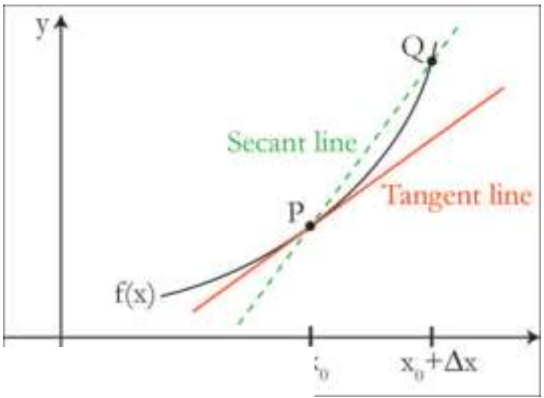
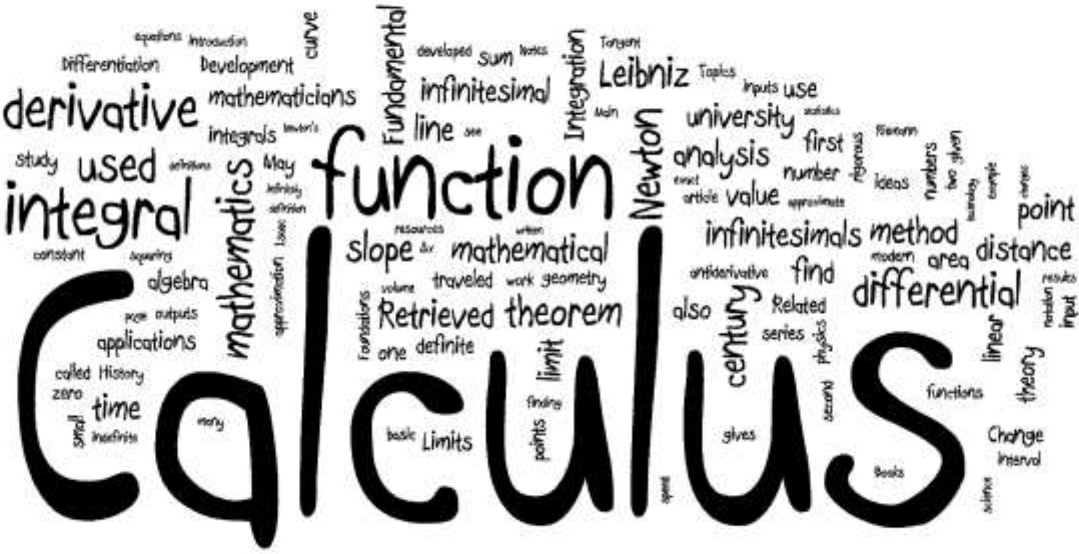
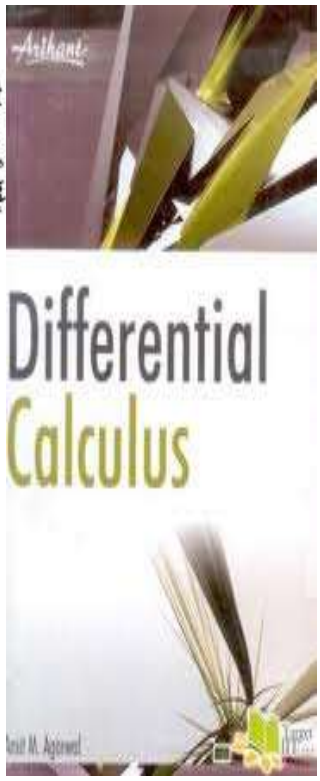
Differential Calculus

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x
\end{aligned}
$$

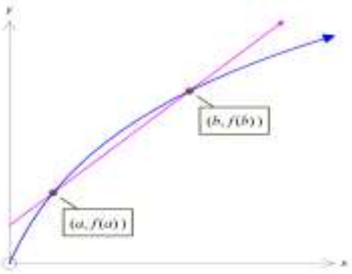






Name: Mr. Wain

Average Rate of Change (AROC)

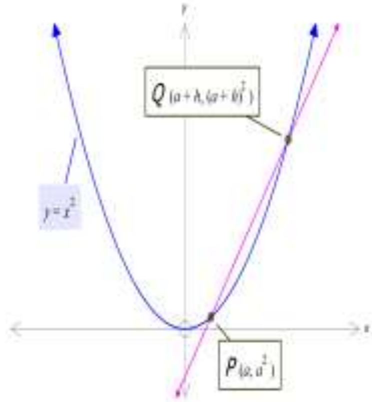


- The *average rate of change* of y over an interval is equal to
  $$\frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}.$$

Example: Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

$$f(x) = x^2 - 2x + 5$$
$$f(1) = (1)^2 - 2(1) + 5 = 4 \quad \& \quad f(5) = (5)^2 - 2(5) + 5 = 20$$
$$AROC = \frac{20 - 4}{5 - 1} = \frac{16}{4} = 4$$

Instantaneous Rate of Change & 1st Principles



- If we look at the graph on the right, $y = x^2$ and wanted to calculate the rate of change at Point *P*, we then calculate the gradient between *P* and *Q*.
- If we bring the point *Q* closer and closer to *P* then the gradient will be approaching the value of the tangent at *P*.
- $m(PQ) = \frac{(a+h)^2 - a^2}{a + h - a} = \frac{a^2 + 2ah + h^2 - a^2}{h} = \frac{2ah + h^2}{h} = 2a + h$
- If *Q* approaches *P* then $h \to 0$, the gradient approaches $2a$.

- The instantaneous rate of change of a function f at point *P* on a graph of $y = f(x)$ is equal to the gradient of the tangent to the graph at *P*. So, to find the instantaneous rate of change at point *P*, we evaluate the derivative of the function at *P*.
  - The instantaneous rate of change of f at $x = a$ is $f'(a)$.

Example: Find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for

(i)
$$f(x) = 3x^2 + 2x + 2$$
$$\frac{f(x+h) - f(x)}{h} =$$
$$\frac{3(x+h)^2 + 2(x+h) + 2 - (3x^2 + 2x + 2)}{h}$$
$$= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 2 - 3x^2 - 2x - 2}{h}$$
$$= \frac{6xh + 3h^2 + 2h}{h}$$
$$= 6x + 3h + 2 \quad \Rightarrow \lim_{h \to 0}(6x + 3h + 2) = 6x + 2$$

(ii)
$$f(x) = 2 - x^3$$
$$\frac{f(x+h) - f(x)}{h} =$$
$$\frac{2 - (x+h)^3 - (2 - x^{3})}{h}$$
$$= \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - 2 + x^3}{h}$$
$$= \frac{-3x^2h - 3xh^2 - h^3}{h}$$
$$= -3x^2 - 3xh - h^2$$
$$\Rightarrow \lim_{h \to 0}(-3x^2 - 3xh - h^2) = -3x^2$$

- **Ex9A** 1, 2, 3, 4 LHS, 8 LHS

The derivative of x^n

- If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ and if $f(x) = ax^n$ then $f'(x) = nax^{n-1}$.

The derivative of a constant

- If $f(x) = c$ then $f'(x) = 0$.

Examples: Find the derivative of the following:

1. $y = 3x^6 - 4x^3$
2. $f(x) = 3x(2x^2 - 7)$
3. $f = 2g^2 - 5$
4. $h = \dfrac{6a^2 + 7a^4}{a}$
5. $y = x + \dfrac{1}{x} + \dfrac{6}{x^3}$
    - **Remember to always to subtract 1 from the power.**
    - **Be careful with + and – signs.**

Solutions

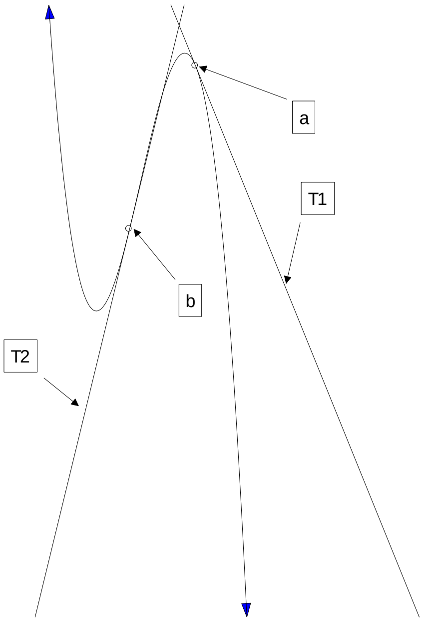
1. $\dfrac{dy}{dx} = 18x^2 - 12x^2$
2. Expand first: $f(x) = 6x^3 - 21x$ therefore $f'(x) = 18x^2 - 21$
3. $\dfrac{df}{dg} = 4g$
4. Simplify first: $h = 6a + 7a^3$ therefore $\dfrac{dh}{da} = 6 + 21a^3$
5. $y = x + x^{-1} + 6x^{-3}$

$\dfrac{dy}{dx} = 1 - x^{-2} - 18x^{-4}$ or $\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2} - \dfrac{18}{x^4}$

- **Ex9B** 1, 2, 4, 5, 6

The Gradient of a Curve

- The gradient of a curve is not constant.
- The gradient of a curve at a certain point is equal to the gradient of the tangent to the curve at that point.
- A tangent is a line that touches another curve at one point only (i.e.it does not cross it).



- The line T_1 is a tangent to the curve at point *a*.
- The line T_2 is a tangent to the curve at point *b*.

- Consider $y = 4x^3 - 8x^2$ and its derivative $\frac{dy}{dx} = 12x^2 - 16x$

- What does all this mean?

- $y = 4x^3 - 8x^2$ is a formula that gives the <u>y-value</u> of the curve at any point x.

- $\frac{dy}{dx} = 12x^2 - 16x$ is a formula that gives the <u>gradient</u> of the curve at any point x.

Example 1: What is the gradient of the curve $y = 4x^3 - 8x^2$ at $x = -2$?

Solution:

$$\frac{dy}{dx} = 12x^2 - 16x$$

$$at\ x = -2$$

$$\frac{dy}{dx} = 12(-2)^2 - 16(-2)$$

$$\frac{dy}{dx} = 80$$

Example 2: What are the co-ordinates of the point(s) of the curve $y = 4x^3 - 8x^2$, where the gradient is -4?

