x+2) = 3(x+2)$$

Multiply out the brackets.

$$4 + x^2 + 2x = 3x + 6$$

Get all the non-zero terms on the left-hand side.

$$x^2 - x - 2 = 0$$

This is a quadratic equation. Solve it by factorising.

$$(x+1)(x-2) = 0$$

So $x+1=0$ or $x-2=0$; that is, $x=-1$ or $x=2$.

These values satisfy the assumption $x \neq -2$, so the solutions of the original equation are $x=-1$ and $x=2$.

You can sometimes use the methods for solving quadratic equations to solve an equation containing a power of x greater than 2, if you start by factorising the equation. This is illustrated in the next example.

Example 17 *Factorising to reveal a quadratic*

Solve the equation $3x^3 - 12x^2 + 3x = 0$.

Solution

Simplify the equation by dividing through by 3.

$$x^3 - 4x^2 + x = 0$$

There is a common factor, x. Take it out.

$$x(x^2 - 4x + 1) = 0$$

This gives two expressions whose product is zero, so at least one of the expressions is zero.

$$x = 0 \quad \text{or} \quad x^2 - 4x + 1 = 0$$

Now solve the quadratic equation. It cannot be factorised using integers, so use the quadratic formula, with $a = 1$, $b = -4$, $c = 1$.

The solutions of the quadratic equation are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{16 - 4}}{2} \\ &= \frac{4 \pm \sqrt{12}}{2} \\ &= \frac{4 \pm 2\sqrt{3}}{2} \\ &= 2 \pm \sqrt{3}. \end{aligned}$$

So the solutions of the original equation are $x = 0$, $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$.

The equation in Example 17 is an example of a *cubic* equation. A **cubic equation** is an equation of the form $ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are constants, with $a \neq 0$. You'll see in Unit 3 that every cubic equation has at most three solutions.

In the next example an equation is transformed into a quadratic equation by making a substitution.

Example 18 *Substituting to reveal a quadratic*

Solve the equation $x^4 - 5x^2 + 4 = 0$.

Solution

Substitute another letter, say X, for x^2.

Let $X = x^2$. The equation becomes

$$X^2 - 5X + 4 = 0$$

This is a quadratic equation in X. It can be solved by factorisation.

$$(X-1)(X-4) = 0$$

$$X = 1 \quad \text{or} \quad X = 4$$

Use the fact that $X = x^2$.

$$x^2 = 1 \quad \text{or} \quad x^2 = 4$$

Take square roots of both sides of each equation.

$$x = \pm 1 \quad \text{or} \quad x = \pm 2.$$

So the solutions of the original equation are $x = 1$, $x = -1$, $x = 2$ and $x = -2$.

The method in Example 18 can be used to solve any equation of the form

$$ax^4 + bx^2 + c = 0,$$

where a, b and c are constants, with $a \neq 0$.

Activity 27 *Solving equations related to quadratic equations*

Solve the following equations.

(a) $\dfrac{4}{x} = \dfrac{3x}{x+1}$ (b) $\dfrac{1}{x-2} = 1 + 4x$ (c) $2x^3 - 2x^2 - 12x = 0$

(d) $x^4 - 2x^2 - 8 = 0$ (e) $u^4 - 4 = 0$

(f) $s^5 - 9s^3 = 0$ (g) $(t^2 - 3)(t^2 - 3t + 2) = 0$

4.9 Sketching quadratic graphs

In Subsection 4.1 you saw how the graph of the equation $y = ax^2 + bx + c$ changes when you change the values of a, b and c. In this subsection you'll revise how to sketch the graph of any equation of this form. Note that a **sketch** of a graph is a diagram that gives an impression of its shape, with key points marked and positioned approximately correctly relative to a pair of coordinate axes. It's different from a **plot** of a graph, which is a more accurate diagram obtained by precisely plotting a reasonably large number of points on the graph. For example, you were asked to plot a graph in Activity 2 on page 123.

Sketching a parabola?

Although a sketch of a graph is not an accurate representation of the graph, it should be sufficiently correct to convey the main properties. The axis scales need not be marked, but there should be some indication of scale – for example, the key points can be labelled with their coordinates. In the case of a parabola, the key points are the points where it crosses the axes, and the vertex. When you're sketching a quadratic graph, it's helpful to remember the properties summarised below. You met these properties in Subsection 4.1.

Properties of the graph of $y = ax^2 + bx + c$, where $a \neq 0$

1. The graph is a parabola with a vertical axis of symmetry.
2. If a is positive it is u-shaped; if a is negative it is n-shaped.
3. It has two, one or no x-intercepts.
4. It has one y-intercept.

Here's a strategy for sketching a quadratic graph. It's illustrated in the next example.

Strategy:
To sketch the graph of $y = ax^2 + bx + c$, where $a \neq 0$

1. Find whether the parabola is u-shaped or n-shaped.
2. Find its intercepts, axis of symmetry and vertex.
3. Plot the features found, and hence sketch the parabola.
4. Label the parabola with its equation, intercepts and the coordinates of the vertex.

It's fine to label a parabola with the coordinates of the points where it crosses the axis, rather than with its intercepts. For example, the parabola in Example 19 is labelled with $(0, 5)$ rather than with 5. Also, you can choose whether or not to draw the axis of symmetry. If you do include it, you should draw it as a dashed line.

Example 19 *Sketching a quadratic graph*

Sketch the graph of $y = -x^2 - 4x + 5$.

Solution

Find whether the parabola is u-shaped or n-shaped.

The coefficient of x^2 is negative, so the graph is n-shaped.

Find the y-intercept.

Putting $x = 0$ gives $y = 5$, so the y-intercept is 5.

Find any x-intercepts.

Putting $y = 0$ gives

$$-x^2 - 4x + 5 = 0$$
$$x^2 + 4x - 5 = 0$$
$$(x - 1)(x + 5) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x + 5 = 0$$
$$x = 1 \quad \text{or} \quad x = -5.$$

So the x-intercepts are 1 and -5.

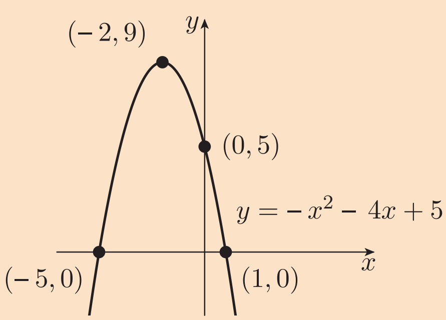
The axis of symmetry lies halfway between the x-intercepts. The number halfway between any two numbers p and q is their mean, $(p + q)/2$.

The value of x halfway between -5 and 1 is $(-5 + 1)/2 = -4/2 = -2$. So the axis of symmetry is the line $x = -2$.

The vertex lies on the axis of symmetry.

The point with x-coordinate -2 on the graph has y-coordinate $y = -(-2)^2 - 4 \times (-2) + 5 = 9$. So the vertex is $(-2, 9)$.

Plot the features found. Hence sketch the parabola as a smooth curve, and label it with its equation, intercepts and vertex.



You can use the method demonstrated in Example 19 to sketch the graph of any equation of the form $y = ax^2 + bx + c$, where $a \neq 0$, provided that there are two x-intercepts or one x-intercept. If there is just one x-intercept, then the vertex of the parabola is at this x-intercept. The graphs of both equations in the activity below have one or two x-intercepts.

Activity 28 *Sketching quadratic graphs*

Sketch the graphs of the following equations.

(a) $y = -2x^2 + 3x - 1$ (b) $y = 2x^2 + 8x + 8$

To sketch the graph of an equation of the form $y = ax^2 + bx + c$ when there are no x-intercepts, you need a different method for finding the equation of the axis of symmetry. One method is to first find any two points on the parabola that have the same y-coordinate, and use the fact that the axis of symmetry lies halfway between them.

The next example demonstrates a neat way to find two such points.

