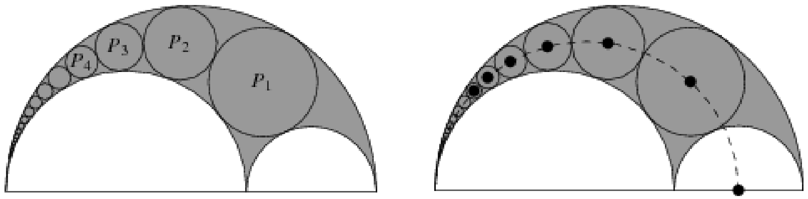
Pappus Chain



Starting with the circle P_1 tangent to the three semicircles forming the arbelos, construct a chain of tangent circles P_i, all tangent to one of the two small interior circles and to the large exterior one. This chain is called the Pappus chain (left figure).

In a Pappus chain, the distance from the center of the first inscribed circle P_1 to the bottom line is twice the circle's radius, from the second circle P_2 is four times the radius, and for the nth circle P_n is $2n$ times the radius. Furthermore, the centers of the circles P_i lie on an ellipse (right figure).

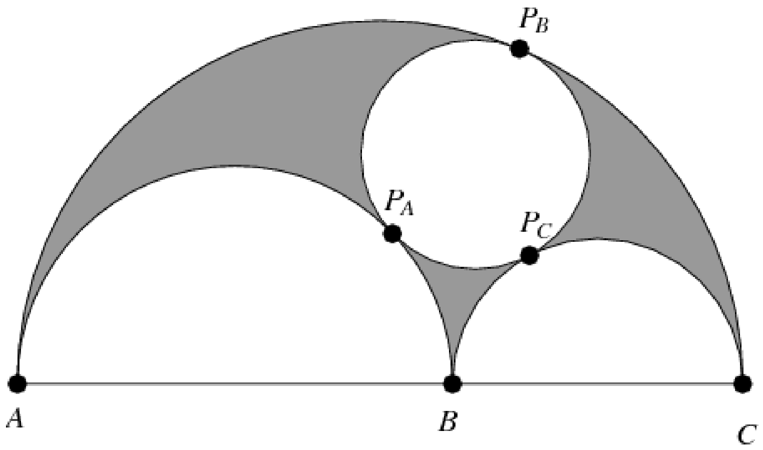
If $r \equiv AB/AC$, then the center and radius of the nth circle P_n in the Pappus chain are

$$x_n = \frac{r(1+r)}{2\left[n^2(1-r)^2+r\right]} \quad (1)$$

$$y_n = \frac{n\,r(1-r)}{n^2(1-r)^2+r} \quad (2)$$

$$r_n = \frac{(1-r)\,r}{2\left[n^2(1-r)^2+r\right]}. \quad (3)$$

This general result simplifies to $r_n = 1/(6+n^2)$ for $r = 2/3$ (Gardner 1979). Further special cases when $AC = 1 + AB$ are considered by Gaba (1940).



The positions of the points of tangency for the first circle are

$$x_A = \frac{r}{(1-r)^2} \quad (4)$$

$$y_A = \frac{r(1-r)}{(1-r)^2} \quad (5)$$

$$x_B = \frac{r(1+r)}{1+r^2} \quad (6)$$

$$y_B = \frac{r(1-r)}{1+r^2} \quad (7)$$

$$x_C = \frac{r^2}{1-2r+2r^2} \quad (8)$$

$$y_C = \frac{r(1-r)}{1-2r+2r^2}. \quad (9)$$

The diameter of the nth circle P_n is given by $(1/n)$th the perpendicular distance to the base of the semicircle. This result was known to Pappus, who referred to it as an ancient theorem (Hood 1961, Cadwell 1966, Gardner 1979, Bankoff 1981). Note that this is also valid for the chain of tangent circles starting with P_1 and tangent to the two interior semicircles of the arbelos. The simplest proof is via inversive geometry.

Eliminating n from the equations for x_n and y_n, the center (x_n, y_n) of the circle P_n, gives

$$4rx^2 - 2r(1+r)x + (1+r)^2 y^2 = 0. \quad (10)$$

Completing the square gives

$$4\,r\left[x-\tfrac{1}{4}(1+r)\right]^2+\left(1+r^2\right)y^2=\tfrac{1}{4}\,r\,(1+r)^2, \tag{11}$$

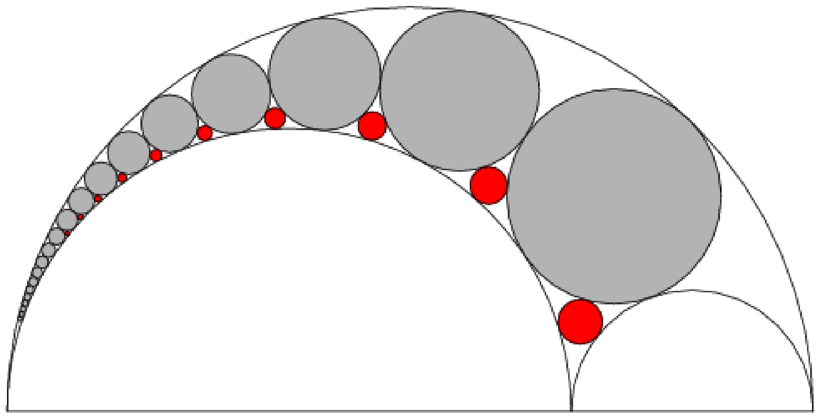
which can be rearranged as

$$\left[\frac{x-\frac{1}{4}(1+r)}{\frac{1}{4}(1+r)}\right]^2+\left(\frac{y}{\frac{1}{2}\sqrt{r}}\right)^2=1, \tag{12}$$

which is simply the equation of an ellipse having center $((1+r)/4,\,0)$ and semimajor and semiminor axes $(1+r)/4$ and $\sqrt{r}\,/2$ respectively. Since

$$c=\sqrt{a^2-b^2}=\tfrac{1}{4}(1-r), \tag{13}$$

$(1+r)/4\pm c=r/2$ and 1/2, so the ellipse has foci at the centers of the semicircles bounding the chain.