Solution:

$\Rightarrow \frac{dy}{dx} = -4$

$\therefore -4 = 12x^2 - 16x$

$0 = 12x^2 - 16x + 4$

$0 = 4(3x-1)(x-1)$

$\therefore x = \frac{1}{3} \quad or \quad x = 1$

$When\ x = \frac{1}{3},\ y = 4\left(\frac{1}{3}\right)^3 - 8\left(\frac{1}{3}\right)^2 = -\frac{20}{27} \quad \therefore \left(\frac{1}{3}, -\frac{20}{27}\right)$

$When\ x = 1,\ y = 4(1)^3 - 8(1)^2 = -4 \quad \therefore (1, -4)$

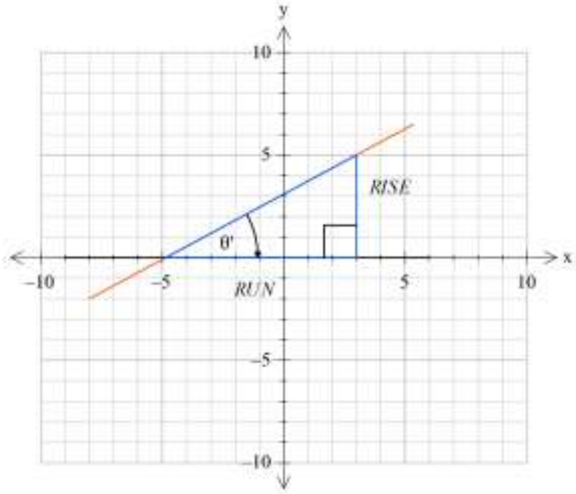
- **Ex 9B** 7, 11, 12, 13, 14, 16, 17 **Ex 9C** 4, 5, 6, 8, 10

Notes: {*x*: *h* ‘ (*x*) > 0 }

Means:	{*x*: }	Find the *x*-values where
	h ‘ (*x*)	The gradient function,
	> 0	that is positive

m = tan *θ*

$\tan\theta = \frac{opp}{adj} = \frac{O}{A}$



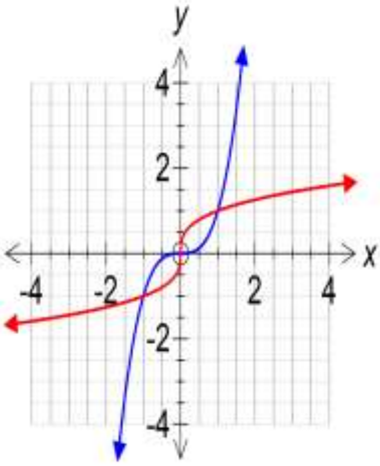
Strictly increasing and strictly decreasing functions

A function f is said to be *strictly increasing* when $a < b$ implies $f(a)<f(b)$ for all a and b in its domain.
The definition does not require f to be differentiable, or to have a non-zero derivative, for all elements of the domain.
If a function is strictly increasing, then it is a one-to-one function and has an inverse that is also strictly increasing.

- **If $f'(x) > 0$ for all x in the interval then the function is strictly increasing.**
- **If $f'(x) < 0$ for all x in the interval then the function is strictly decreasing.**

Strictly Increasing

Example 1: The function $f : R \to R, f(x) = x^3$ is strictly increasing with zero gradient at the origin.

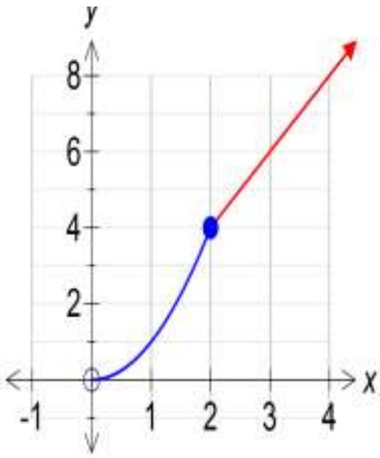


The inverse function $f^{-1} : R \to R, f^{-1}(x) = x^{\frac{1}{3}}$, is also strictly increasing, with a vertical tangent of undefined gradient at the origin.

Example 2: The hybrid function g with domain $[0, \infty)$ and rule:

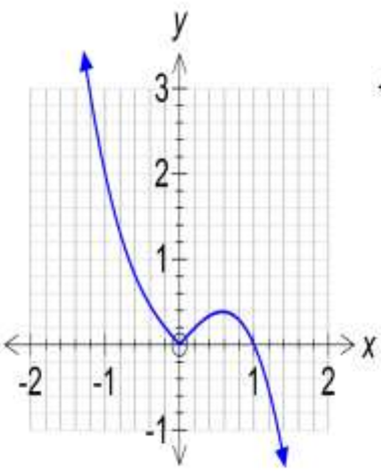
$$g(x) = \begin{cases} x^2 & 0 \le x \le 2 \\ 2x & x > 2 \end{cases}$$

is strictly increasing, and is not differentiable at $x = 2$.



Example 3: Consider
$h : R \to R, h(x) = |x| - x^3$

H is not strictly increasing,
But is strictly increasing over the *interval* $\left[0, \frac{1}{\sqrt{3}}\right]$.

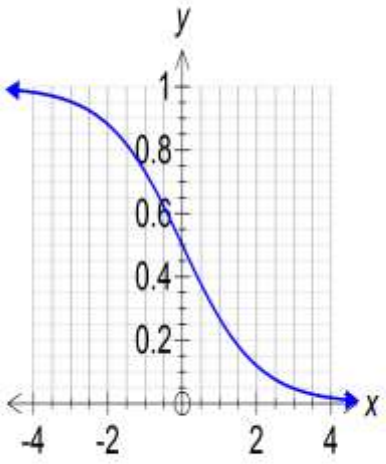


Strictly Decreasing

A function f is said to be *strictly decreasing* when $a < b$ implies $f(a) > f(b)$ for all a and b in its domain.
A function is said to be strictly decreasing over an interval when $a < b$ implies $f(a) > f(b)$ for all a and b in its interval.

Example 4: The function $f : R \to R,\ f(x) = \dfrac{1}{e^x + 1}$

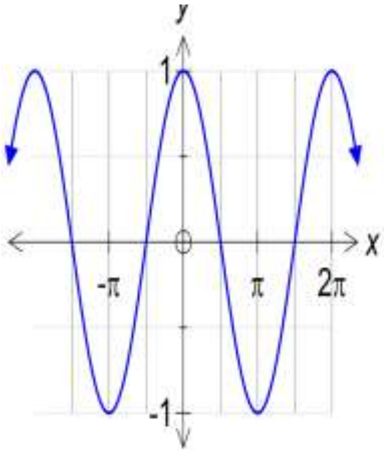
The function is strictly decreasing over R.



Example 5: The function $g : R \to R, g(x) = \cos(x)$

g is not strictly decreasing.

But g is strictly decreasing over the interval $[0, \pi]$.
(also $[-2\pi, -\pi]$ and $[2\pi, 3\pi]$ etc.

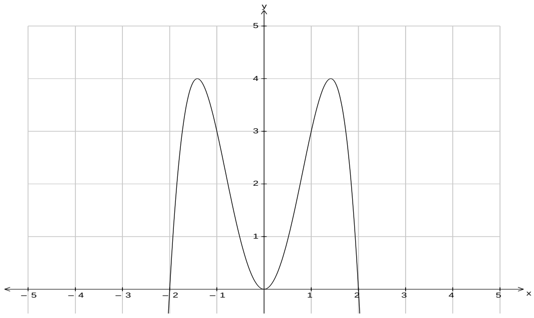


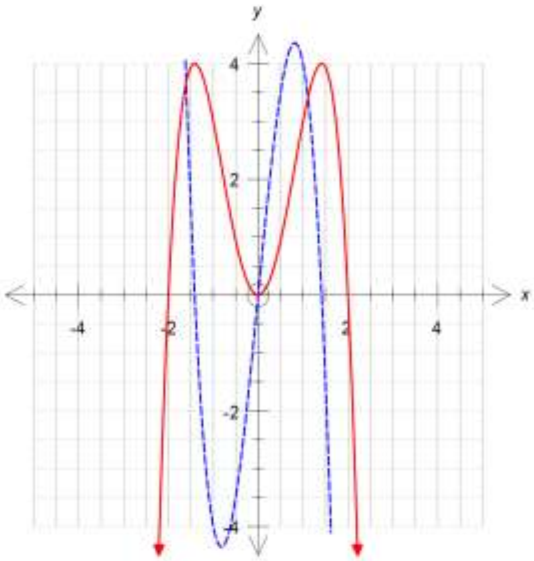
- **Ex9B** 18, 19, 20, 21,

Sketching the Gradient Function

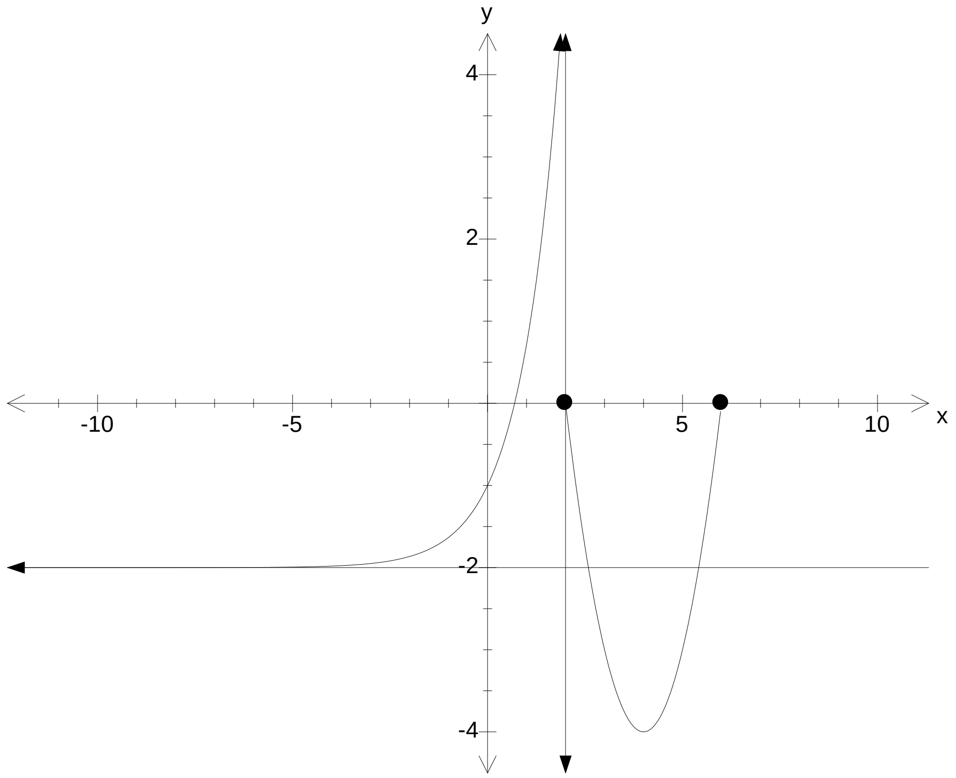
GRAPH OF THE ORIGINAL FUNCTION	GRAPH OF THE GRADIENT FUNCTION
Where the gradient is flat (i.e at all stationary points)	Will cross the x – axis
Where there is a positive gradient (i.e. slopes ↗)	Will be above the x – axis
Where there is a negative gradient (i.e. slopes ↘)	Will be below the x – axis
Where the gradient gets flatter	Gets closer to x – axis
Where the gradient gets steeper	Gets further away from x – axis
At the steepest part of each 'section' of the graph	Will have a 'peak'