Example 20 *Sketching a quadratic graph with no x-intercepts*

Sketch the graph of $y = 2x^2 - 12x + 20$.

Solution

Find whether the parabola is u-shaped or n-shaped.

The coefficient of x^2 is positive, so the graph is u-shaped.

Find the y-intercept.

Putting $x = 0$ gives $y = 20$, so the y-intercept is 20.

Find any x-intercepts.

Putting $y = 0$ gives

$$2x^2 - 12x + 20 = 0$$
$$x^2 - 6x + 10 = 0$$

The discriminant of $x^2 - 6x + 10$ is

$$b^2 - 4ac = (-6)^2 - 4 \times 1 \times 10 = -4.$$

The discriminant is negative, so the quadratic equation above has no solutions, and hence the graph has no x-intercepts.

To find the axis of symmetry, start by taking out the common factor x from the terms in x^2 and x in the equation whose graph you're trying to sketch. It's convenient to take out any numerical common factors too.

The equation $y = 2x^2 - 12x + 20$ can be rearranged as

$$y = 2x(x - 6) + 20.$$

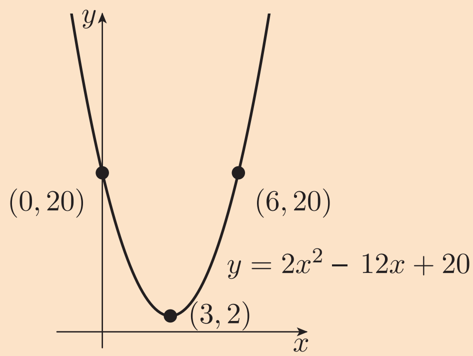
From this form of the equation you can see that the points with x-coordinates 0 and 6 have the same y-coordinate, namely 20. The axis of symmetry lies halfway between these two points.

The points $(0, 20)$ and $(6, 20)$ lie on the graph. The number halfway between 0 and 6 is 3, so the axis of symmetry is the line $x = 3$.

The vertex lies on the axis of symmetry.

The point on the graph with x-coordinate 3 has y-coordinate $y = 2 \times 3^2 - 12 \times 3 + 20 = 2$, so the vertex is $(3, 2)$.

Plot the features found. Hence sketch the parabola as a smooth curve, and label it with its equation, intercepts and vertex.



An alternative way to find the vertex of the graph of an equation of the form $y = ax^2 + bx + c$, where $a \neq 0$, is to complete the square in the quadratic expression. For example, consider again the equation in Example 20:

$$y = 2x^2 - 12x + 20.$$

Completing the square in the right-hand side (which was done for this particular expression in Example 13) gives

$$y = 2(x - 3)^2 + 2.$$

Think about what this form of the equation tells you. Since $(x - 3)^2$ is a square, it can never be negative, no matter what the value of x is. The least value that it can take is 0, and it takes this value when $x = 3$. So the least value that y can take is $y = 2 \times 0 + 2 = 2$, and y takes this value when $x = 3$. This tells you that the vertex is $(3, 2)$.

In general, when you want to sketch a graph, there is no single correct approach. You can use any appropriate means to find the information that you need.

Activity 29 *Sketching a quadratic graph with no x-intercepts*

Sketch the graph of $y = x^2 - 2x + 4$.

4.10 Applications of quadratics

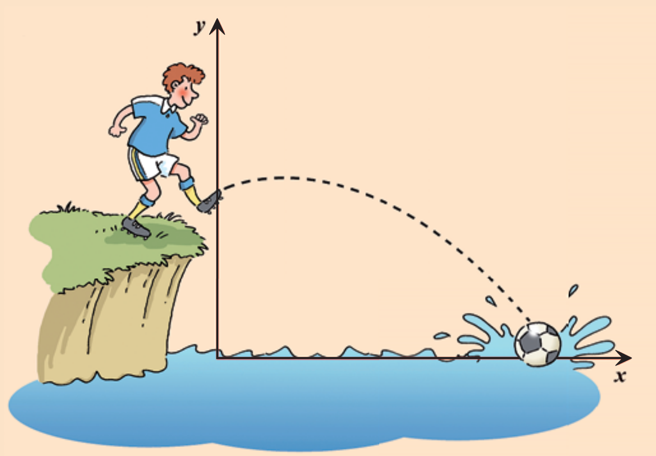
Quadratic equations occur frequently when you model real-life problems involving objects that are falling vertically under the influence of gravity, or have been thrown, particularly when the effects of air resistance are small enough to be ignored. Here's an example.

Example 21 *Solving a quadratic equation in a real-life problem*

A boy standing on a vertical cliff 150 m above sea level kicks a ball into the sea. Let x be the horizontal displacement (in metres) of the ball from the base of the cliff, and y be the vertical displacement (in metres) of the ball measured upwards from sea level. Then the curve followed by the ball can be modelled by the equation

$$y = 150 + x - \frac{x^2}{40}.$$

Find the horizontal distance from the cliff at which the ball hits the sea.



Solution

The ball hits the sea when the vertical displacement y is 0. The corresponding horizontal displacement x satisfies the equation

$$0 = 150 + x - \frac{x^2}{40}.$$

Clear the fraction and simplify the equation.

Rearranging gives

$$0 = 6000 + 40x - x^2$$
$$x^2 - 40x - 6000 = 0$$

Factorise.

$$(x - 100)(x + 60) = 0$$
$$x - 100 = 0 \quad \text{or} \quad x + 60 = 0$$
$$x = 100 \quad \text{or} \quad x = -60.$$

The negative answer is meaningless in the context of the question. So the ball hits the sea 100 m from the cliff.

Activity 30 *Using quadratics*

A ball is thrown vertically upwards. Its vertical displacement s (in metres, measured upwards from the point of release) is modelled by the equation

$$s = 12t - 5t^2,$$

where t is the time in seconds since the moment of release.

(a) How high will the ball be after 2 seconds?

(b) At what times after the ball is thrown will it be at a height of 5 m above its starting position?

(c) After what time will the ball return to its starting position?

(d) By completing the square in the expression for s, find the maximum height that the ball will reach above its starting position.

The techniques that you need to create models like the ones in Example 21 and Activity 30 are taught in the module MST125.

5 Using the computer for graphs and equations

The following activity completes your study of Unit 2.

Activity 31 *Using the computer algebra system*

Work through Sections 2 and 3 of the MST124 *Computer algebra guide*, where you will learn how to use the computer to manipulate algebraic expressions, solve equations and plot lines and curves.

You will need to use your computer while you work through these sections.

Learning outcomes

After studying this unit, you should be able to:

- plot the graph of an equation by constructing a table of values and plotting points
- find the gradient and intercepts of a straight line from its equation
- interpret the gradient and intercepts of a straight line in real-life situations, where possible
- find the equation of a straight line from its gradient and y-intercept, from its gradient and a point on the line, or from two points on the line
- use the equation of a straight line to draw the line
- find the gradient of a line perpendicular to another line whose gradient you know
- solve simultaneous linear equations, and hence find the point of intersection of two straight lines
- factorise a quadratic expression using integers, where possible
- complete the square in a quadratic expression
- solve quadratic equations by factorising, by completing the square and by using the quadratic formula
- sketch the graph of an equation of the form $y = ax^2 + bx + c$
- use the module computer algebra system to manipulate mathematical expressions and plot lines and curves.

Solutions to activities

Solution to Activity 1

(a) When $x = 6$ and $y = 20$,

$$\text{LHS} = y - 2 = 20 - 2 = 18$$
$$\text{and RHS} = 3x = 3 \times 6 = 18.$$

So the point $(6, 20)$ satisfies the equation.