The circles T_n tangent to the first arbelos semicircle and adjacent Pappus circles P_{n-1} and P_n have positions and sizes

$$x_n'=\frac{r(7+r)}{2\left[4+4n(n-1)(1-r)^2+r(r-1)\right]} \tag{14}$$

$$y_n'=\frac{2(2n-1)r(1-r)}{4+4n(n-1)(1-r)^2+r(r-1)} \tag{15}$$

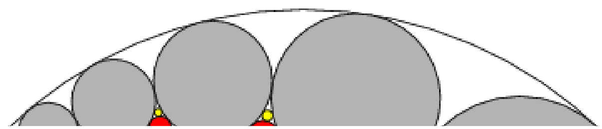
$$r_n'=\frac{r(1-r)}{2\left[4+4n(n-1)(1-r)^2+r(r-1)\right]}. \tag{16}$$

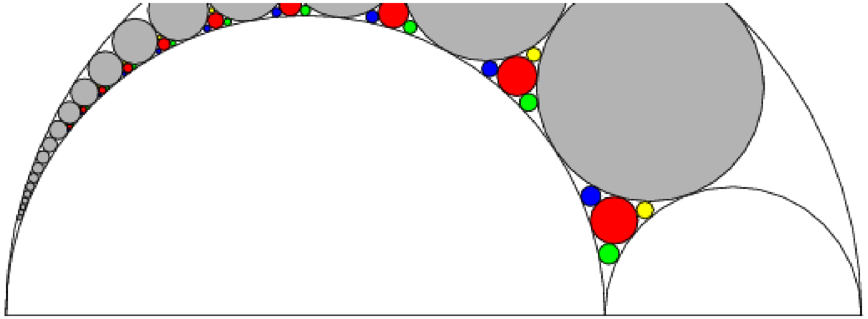
A special case of this problem with $r=1/2$ (giving equal circles forming the arbelos) was considered in a Japanese temple tablet (Sangaku problem) from 1788 in the Tokyo prefecture (Rothman 1998). In this case, the solution simplifies to

$$x_n'=\frac{15}{2\left(15-4n+4n^2\right)} \tag{17}$$

$$y_n'=\frac{2(2n-1)}{15-4n+4n^2} \tag{18}$$

$$r_n'=\frac{1}{2\left(15-4n+4n^2\right)}. \tag{19}$$





Furthermore, the positions and radii of the three tangent circles surrounding this circle can also be found analytically, and are given by

$$x_n^{(1)} = \frac{r(17+r)}{2\left[12+3n(3n-4)(1-r)^2+r(4r-7)\right]} \quad (20)$$

$$y_n^{(1)} = \frac{3(3n-2)(1-r)r}{12+3n(3n-4)(1-r)^2+r(4r-7)} \quad (21)$$

$$r_n^{(1)} = \frac{r(1-r)}{2\left[12+3n(3n-4)(1-r)^2+r(4r-7)\right]} \quad (22)$$

$$x_n^{(2)} = \frac{r(17+r)}{2\left[9+3n(3n-2)(1-r)^2-r(1-r)\right]} \quad (23)$$

$$y_n^{(2)} = \frac{3(3n-1)(1-r)r}{9+3n(3n-2)(1-r)^2-r(1-r)} \quad (24)$$

$$r_n^{(2)} = \frac{r(1-r)}{2\left[9+3n(3n-2)(1-r)^2-r(1-r)\right]} \quad (25)$$

$$x_n^{(3)} = \frac{r(17+7r)}{2\left[9+12n(n-1)(1-r)^2+r(4r-1)\right]} \quad (26)$$

$$y_n^{(3)} = \frac{6(2n-1)(1-r)r}{9+12n(n-1)(1-r)^2+r(4r-1)} \quad (27)$$

$$r_n^{(3)} = \frac{r(1-r)}{2\left[9+12n(n-1)(1-r)^2+r(4r-1)\right]}. \quad (28)$$

If B divides AC in the golden ratio ϕ, then the circles in the chain satisfy a number of other special properties (Bankoff 1955).

In each arbelos, there are two Pappus chains P_i and P_i', with $P_1 = P_1'$. For fixed n, the line connecting the centers of P_n and P_n' passes through the external similitude center S of the two smaller semicircles of the arbelos. The line connecting the point of tangency of P_n and P_{n+1} and the point of tangency of P_n' and P_{n-1}' passes through S as well. Also the line connecting the point of tangency of P_n and the large exterior semicircle (the smaller interior semicircle) and the point of tangency of P_n' and the large exterior semicircle (the smaller interior semicircle) passes through S. This can be proven with circle inversion. In particular, since $P_1 = P_1'$, the common tangent of P_1 and the large exterior semicircle passes through S.

SEE ALSO:
Arbelos, Coxeter's Loxodromic Sequence of Tangent Circles, Pappus's Centroid Theorem, Pappus's Harmonic Theorem, Pappus's Hexagon Theorem, Six Circles Theorem, Soddy Circles, Steiner Chain

Portions of this entry contributed by Floor van Lamoen

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