Example 1: Sketching the Gradient Function $y = 4x^2 - x^4$

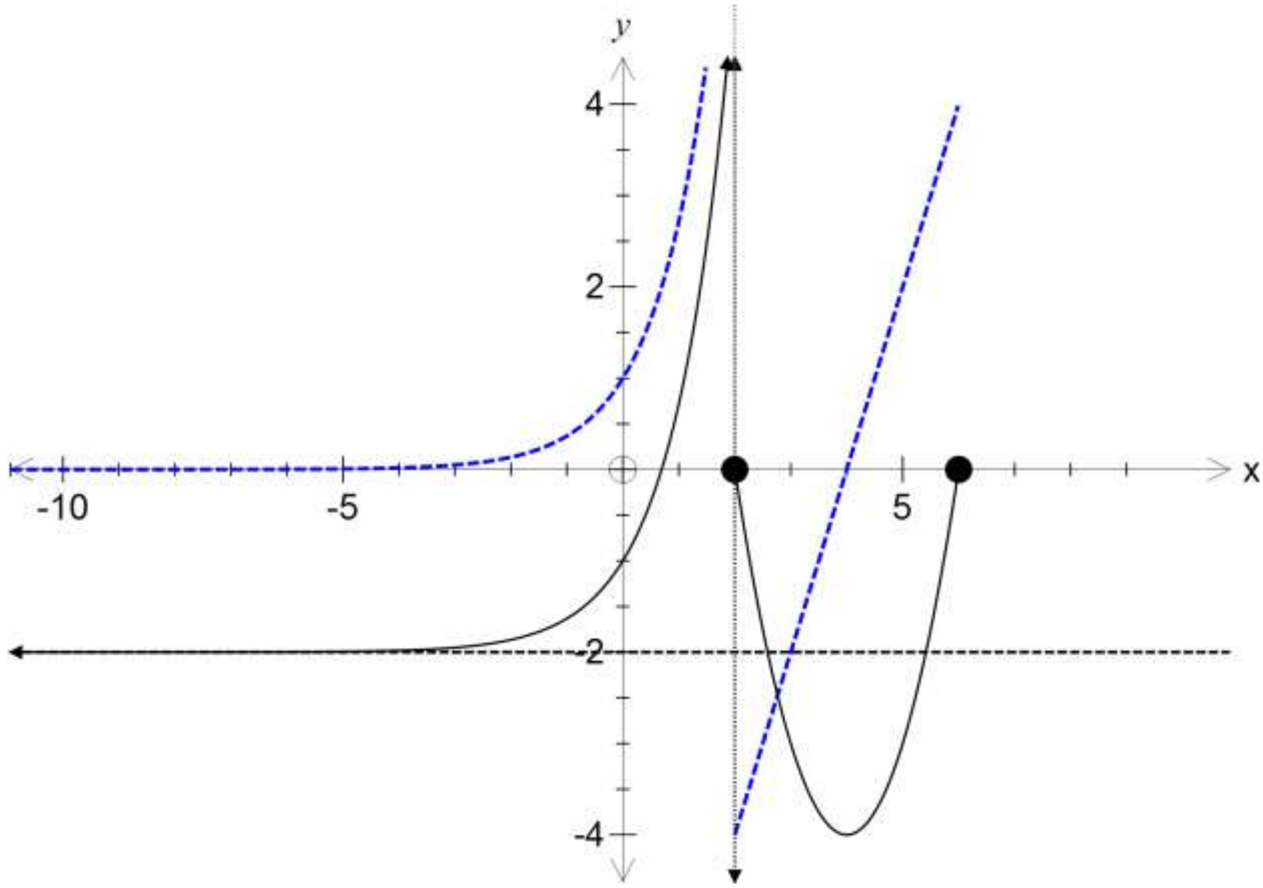




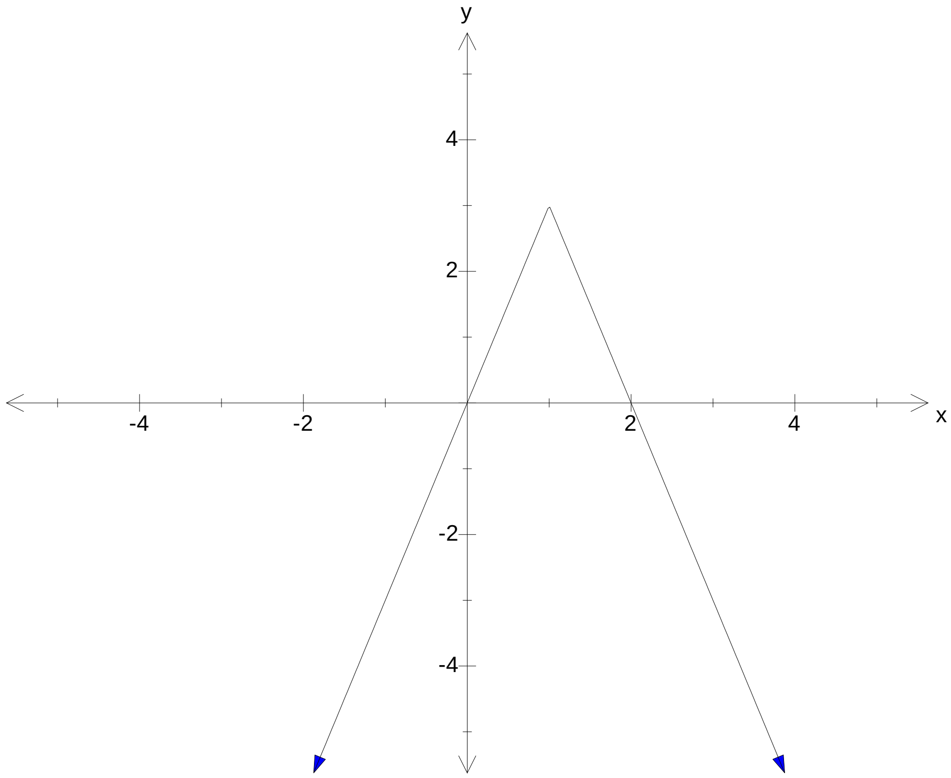
Example 2: Sketch the gradient graph of:



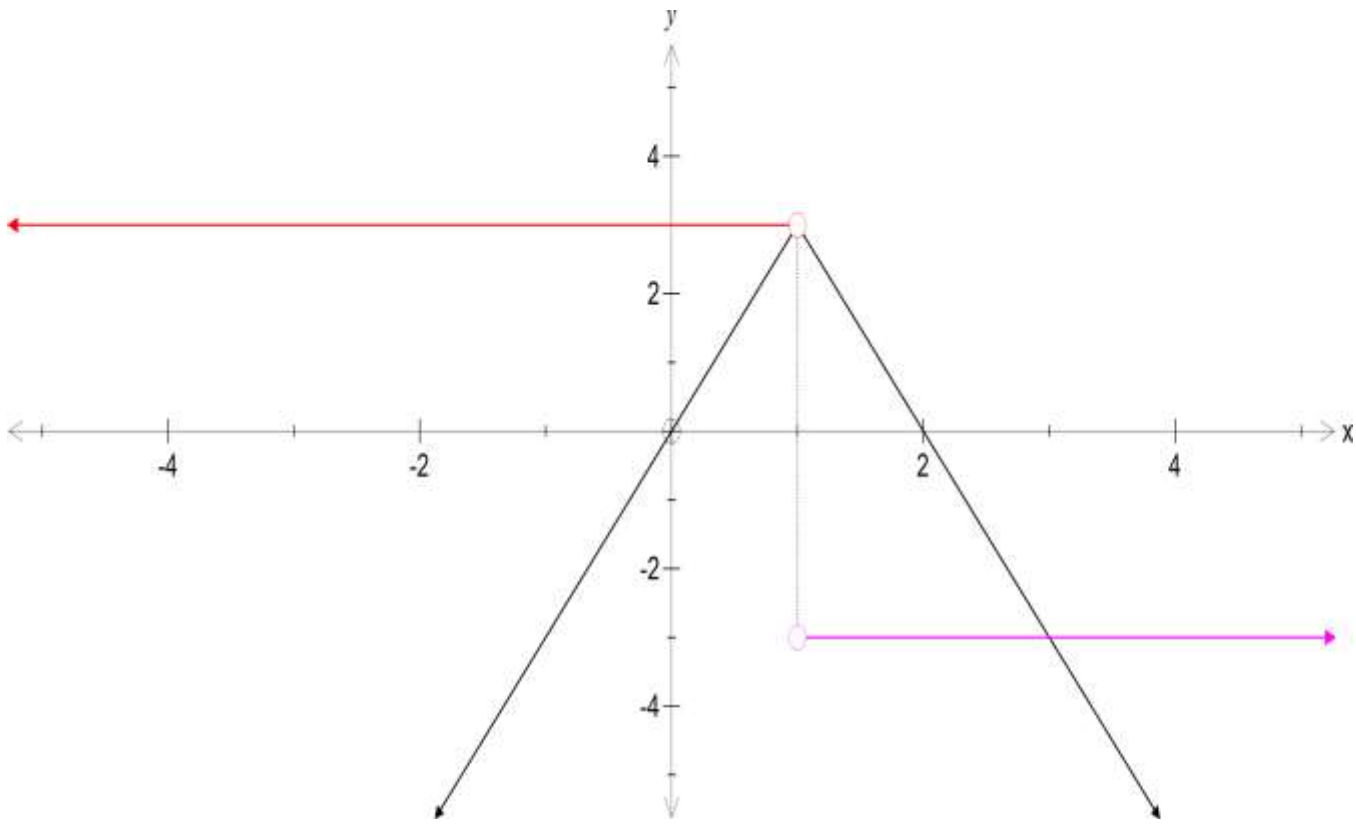
Solution:



Example 3: Sketch the gradient function of:



Solution:



- **Ex9D** 1 acdefhi, 2 acdegi, 3, 5, 6, 7

Chain Rule – The derivative of (function)^n

(The function in a function rule or Composite Function rule).