(b) When $x = -2$ and $y = 8$,

$$\text{LHS} = y - 2 = 8 - 2 = 6$$
$$\text{and RHS} = 3x = 3 \times (-2) = -6.$$

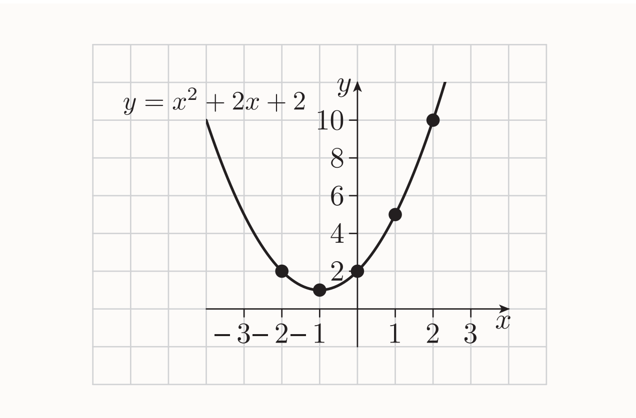
So the point $(-2, 8)$ does not satisfy the equation.

Solution to Activity 2

A table of values for the equation $y = x^2 + 2x + 2$ is as follows.

x	-2	-1	0	1	2
y	2	1	2	5	10

Drawing a smooth curve through these points gives the graph below.



Solution to Activity 3

(a) For every unit moved to the right, the pen tip moves up 3 units, so the gradient is 3.

(b) For every unit moved to the right, the pen tip moves up $\frac{1}{2}$ unit, so the gradient is $\frac{1}{2}$.

(c) For every unit moved to the right, the pen tip moves down 4 units, so the gradient is -4.

(d) For every unit moved to the right, the pen tip moves down $\frac{1}{3}$ unit, so the gradient is $-\frac{1}{3}$.

(e) For every unit moved to the right, the pen tip moves up 1 unit, so the gradient is 1.

(f) For every unit moved to the right, the pen tip moves down 1 unit, so the gradient is -1.

Solution to Activity 4

(a) A is $(-4, 5)$; B is $(1, 3)$; C is $(-1, -2)$; D is $(-6, 0)$.

(b) (i) For the gradient of the line through A and B, take A to be the first point and B the second:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 5}{1 - (-4)} = \frac{-2}{5} = -\frac{2}{5}.$$

(ii) For the gradient of the line through A and D, take A to be the first point and D the second:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 5}{-6 - (-4)} = \frac{-5}{-2} = \frac{5}{2}.$$

(iii) For the gradient of the line through B and C, take B to be the first point and C the second:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 3}{-1 - 1} = \frac{-5}{-2} = \frac{5}{2}.$$

Solution to Activity 5

(a) The coefficient of x is -4, so the gradient is -4. The constant term is 3, so the y-intercept is 3.

To find the x-intercept, put $y = 0$, which gives

$$0 = -4x + 3.$$

Solving this equation gives $4x = 3$, so $x = \frac{3}{4}$. Hence the x-intercept is $\frac{3}{4}$.

(b) Rearranging the equation in the form $y = mx + c$ gives $y = \frac{1}{3}x - \frac{2}{3}$. The coefficient of x is $\frac{1}{3}$, so the gradient is $\frac{1}{3}$.

The constant term is $-\frac{2}{3}$, so the y-intercept is $-\frac{2}{3}$.

To find the x-intercept, put $y = 0$, which gives

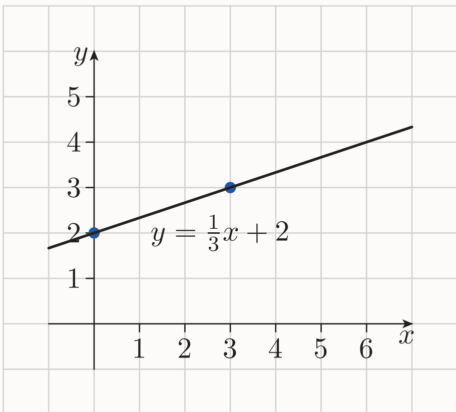
$$0 = \tfrac{1}{3}x - \tfrac{2}{3}.$$

Solving this equation gives $\frac{1}{3}x = \frac{2}{3}$, so $x = 2$. Hence the x-intercept is 2.

Solution to Activity 6

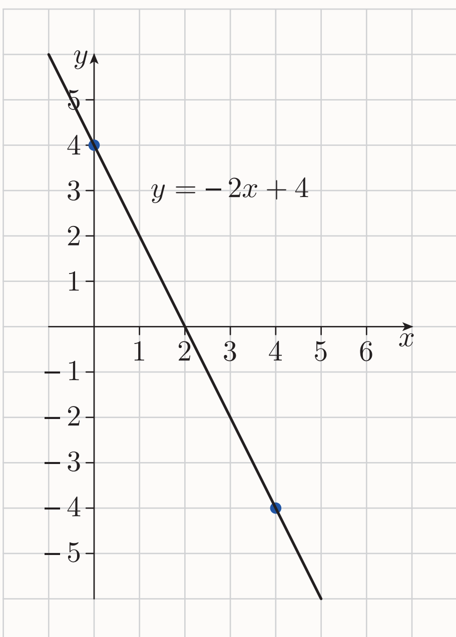
(a) Putting $x = 0$ gives $y = \frac{1}{3} \times 0 + 2 = 2$, so $(0, 2)$ lies on the line. (You might also notice that this point lies on the line since the y-intercept is 2.)

Putting $x = 3$ gives $y = \frac{1}{3} \times 3 + 2 = 1 + 2 = 3$, so $(3, 3)$ lies on the line.

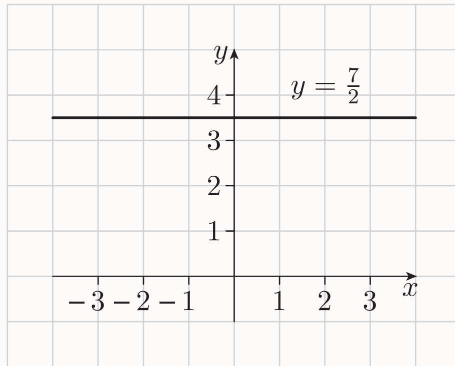


(b) Putting $x = 0$ gives $y = -2 \times 0 + 4 = 4$, so $(0, 4)$ lies on the line.

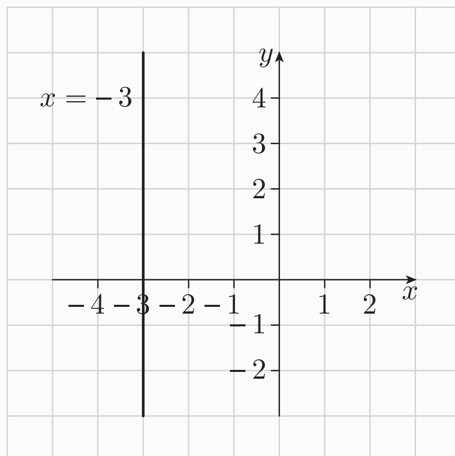
Putting $x = 4$ gives $y = -2 \times 4 + 4 = -8 + 4 = -4$, so $(4, -4)$ lies on the line.



(c) This is a horizontal line, with y-intercept $\frac{7}{2}$.



(d) This is a vertical line, with x-intercept -3.



Solution to Activity 7

(a) Using the equation $y - y_1 = m(x - x_1)$ with $m = 3$, $x_1 = 2$ and $y_1 = 1$ gives

$$y - 1 = 3(x - 2).$$

Expanding the brackets and rearranging gives the equation of the line as

$$y = 3x - 5.$$

(b) The gradient of the line is given by

$$\frac{5 - 3}{4 - 2} = \frac{2}{2} = 1.$$

Using the equation $y - y_1 = m(x - x_1)$ with $m = 1$, $x_1 = 2$ and $y_1 = 3$ gives

$$y - 3 = x - 2.$$

So the equation of the line is

$$y = x + 1.$$

(c) Using the equation $y = mx + c$ with $m = 2$ and $c = 3$ gives the equation of the line as

$$y = 2x + 3.$$

(d) The point at which the line crosses the x-axis is $(2, 0)$. Using the equation $y - y_1 = m(x - x_1)$ with $m = -3$, $x_1 = 2$ and $y_1 = 0$ gives

$$y - 0 = -3(x - 2).$$

So the equation of the line is

$$y = -3x + 6.$$

(e) Each point on a vertical line has the same x-coordinate, so the equation of the line is $x = 1$.

