Survey of Calculus

Exercise: Consider the function $f(x) = x^3 + 3x^2 + 2$. Find the slope and equation of the tangent line at $x = 2$.

Solution: We will use the formula for the slope of the tangent line, which states

$$\text{Slope of Tangent } = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

First, we find $f(x+h)$.

$$\begin{aligned}
f(x+h) &= (x+h)^3 + 3(x+h)^2 + 2 \\
&= (x+h)(x+h)(x+h) + 3(x+h)(x+h) + 2 \\
&= (x^2 + 2xh + h^2)(x+h) + 3(x^2 + 2xh + h^2) + 2 \\
&= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3 + 3x^2 + 6xh + 3h^2 + 2 \quad \text{combining like terms,} \\
&= x^3 + 3x^2h + 3xh^2 + h^2 + 3x^2 + 6xh + 3h^2 + 2.
\end{aligned}$$

Now, we find $f(x+h) - f(x)$. This should cancel out each term in $f(x)$.

$$\begin{aligned}
f(x+h) - f(x) &= x^3 + 3x^2h + 3xh^2 + h^2 + 3x^2 + 6xh + 3h^2 + 2 - (x^3 + 3x^2 + 2) \\
&= x^3 + 3x^2h + 3xh^2 + h^2 + 3x^2 + 6xh + 3h^2 + 2 - x^3 - 3x^2 - 2 \\
&= 3x^2h + 3xh^2 + h^3 + 6xh + 3h^2
\end{aligned}$$

You should be left with terms that each have a common factor of h. Factor this h out to cancel the denominator.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{3x^2h + 3xh^2 + h^3 + 6xh + 3h^2}{h} \\
&= \frac{h(3x^2 + 3xh + h^2 + 6x + 3h)}{h} \\
&= 3x^2 + 3xh + h^2 + 6x + 3h
\end{aligned}$$

Now, we apply the limit.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 6x + 3h \\
&= 3x^2 + 3x(0) + (0)^2 + 6x + 3(0) \\
&= 3x^2 + 6x.
\end{aligned}$$

Therefore, the slope of the tangent line is represented by $3x^2 + 6x$. Since we want to find the slope *at* $x = 2$, we evaluate,

$$3(2^2) + 6(2) = 3(4) + 12 = 24.$$

So the slope of the tangent line at $x = 2$ is 24.

Now, to find the equation of the tangent line, we need two pieces of information: the slope, and an ordered pair on the line. Using $x = 2$, we evaluate $f(2)$,

$$\begin{aligned}
f(2) &= 2^3 + 3(2^2) + 2 \\
&= 8 + 3(4) + 2 \\
&= 22.
\end{aligned}$$

We see that our ordered pair is $(2, 22)$.

Now we have our two pieces of information we need to find the equation of the line. There are two ways to use this information. We can use point-slope form, or slope-intercept form. Usually, point-slope form is easier, but both will give the same answer.

Point-Slope Form

The formula for point-slope form is: $(y - y_1) = m(x - x_1)$, where (x_1, y_1) is the ordered pair of a point on the line. In this case, $x_1 = 2$, $y_1 = 22$, and $m = 24$. Plugging in, we have,

$$(y - 22) = 24(x - 2) \qquad \text{first, distribute 24 across } (x - 2)$$
$$y - 22 = 24x - 48 \qquad \text{now adding 22 to both sides,}$$
$$y = 24x - 26$$

Thus, the equation of the line tangent to f at $x = 2$ is represented by $y = 24x - 26$.

Slope-Intercept Form

To utilize the point-slope formula, recall the form is: $y = mx + b$. We know that $x = 2$, $y = 22$, and $m = 24$, so we are left solving for b in this case:

$$22 = (24)(2) + b$$
$$22 = 48 + b \qquad \text{subtracting 48 from both sides,}$$
$$-26 = b$$

Therefore, the point-slope form of the line tangent to f at $x = 2$ is represented by $y = 24x - 26$.

Notice these results are the same!