Example 1: Find the derivative of $y = -3(14x^2 - x)^4$**.**

Solution:

<u>In words:</u> *find the derivative of "the thing" as a whole, then multiply it by the derivative of the "inside".*

$$\frac{dy}{dx} = -12(14x^2 - x)^3(28x-1)$$

$$\frac{dy}{dx} = -12[x(14x-1)]^3(28x-1)$$

$$\frac{dy}{dx} = -12x^3(14x-1)^3(28x-1)$$

<u>In symbols:</u> *If* $y = (u)^n$ *where* $u = f(x)$, *then* $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Let $u = 14x^2 - x$, $y = -3u^4$

$\frac{du}{dx} = 28x - 1,$ $\quad \frac{dy}{du} = -12u^3 = -12(14x^2 - x)^3$

$\therefore \frac{dy}{dx} = -12(14x^2 - x)^3(28x-1)$ etc…

Example 2: Find $f'(x)$ **if** $f(x) = (2x^4 - 3)^{18}$**.**

$$f'(x) = 18(2x^4 - 3)^{17}(8x^3)$$

$$f'(x) = 144x^3(2x^4 - 3)^{17}$$

Example 3: If $y = \sqrt{x^3 - 3}$ **then find** $\frac{dy}{dx}$**.**

$$y = (x^3 - 3)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^3 - 3)^{-\frac{1}{2}}(3x^2)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{(x^3 - 3)^{\frac{1}{2}}} \times 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2(x^3 - 3)^{\frac{1}{2}}}$$

- **Ex9E** 1, 2, 3, 5

$$\frac{d\ f(g(x))}{dx} = g'(x).f'(g(x))$$

Example 4

$$\frac{d}{dx}\left(f(x^2)\right)$$

$$= 2x.f'(x^2)$$

Example 5

$$\frac{d}{dx}\left((f(x))^3\right)$$

$$= 3f'(x).(f(x))^2$$

Differentiating Rational Powers

Examples: Find the derivative of each of the following with respect to *x*.

(a)

$$y = \frac{2}{\sqrt[5]{x}} + 3x^{\frac{2}{7}}$$

$$y = 2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}$$

$$\frac{dy}{dx} = 2\left(-\frac{1}{5}x^{-\frac{6}{5}}\right) + 3\left(\frac{2}{7}x^{-\frac{5}{7}}\right)$$

$$\frac{dy}{dx} = -\frac{2}{5}x^{-\frac{6}{5}} + \frac{6}{7}x^{-\frac{5}{7}}$$

or

$$= \frac{-2}{5\sqrt[5]{x^6}} + \frac{6}{7\sqrt[7]{x^5}}$$

(b)

$$f(x) = \sqrt[3]{x^2 + 2x}$$

$$f(x) = \left(x^2 + 2x\right)^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}\left(x^2 + 2x\right)^{-\frac{2}{3}} \times (2x + 2) \quad \textit{chain rule}$$

$$f'(x) = \frac{2x + 2}{3\sqrt[3]{\left(x^2 + 2x\right)^2}}$$

- **Ex9F** 2, 3,4, 6, 7

- **<u>Derivatives of Transcendental functions</u>**

The derivative of e^{kx}

- In general: If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$. If $y = ae^{f(x)}$ then $\frac{dy}{dx} = af'(x)e^{f(x)}$

Example 1: Find the derivatives of:

(i) $y = e^{2x}$

(ii) $y = e^{-5x}$

(iii) $y = e^{(x^2+2x)}$

Solutions:

(i) $\frac{dy}{dx} = 2e^{2x}$

(ii) $\frac{dy}{dx} = -5e^{-5x}$

(iii) $\frac{dy}{dx} = (2x+2)e^{(x^2+2x)}$

Example 2: Find $f'(x)$ given $f(x) = x^2e^{4x}$.

Solution:

Product Rule:

$f'(x) = (2x)(e^{4x}) + (4)e^{4x}(x^2)$

$HCF = 2xe^{4x}$

$f'(x) = 2xe^{4x}(1+2x)$

- **Ex9G** 1, 2, 3, 4, 5, 6

Derivative of $\log_e x$

- In general, if $y = \log_e x$ then $\frac{dy}{dx} = \frac{1}{x}$
- If $y = \log_e(h(x))$ then $\frac{dy}{dx} = \frac{1}{h(x)} \times h'(x) = \frac{h'(x)}{h(x)}$

  $y = \log_e|x|$
- If $y = \begin{cases} \log_e x, x > 0 \\ \log_e(-x), x < 0 \end{cases}$

  $dy = \begin{cases} \frac{1}{x}, x > 0 \\ \frac{1}{-x} \times -1 = \frac{1}{x}, x < 0 \end{cases} = \frac{1}{x},\ \textit{for } x \in R \setminus \{0\}$

Examples: Find the derivatives of:

(i) $y = \log_e 3x$

(ii) $y = \log_e(x^2 + x)$

(iii) $y = \log_e x^2 + x$

Solution:

(i) $\frac{dy}{dx} = \frac{1}{3x} \times 3 = \frac{1}{x}$ (note: any rule of the form $y = \log_e(kx)$ has a derivative of $\frac{1}{x}$) , $x \neq 0$

(ii) $\frac{dy}{dx} = \frac{1}{x^2 + x} \times (2x+1) = \frac{2x+1}{x^2 + x}$, $x \neq -1, 0$

(iii) $\frac{dy}{dx} = \frac{1}{x^2} \times 2x + 1 = \frac{2x}{x^2} + 1 = \frac{2}{x} + 1$, $x \neq 0$

Ex9H 1, 2, 3, 4, 5, 6, 7, 8

- **Derivative of the Trigonometric Functions**

- If $y = \sin(kx)$ then $\dfrac{dy}{dx} = k\cos(kx)$
- If $y = \cos(kx)$ then $\dfrac{dy}{dx} = -k\sin(kx)$
- $y = \tan(kx)$ then $\dfrac{dy}{dx} = k\sec^2(kx)$ or $\dfrac{k}{\cos^2(kx)}$

Examples: Find the derivative of the following:

(i) $y = \cos\left(\dfrac{x}{3}\right) = \cos\left(\dfrac{1}{3}x\right)$

(ii) $y = \sin(x^3) =$

(iii) $y = \sin^3 x = (\sin x)^3$

(iv) $y = 3\tan(2x)$

(v) $y = \cos(3x^2 + 2)$

Solution:

(i) $\dfrac{dy}{dx} = -\dfrac{1}{3}\sin\left(\dfrac{x}{3}\right)$

(ii) $\dfrac{dy}{dx} = 3x^2\cos(x^3)$

(iii) $\dfrac{dy}{dx} = 3\cos x\sin^2 x$

(iv) $\dfrac{dy}{dx} = 3\times 2.\sec^2 2x = 6\sec^2 2x$

(v) $\dfrac{dy}{dx} = -6x\sin(3x^2 + 2)$

- **Ex9I** 1, 2, 3, 4, 5, 6

NOTE: Angle MUST be in RADIANS

$$\theta^c = \frac{\pi}{180}\times\theta^o$$

e.g.

$$\sin(x^o) = \sin\left(\frac{\pi x}{180}\right)$$

$$\frac{d}{dx}\left(\sin(x^o)\right) = \frac{d}{dx}\left(\sin\left(\frac{\pi x}{180}\right)\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) \quad or \quad \frac{\pi}{180}\cos(x^o)$$

The Product Rule – The derivative of the product of two functions

Example 1: Find $\frac{dy}{dx}$ (using the product rule) if $y = 3x^2(x^2 - 2x)$.

Solution:

In words: The derivative of the first term multiplied by the second term, ADD the derivative of the second term multiplied by the first term.

In symbols: If $y = u.v$ *then* $\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx}$

In the above example,

$u = 3x^2$ *and* $v = x^2 - 2x$

$\frac{du}{dx} = 6x$ *and* $\frac{dv}{dx} = 2x - 2$

$$\frac{dy}{dx} = 6x(x^2 - 2x) + (2x - 2)(3x^2)$$
$$\frac{dy}{dx} = 6x^3 - 12x^2 + 6x^3 - 6x^2$$
$$\frac{dy}{dx} = 12x^3 - 18x^2$$

Example 2: Find $f'(x)$ if $f(x) = \sqrt{x}(4x^3 - 12)$.

Solution:

Let $u = \sqrt{x}$ *and* $v = 4x^3 - 12$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \quad and \quad \frac{dv}{dx} = 12x^2$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(4x^3 - 12) + 12x^2(x^{\frac{1}{2}})$$
$$\frac{dy}{dx} = \frac{4x^3 - 12}{2x^{\frac{1}{2}}} + 12x^{\frac{5}{2}}$$
$$\frac{dy}{dx} = \frac{4x^3 - 12 + 24x^3}{2x^{\frac{1}{2}}}$$
$$\frac{dy}{dx} = \frac{28x^3 - 12}{2x^{\frac{1}{2}}} = \frac{2(14x^3 - 6)}{2x^{\frac{1}{2}}} = \frac{14x^3 - 6}{\sqrt{x}}$$

Example 3: Find $f'(x)$ if $f(x) = e^{2x}\sin(2x + 1)$.

Solution:

Let $u = e^{2x}$ *and* $v = \sin(2x + 1)$

$$\frac{du}{dx} = 2e^{2x} \quad and \quad \frac{dv}{dx} = 2\cos(2x + 1)$$
$$\frac{dy}{dx} = 2e^{2x}.\sin(2x + 1) + e^{2x}.2\cos(2x + 1)$$
$$\frac{dy}{dx} = 2e^{2x}\left(\sin(2x + 1) + \cos(2x + 1)\right)$$

- **Ex9J** 1, 2, 3, 4, 5, 6, 7, 8

Quotient Rule – The derivative of the quotient of two functions

- Used when you have a problem in fraction form.

Example 1: If $y=\frac{2x+4}{3x-7}$ then find $\frac{dy}{dx}$

Solution:

<u>In words:</u> The derivative of the top term, multiplied by the bottom term, subtract the derivative of the bottom term, multiplied by the top term, all over the bottom term squared.

<u>In symbols:</u> If $y=\frac{u}{v}$ *then* $\frac{dy}{dx}=\frac{v.\frac{du}{dx}-u.\frac{dv}{dx}}{v^2}$,

In the above example:

$u=2x+4\ (top)\quad \frac{du}{dx}=2$

$v=3x-7\ (bottom)\quad \frac{dv}{dx}=3$

$$\frac{dy}{dx}=\frac{2(3x-7)-3(2x+4)}{(3x-7)^2}$$

$$\frac{dy}{dx}=\frac{6x-14-6x-12}{(3x-7)^2}$$

$$\frac{dy}{dx}=\frac{-26}{(3x-7)^2}$$

Example 2: If $y=\frac{x^2-1}{x^2+1}$ then find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx}=\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}=\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

Example 3: Find $\frac{dy}{dx}$ if $y=\frac{e^x}{e^{2x}+1}$.