(f) The y-coordinates of the two points are the same, so the line is horizontal with equation $y = 3$.

Solution to Activity 8

(a) The gradient of the line $y = 3x + 5$ is 3, so the gradient of a line perpendicular to this line is $-\frac{1}{3}$.

(b) The gradient of the required line is $-\frac{1}{3}$ and it passes through the point $(2, 1)$. Using the equation $y - y_1 = m(x - x_1)$ gives

$$y - 1 = -\tfrac{1}{3}(x - 2)$$

$$y - 1 = -\tfrac{1}{3}x + \tfrac{2}{3}.$$

So the required line is

$$y = -\tfrac{1}{3}x + \tfrac{5}{3}.$$

Solution to Activity 9

(a) The line passes through the points $(0, 200)$ and $(40, 500)$, so its gradient is

$$\frac{500 - 200}{40 - 0} = \frac{300}{40} = 7.5\ \text{£/person}.$$

The gradient is the rate of increase of the cost of the room with respect to the number of people attending the meeting. Each additional person attending raises the cost by £7.50.

(b) The line passes through the points $(200, 0.07)$ and $(1000, 0.04)$, so its gradient is

$$\frac{0.04 - 0.07}{1000 - 200} = \frac{-0.03}{800} = -\frac{3}{80\,000}\,\frac{\text{£/kg}}{\text{kg}}.$$

The units can be simplified:

$$\frac{\text{£/kg}}{\text{kg}} = \frac{\text{£}}{\text{kg}} \div \text{kg} = \frac{\text{£}}{\text{kg}} \times \frac{1}{\text{kg}} = \frac{\text{£}}{\text{kg}^2} = \text{£/kg}^2.$$

So the gradient is $-\frac{3}{80\,000}$ £/kg^2. The gradient is the rate of change of the price (in £/kg) with respect to the quantity (in kg). For each extra kg of sand, the price (per kg) decreases by approximately 3×10^{-5} £/kg.

Solution to Activity 10

The vertical intercept is £200. This represents the basic cost of the meeting room without the additional cost for each person attending.

Solution to Activity 11

(a) The gradient of the graph of the cost of the meeting room is 7.5 £/person. From the graph, the vertical intercept is £200. So the equation of the graph is

$$C = 7.5n + 200,$$

where C represents the cost (in £) of the meeting room and n represents the number of people attending. The gradient of the building sand graph is $-\frac{3}{80\,000}$ £/kg^2. The graph passes through the point $(200, 0.07)$, so its equation is given by

$$p - 0.07 = -\frac{3}{80\,000}(q - 200);$$

that is,

$$p = -\frac{3}{80\,000}q + \frac{600}{80\,000} + 0.07$$

$$p = -\frac{3}{80\,000}q + \frac{31}{400}.$$

(b) (i) Using $C = 7.5n + 200$ with $C = 560$ gives

$$560 = 7.5n + 200$$

$$7.5n = 360$$

$$n = \frac{360}{7.5} = 48.$$

So the maximum number of people that can be accommodated is 48.

(ii) Using $p = -\frac{3}{80\,000}q + \frac{31}{400}$ with $q = 500$ gives

$$p = -\frac{3}{80\,000} \times 500 + \frac{31}{400} = \frac{47}{800},$$

so the price per kilogram is approximately £0.059.

Solution to Activity 12

(a) The displacement is 3 km.

(b) The woman remains at the bench for 10 minutes.

(c) On the way to the bench her velocity is the gradient of the first line segment of the graph:

$$\frac{3-0}{30-0} = 0.1\,\text{km/min}.$$

(d) On the way back her velocity is the gradient of the third line segment of the graph:

$$\frac{0-3}{90-40} = -\frac{3}{50} = -0.06\,\text{km/min}.$$

(e) On the way to the bench her speed is 0.1 km/min. On the way back her speed is 0.06 km/min.

(f) Let s be the displacement in kilometres and t be the time in minutes. The equation of the line segment representing the first part of her walk is $s = 0.1t$.

(g) Her displacement after 50 minutes if she hadn't stopped would be $s = 0.1 \times 50 = 5\,\text{km}$.

Solution to Activity 13

(a) The equations are

$$s - 5t = -3, \tag{20}$$
$$s + 3t = 13. \tag{21}$$

Making s the subject of equation (20) gives

$$s = 5t - 3. \tag{22}$$

Substituting for s in equation (21) gives

$$5t - 3 + 3t = 13$$
$$8t = 16$$
$$t = 2.$$

Substituting this value for t in equation (22) gives

$$s = 5 \times 2 - 3 = 7.$$

So the solution is $s = 7$, $t = 2$.

(b) The equations are

$$x + 4y = 2, \tag{23}$$
$$2x + 5y = 3. \tag{24}$$

Making x the subject of equation (23) gives

$$x = 2 - 4y. \tag{25}$$

Substituting for x in equation (24) gives

$$2(2-4y) + 5y = 3$$
$$4 - 8y + 5y = 3$$
$$-3y = -1$$
$$y = \tfrac{1}{3}.$$

Substituting this value for y in equation (25) gives

$$x = 2 - 4 \times \tfrac{1}{3} = \tfrac{2}{3}.$$

So the solution is $x = \frac{2}{3}$, $y = \frac{1}{3}$.

Solution to Activity 14

(a) The equations are

$$s + 6t = 20, \tag{26}$$
$$2s + 7t = 35. \tag{27}$$

Multiplying equation (26) by 2 (to make the coefficients of s the same) gives

$$2s + 12t = 40, \tag{28}$$
$$2s + 7t = 35. \tag{29}$$

Subtracting equation (29) from equation (28) gives

$$12t - 7t = 40 - 35$$
$$5t = 5$$
$$t = 1.$$

Substituting this value for t in equation (26) gives

$$s + 6 \times 1 = 20$$
$$s + 6 = 20$$
$$s = 14.$$

So the solution is $s = 14$, $t = 1$.

(b) The equations are

$$2x + 3y = -5, \tag{30}$$
$$3x - 2y = 12. \tag{31}$$

Multiplying equation (30) by 3 and equation (31) by 2 (to make the coefficients of x the same) gives

$$6x + 9y = -15, \tag{32}$$
$$6x - 4y = 24. \tag{33}$$

Subtracting equation (33) from equation (32) gives

$$9y - (-4y) = -15 - 24$$
$$13y = -39$$
$$y = -3.$$

Substituting this value for y in equation (30) gives

$$2x + 3 \times (-3) = -5$$
$$2x - 9 = -5$$
$$2x = 4$$
$$x = 2.$$

So the solution is $x = 2$, $y = -3$.

(c) The equations are

$$3u - v = -\frac{5}{2}, \qquad (34)$$
$$2u + 5v = 21. \qquad (35)$$

Multiplying equation (34) by 5 (to make the coefficients of v have the same magnitude) gives

$$15u - 5v = -\frac{25}{2}. \qquad (36)$$

Adding equation (36) and equation (35) gives

$$17u = \frac{17}{2}$$
$$u = \tfrac{1}{2}.$$

Substituting this value for u in equation (34) gives

$$3 \times \frac{1}{2} - v = -\frac{5}{2}$$
$$-v = -\frac{5}{2} - \frac{3}{2} = -4$$
$$v = 4.$$

So the solution is $u = \frac{1}{2}$, $v = 4$.