Solution:

$$\frac{dy}{dx}=\frac{(e^{2x}+1).e^x-e^x.2e^{2x}}{(e^{2x}+1)^2}$$

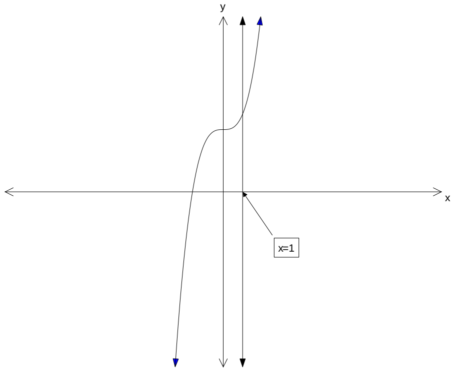
$$\frac{dy}{dx}=\frac{e^x(e^{2x}+1-2e^{2x})}{(e^{2x}+1)^2}$$

$$\frac{dy}{dx}=\frac{e^x(1-e^{2x})}{(e^{2x}+1)^2}$$

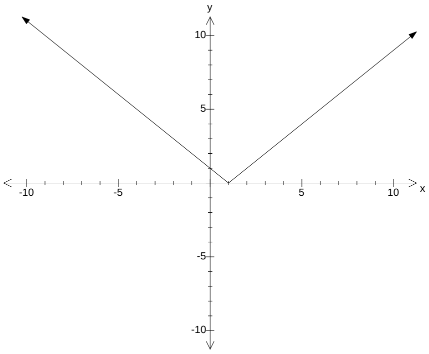
- **Ex9K** 1 aceg, 2 de, 4, 5a, 6a, 7

Continuous functions and Differentiable Functions

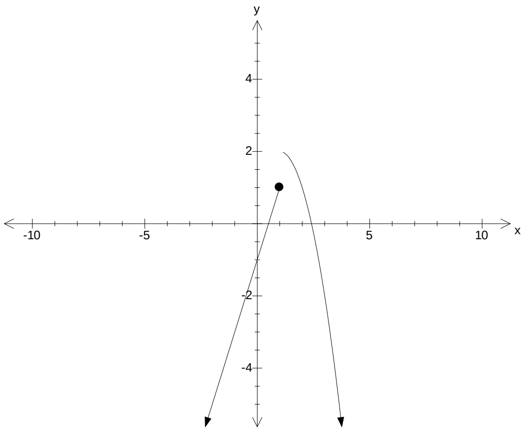
- The graph of a continuous function is one without breaks.
- It is usually a smooth unbroken curve, however it may have sharp corners.
- If the derivative of a function exists at a point on a curve this function is said to be **differentiable** at this point.
- The derivative exists at a point if it is possible to draw a tangent at that point. i.e. the curve must be **smooth** and **continuous**.



At $x = 1$, the graph is continuous and differentiable.



At $x = 1$, the graph is continuous BUT NOT differentiable.



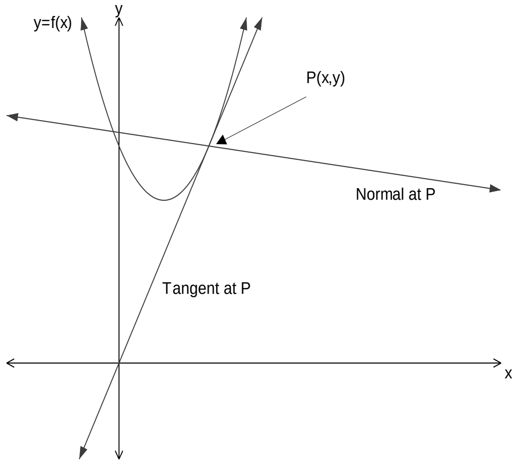
At $x = 1$, the graph is neither continuous or differentiable.

- **Ex9L** 2, 3, 4 **Ex9M** 1, 2, 3, 5

Note: No derivative exists at:

- "CUSP" point
- "END-POINT" (open or closed)
- "HOLE" point

Finding the Equation of a Tangent and a Normal



- P (x, y) is a point on the curve $y = f(x)$.
- The Normal and Tangent are at right angles to each other.
- If m = gradient of the curve at P, then the gradient of the tangent at Point P = m
- The gradient of the normal at point P is $\frac{-1}{m}$
- **The equation of the Tangent is:** $y - y_1 = m(x - x_1)$.
- **The equation of the Normal is:** $y - y_1 = \frac{-1}{m}(x - x_1)$.
- How would you find m if you knew the equation of the curve?
- Find $\frac{dy}{dx}$ and substitute the x - coordinate of P into it.

Example 1: Find the equation of the tangent and of the normal to the curve $y = (2x+1)^9$ at the point $(0,1)$.

Solution: $\frac{dy}{dx} = (9)(2)(2x+1)^8 = 18(2x+1)^8$

At $x = 0$, $\frac{dy}{dx} = 18(2(0)+1)^8 = 18$, so m(tangent at x = 0) = 18 & m(normal at x = 0) = $\frac{-1}{18}$

Equation of the tangent:

$y = 18x + c$

$(0, 1) \therefore 1 = 18(0) + c$

$c = 1$

$y = 18x + 1$

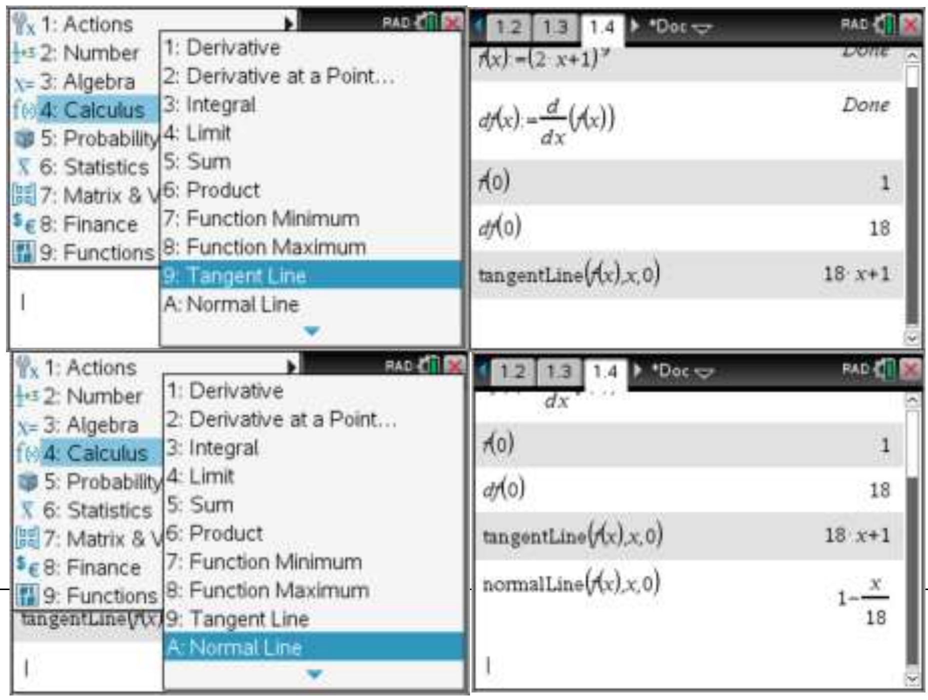
Equation of the normal:

$y = \frac{-1}{18}x + c$

$(0, 1) \therefore 1 = \frac{-1}{18}(0) + c$

$0 = c$

$y = \frac{-x}{18} + 1$



- **Ex10A** 1, 2, 3, 6, 8ac, 9abc, 14, 16

Rates of Change

- What is a rate?
- If you work and earn \$12 an hour your rate of pay = \$12 per hour = \$12/hr.
- This is linked with calculus by …

  If P = total Pay (\$)

  & t = time worked (hr)

  The $P = 12t$

  & $\frac{dP}{dt} = 12$

  $\frac{dP}{dt} =$ rate of change of P with respect to t.

  - For the unit of $\frac{dP}{dt}$, \$ per hour, $\frac{\$}{hr}$.

If you have to find…	**Choose letters for the 2 variables**	**The rate you need is…**	**So you'll need an equation relating…**	**Unit of rate is …**
The rate of change of volume with respect to the radius	V= volume r = radius	$\frac{dV}{dr}$	V and r	$\frac{cm^3}{cm}$
The rate of increase of cost of production of dolls w.r.t the number of dolls	C = cost n = no. of dolls	$\frac{dC}{dn}$	C and n	$\frac{\$}{doll}$
The rate of change of circumference w.r.t height	C = circumference h = height	$\frac{dC}{dh}$	C and h	$\frac{mm}{mm}$
The rate of decrease of amount of water in a draining tank	V = volume t = time	$\frac{dV}{dt}$	V and t	$\frac{m^3}{\text{min}}$

- In the last case, what's missing? w.r.t 2^{nd} "variable" assumes it is time.
- Solving a rate problem is very, very similar to solving max/min prob.
  1. need what rate? (no second variable – assume time)
  2. find a formula.
  3. formula must be in terms of one variable only, if not a relationship between the variables by other info. From question.
  4. find the rate.
  5. substitute given value of second variable, include units
  6. answer all questions. If rate is positive, it is increasing, if the rate is negative, it is decreasing.

Example 1: A spherical balloon is being inflated. Find the rate of increase of volume with respect to the radius when the radius is 10cm.

Solution:

1. $\frac{dV}{dr}$
2. $V = \frac{4}{3}\pi r^3$
3. ✓
4. $\frac{dV}{dr} = 4\pi r^2$
5. when r = 10 cm
   $\frac{dV}{dr} = 4\pi(10)^2 = 400\pi\ cm^3/cm$
6. volume of the sphere is increasing at a rate of $400\pi\ cm^3/cm$.

Example 2: The amount of water in a tank (A litres) at any time (seconds) is given by $A = \frac{3}{t}$. Find the rate of change of A when $t = 5s$.

Solution:

1. need $\frac{dA}{dt}$
2. $A = \frac{3}{t}$
3. ✓
   $A = 3t^{-1}$
4. $\frac{dA}{dt} = -3t^{-2}$
   $\frac{dA}{dt} = \frac{-3}{t^2}$
5. when $t = 5$
   $\frac{dA}{dt} = \frac{-3}{5^2} = \frac{-3}{25}\ l/s$
6. A is changing at a rate of $\frac{-3}{25}\ l/s$, when $t = 5$

OR

A is decreasing at a rate of $\frac{3}{25}\ l/s$, when $t = 5$

Example 3: A balloon develops a microscopic leak. It's volume $V(cm^3)$ at time, $t(s)$ is:

$$V = 600 - 10t - \frac{t^2}{100}, t > 0$$

(i) At what rate is the volume changing when $t = 10$ seconds ?

(ii) What is the average rate of change of volume in the first 10 seconds?

(iii) What is the average rate of change of volume in the time interval from $t = 10$ to $t = 20$ seconds?

Solution:

(i) need $\frac{dV}{dt}$ at $t = 10$

$$\frac{dV}{dt} = -10 - \frac{2t}{100} = -10 - \frac{t}{50}$$

$$at\ t = 10, \quad \frac{dV}{dt} = -10 - \frac{10}{50} = -10.2\ cm^3/s$$

i.e. the volume is decreasing at a rate of 10.2 cm^3/s.