Solution to Activity 15

(a) The equations are

$$y - 3x = -2,$$
$$2y + x = 10,$$

which can be rearranged as

$$y = 3x - 2, \qquad (37)$$
$$y = -\tfrac{1}{2}x + 5. \qquad (38)$$

So the gradient of the line in equation (37) is 3 and the gradient of the line in equation (38) is $-\frac{1}{2}$. Since the gradients are different, the lines must intersect and there is exactly one solution.

The equations can be solved by eliminating y from equations (37) and (38) to give

$$3x - 2 = -\tfrac{1}{2}x + 5$$
$$\tfrac{7}{2}x = 7$$
$$x = 2.$$

Substituting this value for x in equation (37) gives

$$y = 3 \times 2 - 2$$
$$y = 4.$$

So the solution is $x = 2$, $y = 4$.

(b) The equations are

$$y = 4x - 5,$$
$$2y - 8x = 10.$$

Writing the equations with both unknowns on the LHS, and dividing the second equation by the common factor 2 gives

$$y - 4x = -5$$
$$y - 4x = 5.$$

Since $y - 4x = -5$ and $y - 4x = 5$ cannot both be true, the equations have no solutions.

(c) The equations are

$$4y = 2x + 6,$$
$$2y - x = 3.$$

Writing the equations with both unknowns on the LHS, and dividing the first equation by the common factor 2 gives

$$2y - x = 3,$$
$$2y - x = 3.$$

The two equations are identical, so their graphs are identical. So all values of x and y that satisfy the first equation also satisfy the second equation, and hence there are infinitely many solutions.

Solution to Activity 16

(The effects that you should have seen are described in the text that follows this activity.)

Solution to Activity 17

(a) $x^2 + 5x + 6 = (x + 2)(x + 3)$

(b) $x^2 - 8x + 15 = (x - 3)(x - 5)$

(c) $x^2 + 4x - 5 = (x - 1)(x + 5)$

(d) $x^2 - 2x - 35 = (x + 5)(x - 7)$

(e) $x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$

(f) $x^2 - 6x + 8 = (x - 2)(x - 4)$

(g) $y^2 - 7y - 18 = (y + 2)(y - 9)$

(h) $u^2 + 4u + 4 = (u + 2)(u + 2) = (u + 2)^2$

(i) $p^2 - 4p - 12 = (p + 2)(p - 6)$

(j) $s^2 + s - 30 = (s - 5)(s + 6)$

(k) $v^2 + 5v - 50 = (v - 5)(v + 10)$

(l) $r^2 - 10r + 16 = (r - 2)(r - 8)$

Solution to Activity 18

(a) $5x^2 + 13x - 6 = (5x - 2)(x + 3)$

(b) $3x^2 + 16x + 5 = (3x + 1)(x + 5)$

(c) $6x^2 - 11x + 3 = (2x - 3)(3x - 1)$

(d) $5x^2 - 8x - 21 = (5x + 7)(x - 3)$

(e) $18x^2 + 9x - 2 = (6x - 1)(3x + 2)$

(f) $4x^2 - 8x + 3 = (2x - 3)(2x - 1)$

(g) $4p^2 - 19p - 5 = (4p + 1)(p - 5)$

(h) $6u^2 + 11u - 35 = (2u + 7)(3u - 5)$

(i) $4t^2 + 4t + 1 = (2t + 1)(2t + 1) = (2t + 1)^2$

(j) $9v^2 - 12v + 4 = (3v - 2)(3v - 2) = (3v - 2)^2$

(k)
$$\begin{aligned} -4s^2 + 4s + 3 &= -(4s^2 - 4s - 3) \\ &= -(2s + 1)(2s - 3) \end{aligned}$$

(l)
$$\begin{aligned} 12y^2 - 10y - 2 &= 2(6y^2 - 5y - 1) \\ &= 2(6y + 1)(y - 1) \end{aligned}$$

(In part (l), if you forget to take the common factor 2 out of the quadratic, then you obtain

$$12y^2 - 10y - 2 = (12y + 2)(y - 1)$$

or

$$12y^2 - 10y - 2 = (6y + 1)(2y - 2).$$

These answers are fine, though they can be factorised further by taking the common factor 2 out of one of the brackets, which gives the answer above. It's better to take the common factor out at the beginning, as this makes it easier to factorise.)

Solution to Activity 19

(a) $x^2 - 4 = x^2 - 2^2 = (x + 2)(x - 2)$

(b) $x^2 - 4x = x(x - 4)$

(c) $2x^2 + 5x = x(2x + 5)$

(d) $x^2 - 25 = x^2 - 5^2 = (x + 5)(x - 5)$

(e) $9y^2 - 4 = (3y)^2 - 2^2 = (3y + 2)(3y - 2)$

(f) $25t^2 - 1 = (5t)^2 - 1^2 = (5t + 1)(5t - 1)$

(g) $9u^2 - u = u(9u - 1)$

(h) $p^2 + p = p(p + 1)$

Solution to Activity 20

(a)
$$\begin{aligned} &x^2 + 4x - 21 = 0 \\ &(x - 3)(x + 7) = 0 \\ &x - 3 = 0 \quad \text{or} \quad x + 7 = 0 \\ &x = 3 \quad \text{or} \quad x = -7 \end{aligned}$$

(b)
$$\begin{aligned} &3x^2 - 18x + 24 = 0 \\ &x^2 - 6x + 8 = 0 \\ &(x - 2)(x - 4) = 0 \\ &x - 2 = 0 \quad \text{or} \quad x - 4 = 0 \\ &x = 2 \quad \text{or} \quad x = 4 \end{aligned}$$

(c)
$$\begin{aligned} &-t^2 - 6t - 9 = 0 \\ &t^2 + 6t + 9 = 0 \\ &(t + 3)(t + 3) = 0 \\ &t + 3 = 0 \quad \text{or} \quad t + 3 = 0 \\ &t = -3 \end{aligned}$$

(d)
$$\begin{aligned} &8x + 4x^2 - 5 = 0 \\ &4x^2 + 8x - 5 = 0 \\ &(2x - 1)(2x + 5) = 0 \\ &2x - 1 = 0 \quad \text{or} \quad 2x + 5 = 0 \\ &x = \tfrac{1}{2} \quad \text{or} \quad x = -\tfrac{5}{2} \end{aligned}$$

(e)
$$\begin{aligned} &3x^2 - x = 0 \\ &x(3x - 1) = 0 \\ &x = 0 \quad \text{or} \quad 3x - 1 = 0 \\ &x = 0 \quad \text{or} \quad x = \tfrac{1}{3} \end{aligned}$$

(f)
$$\begin{aligned} &x^2 - 16 = 0 \\ &(x - 4)(x + 4) = 0 \\ &x - 4 = 0 \quad \text{or} \quad x + 4 = 0 \\ &x = 4 \quad \text{or} \quad x = -4 \end{aligned}$$

(Because the quadratic equation in this part has no term in x, there's an even simpler way of solving it:

$$x^2 - 16 = 0$$
$$x^2 = 16$$
$$x = 4 \quad \text{or} \quad x = -4.)$$

(g)
$$\tfrac{3}{2}u^2 + u - \tfrac{1}{2} = 0$$
$$3u^2 + 2u - 1 = 0$$
$$(3u - 1)(u + 1) = 0$$
$$3u - 1 = 0 \quad \text{or} \quad u + 1 = 0$$
$$u = \tfrac{1}{3} \quad \text{or} \quad u = -1$$

(h)
$$x^2 + \tfrac{9}{4}x + \tfrac{1}{2} = 0$$
$$4x^2 + 9x + 2 = 0$$
$$(4x + 1)(x + 2) = 0$$
$$4x + 1 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = -\tfrac{1}{4} \quad \text{or} \quad x = -2$$

(i)
$$10(x^2 + 1) = 29x$$
$$10x^2 + 10 = 29x$$
$$10x^2 - 29x + 10 = 0$$
$$(5x - 2)(2x - 5) = 0$$
$$5x - 2 = 0 \quad \text{or} \quad 2x - 5 = 0$$
$$x = \tfrac{2}{5} \quad \text{or} \quad x = \tfrac{5}{2}$$

Solution to Activity 21

(a)
$$\begin{aligned} x^2 + 8x &= (x + 4)^2 - 4^2 \\ &= (x + 4)^2 - 16 \end{aligned}$$

(b)
$$\begin{aligned} x^2 - 4x &= (x - 2)^2 - (-2)^2 \\ &= (x - 2)^2 - 4 \end{aligned}$$

(c)
$$\begin{aligned} x^2 + x &= \left(x + \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 \\ &= \left(x + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} \end{aligned}$$

(d)
$$\begin{aligned} t^2 - 3t &= (t - \tfrac{3}{2})^2 - (-\tfrac{3}{2})^2 \\ &= (t - \tfrac{3}{2})^2 - \tfrac{9}{4} \end{aligned}$$

Solution to Activity 22

(a)
$$\begin{aligned} x^2 + 2x + 2 &= (x + 1)^2 - 1 + 2 \\ &= (x + 1)^2 + 1 \end{aligned}$$

(b)
$$\begin{aligned} x^2 + 3x - 1 &= (x + \tfrac{3}{2})^2 - \tfrac{9}{4} - 1 \\ &= (x + \tfrac{3}{2})^2 - \tfrac{13}{4} \end{aligned}$$

(c)
$$\begin{aligned} y^2 + \tfrac{1}{2}y - \tfrac{1}{4} &= (y + \tfrac{1}{4})^2 - \tfrac{1}{16} - \tfrac{1}{4} \\ &= (y + \tfrac{1}{4})^2 - \tfrac{5}{16} \end{aligned}$$

Solution to Activity 23

(a)
$$\begin{aligned} 3x^2 + 6x + 5 &= 3(x^2 + 2x) + 5 \\ &= 3\left((x + 1)^2 - 1\right) + 5 \\ &= 3(x + 1)^2 - 3 + 5 \\ &= 3(x + 1)^2 + 2 \end{aligned}$$

(b)
$$\begin{aligned} 2y^2 - 5y + 4 &= 2\left(y^2 - \tfrac{5}{2}y\right) + 4 \\ &= 2\left(\left(y - \tfrac{5}{4}\right)^2 - \tfrac{25}{16}\right) + 4 \\ &= 2\left(y - \tfrac{5}{4}\right)^2 - \tfrac{25}{8} + 4 \\ &= 2\left(y - \tfrac{5}{4}\right)^2 + \tfrac{7}{8} \end{aligned}$$

(c)
$$\begin{aligned} -x^2 + x - \tfrac{1}{2} &= -(x^2 - x) - \tfrac{1}{2} \\ &= -\left((x - \tfrac{1}{2})^2 - \tfrac{1}{4}\right) - \tfrac{1}{2} \\ &= -(x - \tfrac{1}{2})^2 + \tfrac{1}{4} - \tfrac{1}{2} \\ &= -(x - \tfrac{1}{2})^2 - \tfrac{1}{4} \end{aligned}$$

Solution to Activity 24

(a) First complete the square on the left-hand side.

$$x^2 + 4x + 1 = 0$$
$$(x + 2)^2 - 4 + 1 = 0$$
$$(x + 2)^2 - 3 = 0$$

Now solve this equation.

$$(x + 2)^2 = 3$$
$$x + 2 = \sqrt{3} \quad \text{or} \quad x + 2 = -\sqrt{3}$$
$$x = \sqrt{3} - 2 \quad \text{or} \quad x = -\sqrt{3} - 2$$

So the solutions are $x = \sqrt{3} - 2$ and $x = -\sqrt{3} - 2$.

(b) First complete the square on the left-hand side.

$$3t^2 - 12t + 11 = 0$$
$$3(t^2 - 4t) + 11 = 0$$
$$3((t - 2)^2 - 4) + 11 = 0$$
$$3(t - 2)^2 - 1 = 0$$

Now solve this equation.

$$(t - 2)^2 = \frac{1}{3}$$
$$t - 2 = \frac{1}{\sqrt{3}} \quad \text{or} \quad t - 2 = -\frac{1}{\sqrt{3}}$$
$$t = \frac{1}{\sqrt{3}} + 2 \quad \text{or} \quad t = -\frac{1}{\sqrt{3}} + 2$$

So the solutions are $t = 1/\sqrt{3} + 2$ and $t = -1/\sqrt{3} + 2$.

(c) First complete the square on the left-hand side.

$$2x^2 + 3x - 3 = 0$$
$$2\left(x^2 + \frac{3}{2}x\right) - 3 = 0$$
$$2\left(\left(x + \frac{3}{4}\right)^2 - \frac{9}{16}\right) - 3 = 0$$
$$2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8} - 3 = 0$$
$$2\left(x + \frac{3}{4}\right)^2 - \frac{33}{8} = 0$$

Now solve this equation.

$$\left(x + \frac{3}{4}\right)^2 = \frac{33}{16}$$
$$x + \frac{3}{4} = \frac{\sqrt{33}}{4} \quad \text{or} \quad x + \frac{3}{4} = -\frac{\sqrt{33}}{4}$$
$$x = \frac{\sqrt{33}}{4} - \frac{3}{4} \quad \text{or} \quad x = -\frac{\sqrt{33}}{4} - \frac{3}{4}$$

So the solutions are

$$x = \frac{\sqrt{33} - 3}{4} \quad \text{and} \quad x = -\frac{\sqrt{33} + 3}{4}.$$

Solution to Activity 25

(a) The equation is

$$x^2 - 6x - 1 = 0,$$

so $a = 1$, $b = -6$ and $c = -1$.

The quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 1 \times (-1)}}{2 \times 1} \\ &= \frac{6 \pm \sqrt{36 + 4}}{2} \\ &= \frac{6 \pm \sqrt{40}}{2} \\ &= \frac{6 \pm 2\sqrt{10}}{2} \\ &= 3 \pm \sqrt{10}. \end{aligned}$$

The solutions are $x = 3 + \sqrt{10}$ and $x = 3 - \sqrt{10}$.

(b) The equation is

$$9x^2 + 15x - 6 = 0,$$

which can be simplified to

$$3x^2 + 5x - 2 = 0.$$

So $a = 3$, $b = 5$ and $c = -2$.

The quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-2)}}{2 \times 3} \\ &= \frac{-5 \pm \sqrt{25 + 24}}{6} \\ &= \frac{-5 \pm \sqrt{49}}{6} \\ &= \frac{-5 \pm 7}{6} \\ &= \frac{-5 + 7}{6} \quad \text{or} \quad \frac{-5 - 7}{6} \\ &= \frac{2}{6} \quad \text{or} \quad \frac{-12}{6} \\ &= \frac{1}{3} \quad \text{or} \quad -2. \end{aligned}$$

Alternatively, factorisation gives

$$3x^2 + 5x - 2 = 0$$
$$(3x - 1)(x + 2) = 0$$
$$3x - 1 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = \frac{1}{3} \quad \text{or} \quad x = -2.$$

The solutions are $x = \frac{1}{3}$ and $x = -2$.

(c) The equation is

$$9x^2 + 6x = 11,$$

which can be rearranged as

$$9x^2 + 6x - 11 = 0.$$

So $a = 9$, $b = 6$ and $c = -11$.

The quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4 \times 9 \times (-11)}}{2 \times 9} \\ &= \frac{-6 \pm \sqrt{36 + 396}}{18} \\ &= \frac{-6 \pm \sqrt{432}}{18} \\ &= \frac{-6 \pm 12\sqrt{3}}{18} \\ &= \frac{-1 \pm 2\sqrt{3}}{3}. \end{aligned}$$

The solutions are $x = \frac{1}{3}(-1 + 2\sqrt{3})$ and $x = \frac{1}{3}(-1 - 2\sqrt{3})$.