(ii) average rate of change of V: $\frac{V_2 - V_1}{t_2 - t_1}$

$$t_1 = 0, V_1 = 600 \qquad and \quad t_2 = 10, V_2 = 499$$

$$A.R.O.C = \frac{499 - 600}{10 - 0} = \frac{-101}{10} = -10.1\,cm^3/s$$

The volume is decreasing at an average rate of 10.1 cm^3/s.

(iii) average rate of change of V: $\frac{V_2 - V_1}{t_2 - t_1}$

$$t_1 = 10, V_1 = 499 \qquad and \quad t_2 = 20, V_2 = 396$$

$$A.R.O.C = \frac{396 - 499}{20 - 10} = \frac{-103}{10} = -10.3\,cm^3/s$$

The volume is decreasing at an average rate of 10.3 cm^3/s.

- Part (i) above is **an INSTANTANEOUS rate of change**,
- Part (ii) & (iii) is an AVERAGE rate of change, i.e. and average of a number of instantaneous rates.

Particular Case

Displacement – Velocity – Acceleration

	Symbol	Units	Definition
Displacement	x, $x(t)$, $s(t)$, d	m, km, …	The distance from a fixed point O
Velocity	v, $\frac{dx}{dt}$, $\frac{ds}{dt}$	m/s, ms^{-1}, km/h, …	The rate of change of displacement
Acceleration	a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$	m/s^2, ms^{-2}, km/h^2	The rate of change of velocity

- Original displacement/velocity/acceleration occurs at $t = 0$

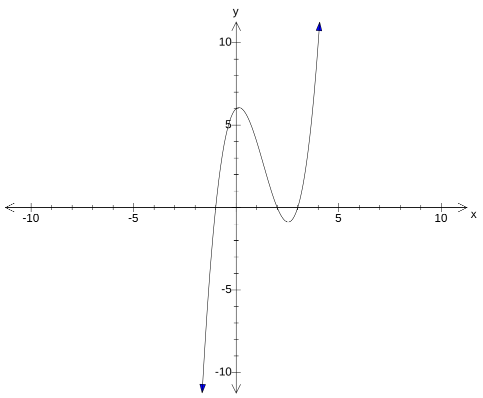
NOTE: If you were asked to find the average rate of velocity, it would be done as an average rate of change (i.e. $\frac{x_2 - x_1}{t_2 - t_1}$) using the displacement values not the velocity values. (If the velocity values were used then you get the average acceleration!)

- **Ex10B** 1, 2, 4, 8, 10, 12, 13

Finding the Stationary Points of a Curve

Example 1: Sketch the graph of $f(x) = (x+1)(x-2)(x-3)$ and determine the coordinates of all turning points (2 d.p.).

- ***Solution:***
- 1.Find x and y intercepts:
- X-Int ($y = 0$)   Y-Int ($x = 0$)
- $(x+1)(x-2)(x-3) = 0$, $x = -1, 2, 3$ $f(0) = (0+1)(0-2)(0-3) = 1 \times -2 \times -3 = 6$
-
- 2. Stationary Points ($\frac{dy}{dx} = 0$):



$f(x) = (x+1)(x^2 - 5x + 6)$

$f(x) = x^3 - 5x^2 + 6x + x^2 - 5x + 6$

$f(x) = x^3 - 4x^2 + x + 6$

$\therefore f'(x) = 3x^2 - 8x + 1$

- $f'(x) = 0$

$3x^2 - 8x + 1 = 0$

$x = 0.13, 2.54$ (*quadratic formula*)

$\Rightarrow (0.13, 6.06)$ & $(2.54, -0.88)$

- 3 Type of Stationary Points:
    - Local Minimum
    - Local Maximum;
    - Point of Inflection.

Example 2: Using the above example, determine the *nature* of the Turning Points.

Solution:

x	0	~0.13	1	~2.54	3
$f'(x)$	1	0	-4	0	4
Slope	/	-	\	-	/
Nature of T.P		Local Maximum		Local Minimum	

- Consider the 3 graphs: $y = x^3$ $y = x^3 - x$ $y = x^3 + x$
- **Graphs similar but different number of stationary points.**

- **Ex10C** 1 LHS, 2, 3, 5, 7, 10;
- **Ex10D** 1 cef, 2 adf, 4, 10, 12, 13, 17, 18, 22, 24, 25, 26

NOTE: 2nd derivatives can be used, *f* "(*x*) < = Max, *f* "(*x*) > 0 = Min, *f* "(*x*) = 0 = inconclusive
e.g. from above *f* "(*x*) = 6*x* – 8 , *f* "(*0*) = -8, *f* "(*0.13*) = -ve, *f* "(*x*) = -2 and *f* "(3) = 10

Maxima/Minima Problems

Solving a maximum/minimum problem

SETP 1: Need to Maximise/minimise what? Call it “A”

STEP 2: Write the formula for A = ………, making up **variables** where necessary, maybe a diagram could help.

STEP 3: Can you write another equation?
A = must be written as A = …(with **only 1 variable** on the Right Hand Side).

STEP 4: Differentiate A = (i.e. $\frac{dA}{dx}$) and equate to zero, $\frac{dA}{dx} = 0$ and solve for x.

STEP 5: Test for the type of stationary point obtained.

x					
$\frac{dA}{dx}$					

Are any answers impossible (i.e. a negative length)

STEP 6: Answer the question in words.

Example 1: Four square corners are removed from a sheet of card of dimensions 21cm by 30 cm. The sheet is folded to form an open rectangular container. Find the dimensions (to 1 d.p.), such that the total volume of the container is a maximum.

Solution: *(using the 6 steps from the photocopy sheet).*

1. Need to maximise the volume of the container, V.
2.

$$V = L \times W \times H$$

3. Right Hand side has 3 variables (must be in terms of only 1 variable)

$L = 30 - 2H$
$W = 21 - 2H$
$V = (30 - 2H)(21 - 2H)H$

4. Maximise ➔ let the derivative = 0 i.e. $\frac{dy}{dx}=0$.

Expand $V=$

$$V=(630-60H-42H+4H^2)H$$

$$V=(630-102H+4H^2)H$$

$$V=630H-102H^2+4H^3$$

$$\frac{dV}{dH}=630-204H+12H^2$$

$$6(105-34H+2H^2)=0$$

$$2H^2-34H+105=0$$

Quad. Formula...

$$H=\frac{17\pm\sqrt{79}}{2}, H=4.1, H=12.9$$

5.

(a) $V=(30-2H)(21-2H)H$

Must ignore H = 12.9 why? (hint: what is the implied domain of H?)

(b) H = 4.1 cm

H	4	4.1	5
$\frac{dV}{dH}$	6	0	-90
	/	-	\

Therefore a local maximum

6.

$L=30-2H$ $\quad$ $W=21-2H$

$L=30-2\left(\frac{17-\sqrt{79}}{2}\right)$ $\quad$ $W=21-2\left(\frac{17-\sqrt{79}}{2}\right)$

$L=21.9cm$ $\quad$ $W=12.9cm$

The maximum volume is obtained with the dimensions:

H = 4.1 cm

L = 21.9 cm

& W = 12.9 cm

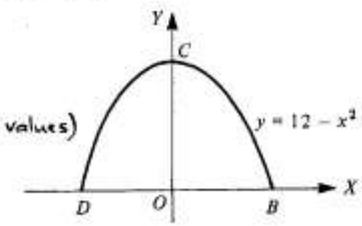
- The maximum/minimum value of a function DOES NOT NECESSARILY OCCUR AT A TURNING POINT. It depends on the feasible Domain caused by the Physical constraints.

- **Ex10F** 1, 2, 3, 4, 6, 7, 12, 14, 16, 17 **Worksheet**

Maximum-Minimum Problems

1. The equation of the parabola drawn in the diagram is

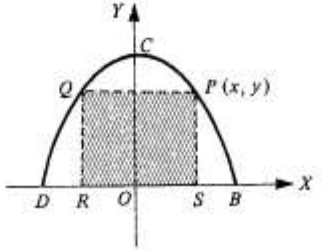
$$y = 12 - x^2 \quad 0 \leqslant y \leqslant 12.$$



(a) Find the coordinates of the points B, C and D. (exact values)

(b) State the domain of the function.

(c) If $PQRS$ is a rectangle with side RS on the x-axis and the vertices of the side PQ on the parabola (as illustrated), find the area of the rectangle, A, in terms of x.

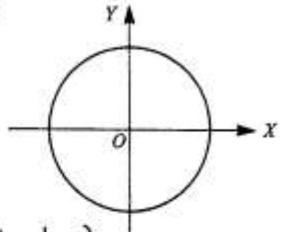


(d) Find $\frac{dA}{dx}$.

(e) Find the value of x for which the area, A, is a maximum. **Hence** find the maximum area of the rectangle $PQRS$.

2. The equation of the circle drawn in the diagram is

$$x^2 + y^2 = 40.$$



(a) (i) Express y in terms of x.

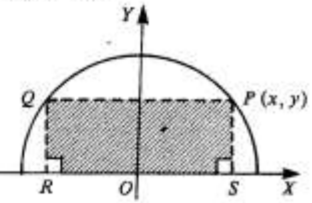
(ii) Show that the equation of the upper semicircle is

$$y = \sqrt{40 - x^2}$$

(iii) For the function defined in part (ii), find the domain. (exact values)

(b) (i) If $P(x, y)$ is any point on the semicircle and rectangle $PQRS$ is inscribed in the semicircle, show that the area, A, of the rectangle is modelled by the function

$$A = 2x\sqrt{40 - x^2}$$

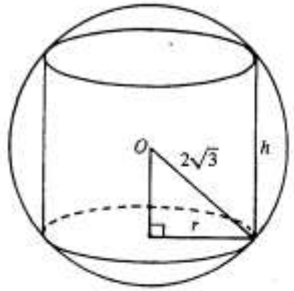


(ii) Find $\frac{dA}{dx}$, and find the exact value of x for which area (A) is a maximum.

Justify your answer by giving reasons.

Hence calculate the area of the largest rectangle that can be inscribed in the semicircle

3. A right circular cylinder has to be designed to fit inside a sphere of radius $2\sqrt{3}$ cm so that the bottom and top touch the sphere completely on the circular rim, as shown.



(a) If the radius and height of the cylinder are r cm and h cm respectively, find an equation for

(i) r^2 in terms of h

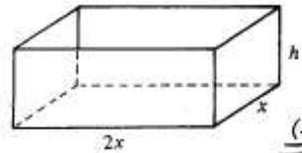
(ii) V in terms of h, where V denotes the volume of the cylinder.

(b) (i) Find $\frac{dV}{dh}$, and find the value of h for which V is a maximum.