(d) The equation is

$$t^2 + \tfrac{5}{2}t + \tfrac{3}{2} = 0,$$

which can be simplified as

$$2t^2 + 5t + 3 = 0.$$

So $a = 2$, $b = 5$ and $c = 3$.

The quadratic formula gives

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times 3}}{2 \times 2} \\ &= \frac{-5 \pm \sqrt{25 - 24}}{4} \\ &= \frac{-5 \pm 1}{4} \\ &= \frac{-5 + 1}{4} \quad \text{or} \quad \frac{-5 - 1}{4} \\ &= \frac{-4}{4} \quad \text{or} \quad \frac{-6}{4} \\ &= -1 \quad \text{or} \quad -\frac{3}{2}. \end{aligned}$$

Alternatively, factorisation gives

$$\begin{aligned} &2t^2 + 5t + 3 = 0 \\ &(t + 1)(2t + 3) = 0 \\ &t + 1 = 0 \quad \text{or} \quad 2t + 3 = 0 \\ &t = -1 \quad \text{or} \quad t = -\frac{3}{2}. \end{aligned}$$

The solutions are $t = -1$ and $t = -\frac{3}{2}$.

(e) The equation is $u^2 = 4u - 4$, which can be rearranged as $u^2 - 4u + 4 = 0$. So $a = 1$, $b = -4$ and $c = 4$.

The quadratic formula gives

$$\begin{aligned} u &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 4}}{2 \times 1} \\ &= \frac{4 \pm \sqrt{16 - 16}}{2} \\ &= \frac{4 \pm \sqrt{0}}{2} \\ &= 2. \end{aligned}$$

Alternatively, factorisation gives

$$\begin{aligned} &u^2 - 4u + 4 = 0 \\ &(u - 2)(u - 2) = 0 \\ &u - 2 = 0 \quad \text{or} \quad u - 2 = 0 \\ &u = 2. \end{aligned}$$

The only solution is $u = 2$.

Solution to Activity 26

(a) The equation is

$$4x^2 - 20x + 25 = 0,$$

so $a = 4$, $b = -20$ and $c = 25$.

The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-20)^2 - 4 \times 4 \times 25 \\ &= 400 - 400 \\ &= 0. \end{aligned}$$

Since the discriminant is 0, there is one solution.

The equation can be solved by factorising, as follows. Since there is only one solution, the two linear expressions in the factorisation must be the same (or one linear expression must be the other multiplied through by some number).

$$\begin{aligned} &4x^2 - 20x + 25 = 0 \\ &(2x - 5)(2x - 5) = 0 \\ &(2x - 5)^2 = 0 \\ &2x - 5 = 0 \\ &x = \frac{5}{2} \end{aligned}$$

The only solution is $x = \frac{5}{2}$.

(b) The equation is

$$2x^2 + 6x + 5 = 0,$$

so $a = 2$, $b = 6$ and $c = 5$.

The discriminant is

$$b^2 - 4ac = 6^2 - 4 \times 2 \times 5 = 36 - 40 = -4.$$

Since the discriminant is negative, there are no real solutions.

(c) The equation can be rearranged as follows:

$$4x^2 = 4x + 5$$
$$4x^2 - 4x - 5 = 0.$$

So $a = 4$, $b = -4$ and $c = -5$.

The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4 \times 4 \times (-5) \\ &= 16 + 80 \\ &= 96. \end{aligned}$$

Since the discriminant is positive, there are two real solutions.

The equation cannot be factorised using integers, so we solve it by using the quadratic formula. This gives

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-(-4) \pm \sqrt{96}}{2 \times 4} \\ &= \frac{4 \pm 4\sqrt{6}}{8} \\ &= \tfrac{1}{2}(1 \pm \sqrt{6}). \end{aligned}$$

The solutions are $x = \frac{1}{2}(1 + \sqrt{6})$ and $x = \frac{1}{2}(1 - \sqrt{6})$.

(The value of $b^2 - 4ac$ was already worked out before the quadratic formula was used, so this value was substituted in, instead of working it out again.)

Solution to Activity 27

(a) The equation is $\dfrac{4}{x} = \dfrac{3x}{x+1}$.

Assume that $x \neq 0$ and $x \neq -1$.

Multiply through by $x(x + 1)$ to clear the fractions.

$$4(x + 1) = 3x \times x$$

Rearrange this equation.

$$4x + 4 = 3x^2$$
$$3x^2 - 4x - 4 = 0$$
$$(3x + 2)(x - 2) = 0$$
$$3x + 2 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -\frac{2}{3} \quad \text{or} \quad x = 2$$

Neither value is 0 or -1, so the solutions of the original equation are $x = -\frac{2}{3}$ and $x = 2$.

(b) The equation is $\dfrac{1}{x-2} = 1 + 4x$.

Assume that $x \neq 2$.

Multiply through by $x - 2$ to clear the fraction.

$$1 = (1 + 4x)(x - 2)$$

Rearrange this equation.

$$1 = 4x^2 - 7x - 2$$
$$4x^2 - 7x - 3 = 0$$

This quadratic cannot be factorised, so use the quadratic formula with $a = 4$, $b = -7$, $c = -3$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 4 \times (-3)}}{2 \times 4} \\ &= \frac{7 \pm \sqrt{49 + 48}}{8} \\ &= \frac{7 \pm \sqrt{97}}{8} \end{aligned}$$

Neither value is 2, so the solutions of the original equation are $x = \frac{1}{8}(7 + \sqrt{97})$ and $x = \frac{1}{8}(7 - \sqrt{97})$.

(c) The equation is $2x^3 - 2x^2 - 12x = 0$.

Take out the common factor $2x$.

$$2x(x^2 - x - 6) = 0$$

So $2x = 0$ or $x^2 - x - 6 = 0$.

The equation $2x = 0$ gives $x = 0$.

Factorise the quadratic $x^2 - x - 6$.

$$(x - 3)(x + 2) = 0$$

So $x - 3 = 0$ or $x + 2 = 0$, giving $x = 3$ or $x = -2$.

So the solutions of $2x^3 - 2x^2 - 12x = 0$ are $x = 0$, $x = 3$ and $x = -2$.

(d) The equation is $x^4 - 2x^2 - 8 = 0$.

It includes only even powers of x, so let $X = x^2$. The equation becomes

$$X^2 - 2X - 8 = 0.$$

Factorise the quadratic.

$$(X-4)(X+2) = 0$$

So $X - 4 = 0$ or $X + 2 = 0$, giving $X = 4$ or $X = -2$.

Now $X = x^2$, so x satisfies $x^2 = 4$ or $x^2 = -2$.

The equation $x^2 = -2$ has no real solutions. The equation $x^2 = 4$ gives $x = \pm 2$.

So the solutions are $x = -2$ and $x = 2$.

(e) The equation is $u^4 - 4 = 0$, which can be rearranged as $u^4 = 4$. Taking the square root of both sides gives $u^2 = \pm 2$.

There are no real solutions satisfying $u^2 = -2$. The equation $u^2 = 2$ gives two solutions, $u = \sqrt{2}$ and $u = -\sqrt{2}$.

(f) The equation is $s^5 - 9s^3 = 0$.

Taking out the common factor s^3 gives $s^3(s^2 - 9) = 0$.

So $s^3 = 0$ or $s^2 - 9 = 0$. The first equation has just one solution, $s = 0$. The second equation has two solutions, $s = \pm 3$.

So the solutions are $s = 0$, $s = 3$ and $s = -3$.

(g) The equation is $(t^2 - 3)(t^2 - 3t + 2) = 0$.

So $t^2 - 3 = 0$ or $t^2 - 3t + 2 = 0$.

If $t^2 - 3 = 0$, then $t^2 = 3$, giving $t = \pm\sqrt{3}$.

If $t^2 - 3t + 2 = 0$, then $(t-1)(t-2) = 0$, giving $t = 1$ or $t = 2$.