Justify your answer by giving reasons.

(ii) Find the radius (r) of the cylinder of maximum volume, and hence determine this volume. (exact values)

4. A cuboid shaped tank is open at the top and the internal dimensions of its base are x metres and $2x$ metres. (See diagram.)



(a) If the volume V cubic metres of the tank is fixed, show that

$$h = \frac{V}{2x^2}.$$

(b) Show that the internal surface area, A m^2, of the tank is modelled by the function

$$A = 2x^2 + \frac{3V}{x}.$$

(c) Find:

(i) $\frac{dA}{dx}$

(ii) the value of x for which $\frac{dA}{dx} = 0$

Answers:

1(a) $B(2\sqrt{3}, 0)$, $C(0, 12)$, $D(-2\sqrt{3}, 0)$

(b) $[-2\sqrt{3}, 2\sqrt{3}]$

(c) $2x(12 - x^2)$

(d) $24 - 6x^2$

(e) 2 units, 32 sq.units

2(a)(i) $y = \pm\sqrt{40 - x^2}$

(iii) $[-2\sqrt{10}, 2\sqrt{10}]$

b)(ii) $\frac{4(20 - x^2)}{\sqrt{(40 - x^2)}}$

2 b) (ii) $2\sqrt{5}$ units, 40 sq.units

3a) (i) $r^2 = 12 - \frac{h^2}{4}$

3a)(ii) $12\pi h - \frac{\pi h^3}{4}$

b)(i) 4 cm (ii) $2\sqrt{2}$ cm, 32π cm^3

4 c)(i) $\frac{-3V}{x^2} + 4x$ (ii) $\sqrt[3]{\frac{3V}{4}}$

Absolute Maximum/Minimum Problems

Example: Let A be the function that models the total enclosed area when a 100 cm piece of wire is cut into two pieces, where one piece is used to form the perimeter of a square, and the other piece is used to form the circumference of a circle.

(a) Show that A can be modelled by, $A:[0,100]\to R, A(x)=\frac{x^2}{16}+\frac{(100-x)^2}{4\pi}$, where x cm is the length of the piece of wire used to form the perimeter of the square.
(b) For what values of x, is A is a maximum and a minimum?
(c) What is the minimum area?

Solution:

$$Square: \frac{x}{4}\times\frac{x}{4}=\frac{x^2}{16}$$

$$Circle: C=2\pi r \quad \therefore 100-x=2\pi r \quad \therefore r=\frac{100-x}{2\pi}$$

$$A=\pi r^2=\pi\left(\frac{100-x}{2\pi}\right)^2=\frac{(100-x)^2}{4\pi}$$

$$\Rightarrow A:[0,100]\to R, A(x)=\frac{x^2}{16}+\frac{(100-x)^2}{4\pi}$$

$$A'(x)=\frac{x}{8}-\frac{-(100-x)}{2\pi},$$

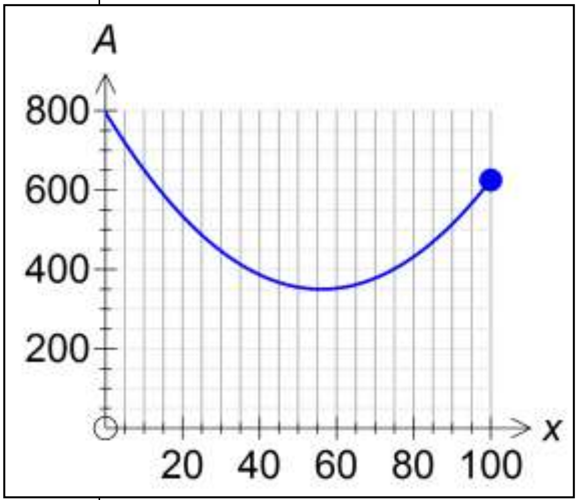
$$\frac{x}{8}-\frac{-(100-x)}{2\pi}=0$$

$$2\pi x-800+8x=0$$

$$x(2\pi+8)=800$$

$$x=\frac{800}{2\pi+8}=\frac{400}{\pi+4}\approx 56cm(minimum), A=350.1cm^2$$

$$Maximum\ at\ x=0, A=\frac{2500}{\pi}\approx 795.8cm^2$$



- **Ex10E** 1, 3, 5, 7, 9, 10, 11, 13, 14, 16

Calculator use: *fmax*(*a*(*x*), *x*) | 0 ≤ *x* ≤100

Families of functions

Example 1: Consider the family of functions of the form $f(x)=(x-a)^2(x-b)$, where *a* and *b* are positive constants with $b > a$.

a Find the derivative of $f(x)$ with respect to *x*.

b Find the coordinates of the stationary points of the graph of $y=f(x)$**.**

c Show that the stationary point at (*a*, 0) is always a local minimum.

d Find the values of *a* and *b* if the stationary points occur where $x = 3$ and $x = 4$.

Solution:

a

$$f'(x)=2\times1\times(x-a)\times(x-b)+(x-a)^2\times1=(x-a)(2(x-b)+(x-a))=(x-a)(3x-2b-a)$$

b

$$f'(x)=(x-a)(3x-2b-a)=0\Rightarrow x=a, x=\frac{a+2b}{3}$$

$$f(a)=0 \quad \Rightarrow (a,0)$$

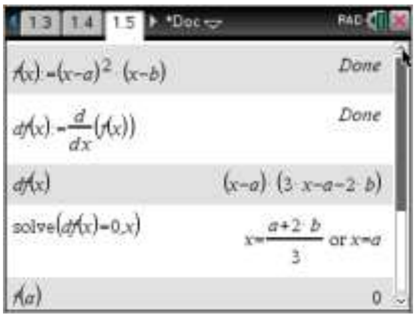
$$f\left(\frac{a+2b}{3}\right)=\frac{4(a-b)^3}{27} \quad \Rightarrow \left(\frac{a+2b}{3},\frac{4(a-b)^3}{27}\right)$$

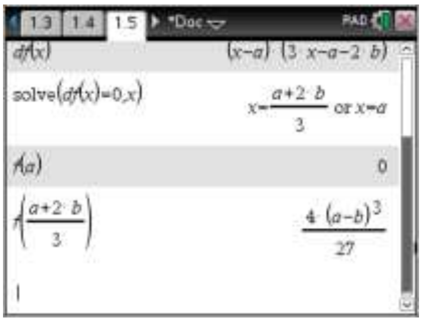
c

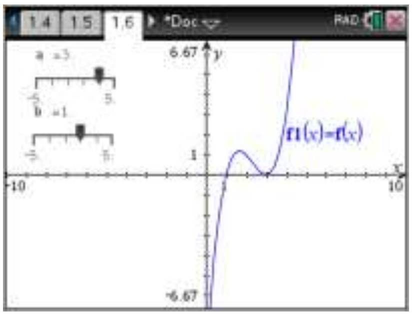
look at graph, $x <$ a, $f'(x)>0$**,** and if $a<x<\frac{a+2b}{3}$, *then* $f'(x)<0$

d

Since $a < b$, we must have $a = 3$ and $\frac{a+2b}{3}=4\Rightarrow b=\frac{9}{2}$







Example 2: The graph of $y=x^3-3x^2$, is translated by *a* units in the positive direction of the *x*-axis and *b* units in the positive direction of the *y*-axis (where *a* and *b* are positive constants).

a Find the coordinates of the turning points of the graph $y=x^3-3x^2$**.**

b Find the coordinates of the stationary points of its image.

Solution:

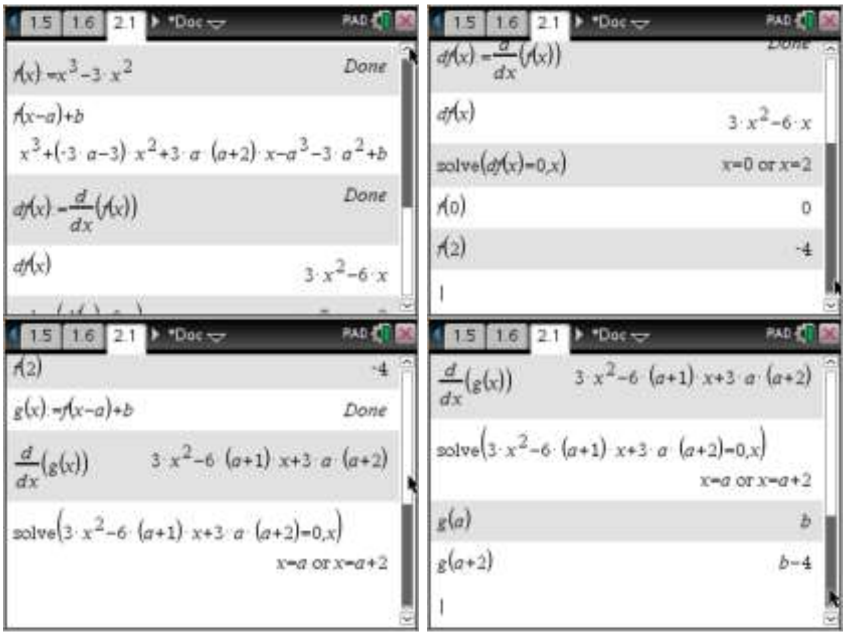
a

$$\frac{dy}{dx}=3x^2-6x \quad \Rightarrow \quad 3x^2-6x=0 \quad \Rightarrow \quad 3x(x-2)=0 \quad \Rightarrow \quad x=0, x=2$$

$$x=0, y=0 \quad \Rightarrow \quad (0,0)$$

$$x=2, y-4 \quad \Rightarrow \quad (2,-4)$$

b The turning points of the image are: (a, b) and (a + 2, b - 4).



- **Ex10G** 1, 2, 3, 5, 7, 9

Past Exam Questions
2008 Exam 1

Question 1

a. Let $y = (3x^2 - 5x)^5$. Find $\frac{dy}{dx}$.

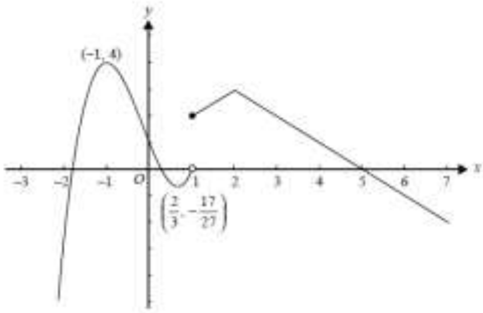
b. Let $f(x) = xe^{3x}$. Evaluate $f'(0)$.