So the solutions are $t = 1$, $t = 2$, $t = \sqrt{3}$ and $t = -\sqrt{3}$.

Solution to Activity 28

(a) The equation is $y = -2x^2 + 3x - 1$.

Since the coefficient of x^2 is negative, the graph is n-shaped.

Putting $x = 0$ gives $y = -2 \times 0^2 + 3 \times 0 - 1 = -1$, so the y-intercept is -1.

To find the x-intercepts (if any), solve

$$0 = -2x^2 + 3x - 1.$$

Simplify by multiplying through by -1:

$$2x^2 - 3x + 1 = 0.$$

Factorising gives

$$(2x-1)(x-1) = 0.$$

So $2x - 1 = 0$ or $x - 1 = 0$, giving $x = \frac{1}{2}$ or $x = 1$. The x-intercepts are $\frac{1}{2}$ and 1.

The axis of symmetry lies halfway between the points $(\frac{1}{2}, 0)$ and $(1, 0)$, so its equation is

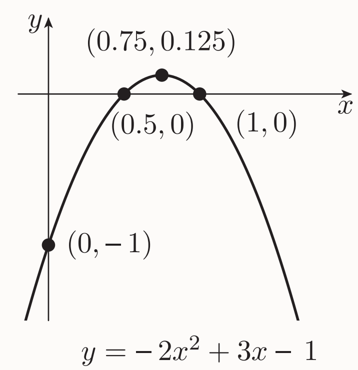
$$x = \frac{1/2 + 1}{2} = \frac{3}{4}.$$

The vertex lies on the axis of symmetry, so it has y-coordinate

$$y = -2\left(\frac{3}{4}\right)^2 + 3 \times \frac{3}{4} - 1 = \frac{1}{8}.$$

So the vertex is $(\frac{3}{4}, \frac{1}{8})$.

This is shown in the following graph.



(b) The equation is $y = 2x^2 + 8x + 8$.

Since the coefficient of x^2 is positive, the graph is u-shaped.

Putting $x = 0$ gives $y = 2 \times 0^2 + 8 \times 0 + 8 = 8$, so the y-intercept is 8.

To find the x-intercepts (if any), solve

$$0 = 2x^2 + 8x + 8.$$

Simplify by factoring out the common factor 2:

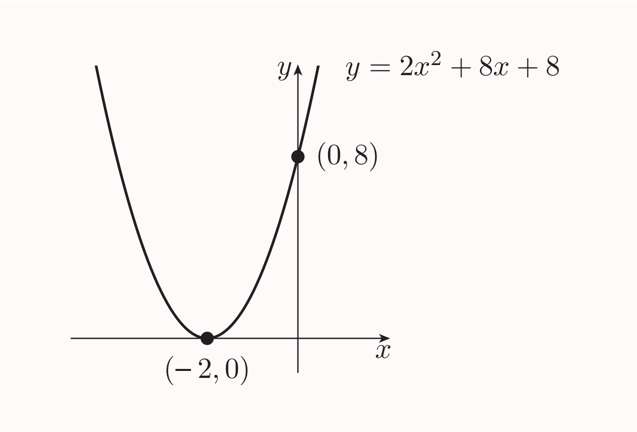
$$x^2 + 4x + 4 = 0.$$

Factorising gives

$$(x+2)(x+2) = 0.$$

So $x + 2 = 0$, giving $x = -2$ as the only x-intercept. Hence $(-2, 0)$ is the only point of the curve on the x-axis, so this is the vertex.

The graph is shown below.



Solution to Activity 29

The equation is $y = x^2 - 2x + 4$.

Since the coefficient of x^2 is positive, the graph is u-shaped.

Putting $x = 0$ gives $y = 0^2 - 2 \times 0 + 4 = 4$, so the y-intercept is 4 and the point $(0, 4)$ lies on the curve.

To find the x-intercepts (if any), solve

$$0 = x^2 - 2x + 4.$$

The discriminant of this quadratic is

$$b^2 - 4ac = (-2)^2 - 4 \times 1 \times 4 = 4 - 16 = -12,$$

which is negative. So the quadratic equation above has no real solutions and there are no x-intercepts.

Rearrange the equation $y = x^2 - 2x + 4$ as

$$y = x(x - 2) + 4.$$

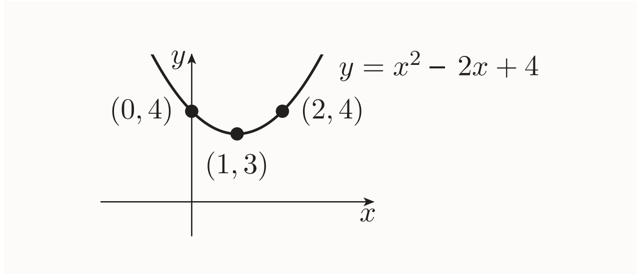
It follows that $x = 0$ and $x = 2$ have the same y-coordinate 4, so the points $(0, 4)$ and $(2, 4)$ lie on the graph.

The axis of symmetry lies halfway between the points $(0, 4)$ and $(2, 4)$, so its equation is

$$x = \frac{1}{2}(0 + 2) = 1.$$

The vertex lies on the axis of symmetry, so it has y-coordinate $y = 1^2 - 2 \times 1 + 4 = 3$.

So the vertex is $(1, 3)$.



Solution to Activity 30

(a) Substitute $t = 2$ into $s = 12t - 5t^2$ to give

$$s = 12 \times 2 - 5 \times 2^2 = 24 - 20 = 4.$$

So the ball will be at height 4 m after 2 seconds.

(b) To find when the ball is at height 5 m, solve

$$5 = 12t - 5t^2.$$

Rearranging gives

$$5t^2 - 12t + 5 = 0.$$

Use the quadratic formula with $a = 5$, $b = -12$ and $c = 5$ to give

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 5 \times 5}}{2 \times 5} \\ &= \frac{12 \pm \sqrt{44}}{10} \\ &= \frac{12 \pm 2\sqrt{11}}{10} \\ &= 0.54 \quad \text{or} \quad 1.86 \text{ (to 2 d.p.)}. \end{aligned}$$

So the ball is at height 5 m on two occasions, namely 0.54 s and 1.86 s after it was thrown (once on the way up and once on the way down).

(c) The ball returns to its starting position when its vertical displacement is zero, that is, when $s = 0$. Substituting $s = 0$ in $s = 12t - 5t^2$ gives

$$0 = 12t - 5t^2.$$

Factorising gives

$$t(12 - 5t) = 0.$$

So $t = 0$ or $12 - 5t = 0$, giving $t = 0$ or $t = \frac{12}{5} = 2.4$.

The solution $t = 0$ corresponds to the time when the ball is at its starting position, at the beginning of its motion, so the ball returns to this position 2.4 s later.

(d) Completing the square in the quadratic expression for s gives

$$\begin{aligned} s &= -5\left(t^2 - \frac{12}{5}t\right) \\ &= -5\left(\left(t - \frac{6}{5}\right)^2 - \frac{36}{25}\right) \\ &= -5\left(t - \frac{6}{5}\right)^2 + \frac{36}{5}. \end{aligned}$$

The maximum value of s occurs when $5\left(t - \frac{6}{5}\right)^2$ is zero, that is, when $t = \frac{6}{5} = 1.2$. So the maximum height that a ball will reach above its starting position is 1.2 m.

When $t = \frac{6}{5}$ the value of s is $\frac{36}{5} = 7.2$, so the maximum height of the ball is 7.2 m.

Acknowledgements

Grateful acknowledgement is made to the following sources:

Page 147: Taken from:
www.metoffice.gov.uk/media/image/4/1/forecast_chart.jpg

Page 156: Credit: Jodrell Bank Centre for Astrophysics, University of Manchester

Page 156: Michael Mags.
http://en.wikipedia.org/wiki/File:Bouncing_ball_strobe_edit.jpg