2 + 3 = 5 marks

Question 6

a. The graph of the function f is shown, where

$$f(x) = \begin{cases} 2x^3 + x^2 - 4x + 1 & \text{if } x \in (-\infty, 1) \\ -|x-2| + 3 & \text{if } x \in [1, \infty) \end{cases}$$

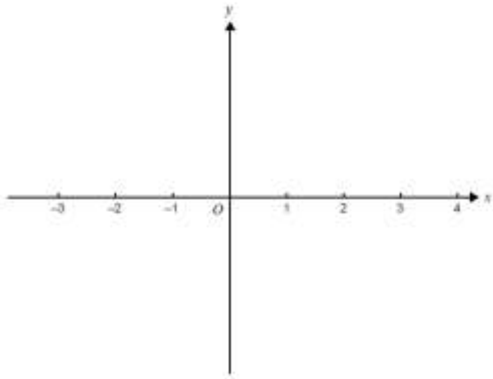


The stationary points of the function f are labelled with their coordinates.

Write down the domain of the derivative function f'.

b. By referring to the graph in part a., sketch the graph of the function with rule $y = |2x^3 + x^2 - 4x + 1|$, for $x < 1$, on the set of axes below.

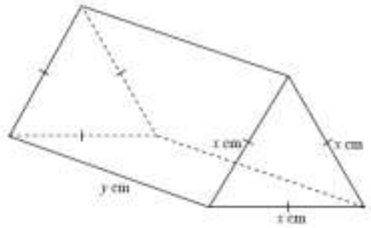
Label stationary points with their coordinates. (Do not attempt to find x-axis intercepts.)



1 + 2 = 3 marks

Question 9

A plastic brick is made in the shape of a right triangular prism. The triangular end is an equilateral triangle with side length x cm and the length of the brick is y cm.



The volume of the brick is 1000 cm^3.

a. Find an expression for y in terms of x.

b. Show that the total surface area, A cm^2, of the brick is given by

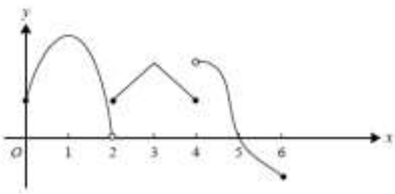
$$A = \frac{4000\sqrt{3}}{x} + \frac{\sqrt{3}x^2}{2}$$

c. Find the value of x for which the brick has minimum total surface area. (You do not have to find this minimum.)

2008 Exam 2

Question 22

The graph of the function f with domain $[0, 6]$ is shown below.



Which one of the following is **not** true?

A. The function is not continuous at $x = 2$ and $x = 4$.

B. The function exists for all values of x between 0 and 6.

C. $f(x) = 0$ for $x = 2$ and $x = 5$.

D. The function is positive for $x \in [0, 5)$.

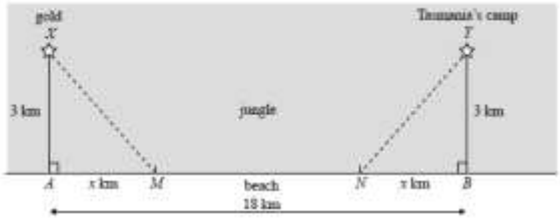
E. The gradient of the function is not defined at $x = 4$.

Question 3

Tasmania Jones is in the jungle, digging for gold. He finds the gold at X which is 3 km from a point A. Point A is on a straight beach.

Tasmania's camp is at Y which is 3 km from a point B. Point B is also on the straight beach.

$AB = 18$ km and $AM = NB = x$ km and $AX = BY = 3$ km.



While he is digging up the gold, Tasmania is bitten by a snake which injects toxin into his blood. After he is bitten, the concentration of the toxin in his bloodstream increases over time according to the equation

$$y = 50 \log_e(1 + 2t)$$

where y is the concentration, and t is the time in hours after the snake bites him.

The toxin will kill him if its concentration reaches 100.

a. Find the time, to the nearest minute, that Tasmania has to find an antidote (that is, a cure for the toxin).

2 marks

Tasmania has an antidote to the toxin at his camp. He can run through the jungle at 5 km/h and he can run along the beach at 13 km/h.

b. Show that he will not get the antidote in time if he runs directly to his camp through the jungle.

1 mark

In order to get the antidote, Tasmania runs through the jungle to M on the beach, runs along the beach to N and then runs through the jungle to the camp at T. M is x km from A and N is x km from B. (See diagram.)

c. Show that the time taken to reach the camp, T hours, is given by

$$T = 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$$

2 marks

d. Find the value of x which allows Tasmania to get to his camp in the minimum time.

2 marks

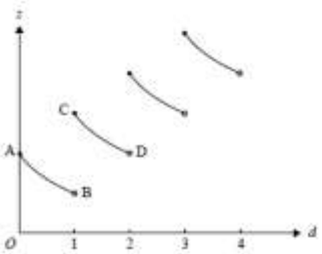
e. Show that he gets to his camp in time to get the antidote.

1 mark

At his camp, Tasmania Jones takes a capsule containing 16 units of antidote to the toxin. After taking the capsule the quantity of antidote in his body decreases over time.

At exactly the same time on successive days, he takes another capsule containing 16 units of antidote and again the quantity of antidote decreases in his body.

The graph of the quantity of antidote z units in his body at time d days after taking the first capsule looks like this. Each section of the curve has exactly the same shape as curve AB.



The equation of the curve AB is $z = \frac{16}{d+1}$

f. Write down the coordinates of the points A and C.

2 marks

g. Find the equation of the curve CD.

2 marks

Tasmania will no longer be affected by the snake toxin when he first has 50 units of the antidote in his body.

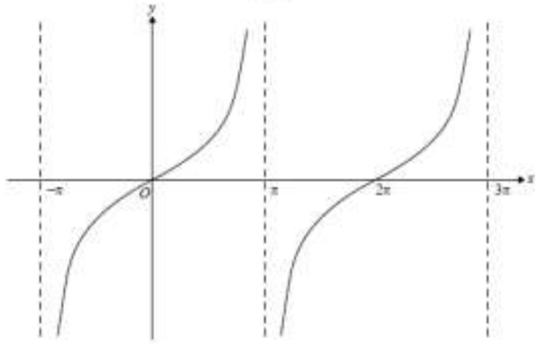
h. Assuming he takes a capsule at the same time each day, on how many days does he need to take a capsule so that he will no longer be affected by the snake toxin?

1 mark

Total 13 marks

Question 4

The graph of $f: (-\pi, \pi) \cup (\pi, 3\pi) \to R$, $f(x) = \tan\left(\frac{x}{2}\right)$ is shown below.



a. i. Find $f'\left(\frac{\pi}{2}\right)$.

ii. Find the equation of the normal to the graph of $y = f(x)$ at the point where $x = \frac{\pi}{2}$.

iii. Sketch the graph of this normal on the axes above. Give the exact axis intercepts.

1 + 2 + 3 = 6 marks

b. Find the exact values of $x \in (-\pi, \pi) \cup (\pi, 3\pi)$ such that $f'(x) = f'\left(\frac{\pi}{2}\right)$.

2 marks

Let $g(x) = f(x - a)$.

c. Find the exact value of $a \in (-1, 1)$ such that $g(1) = 1$.

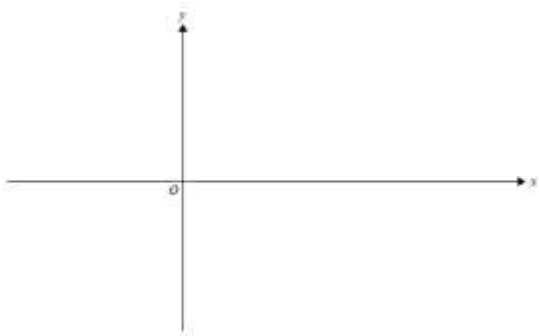
2 marks

Let $h: (-\pi, \pi) \cup (\pi, 3\pi) \to R$, $h(x) = \sin\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right) + 2$.

d. i. Find $h'(x)$.

ii. Solve the equation $h'(x) = 0$ for $x \in (-\pi, \pi) \cup (\pi, 3\pi)$. (Give exact values.)

e. Sketch the graph of $y = h(x)$ on the axes below.
- Give the exact coordinates of any stationary points.
- Label each asymptote with its equation.
- Give the exact value of the y-intercept.



2 marks

Total 15 marks

2009 Exam 1

Question 1

a. Differentiate $x\log_e(x)$ with respect to x.

2 marks

b. For $f(x) = \frac{\cos(x)}{2x+2}$ find $f'(\pi)$.

3 marks

2009 Exam 2

Question 7

For $y = e^{2x}\cos(3x)$ the rate of change of y with respect to x when $x = 0$ is

A. 0
B. 2
C. 3
D. −6
E. −1

Question 8

For the function $f: R \to R, f(x) = (x+5)^2(x-1)$, the subset of R for which the gradient of f is negative is

A. $(-\infty, 1)$
B. $(-5, 1)$
C. $(-5, -1)$
D. $(-\infty, -5)$
E. $(-5, 0)$

Question 9

The tangent at the point (1, 5) on the graph of the curve $y = f(x)$ has equation $y = 3 + 2x$.
The tangent at the point (3, 8) on the curve $y = f(x-2) + 3$ has equation

A. $y = 2x - 4$
B. $y = x + 5$
C. $y = -2x + 14$
D. $y = 2x + 4$
E. $y = 2x + 2$

Question 15

For $y = \sqrt{1-f(x)}$, $\frac{dy}{dx}$ is equal to

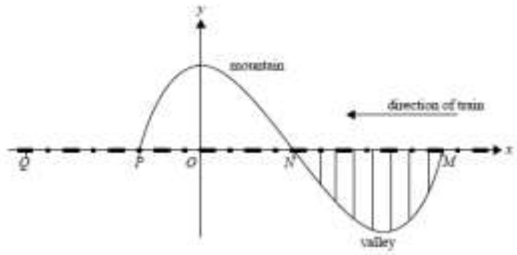
A. $\frac{2f'(x)}{\sqrt{1-f(x)}}$
B. $\frac{-1}{2\sqrt{1-f'(x)}}$
C. $\frac{1}{2}\sqrt{1-f'(x)}$
D. $\frac{3}{2(1-f'(x))}$
E. $\frac{-f'(x)}{2\sqrt{1-f(x)}}$

Question 21

A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at (2, 0) and (−3, 0). The maximum value of the derivative function is 10.
The value of x for which the graph of $y = f(x)$ has a local maximum is

A. −2
B. 2
C. −3
D. 3
E. $-\frac{1}{2}$

Question 2



A train is travelling at a constant speed of w km/h along a straight level track from M towards Q.
The train will travel along a section of track $MNPQ$.

Section MN passes along a bridge over a valley.
Section NP passes through a tunnel in a mountain.
Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of

$$y = \frac{1}{200}(ax^3 + bx^2 + c) \text{ where } a, b \text{ and } c \text{ are real numbers.}$$

All measurements are in kilometres.

a. The curve defined from M to P passes through $N(2, 0)$. The gradient of the curve at N is −0.06 and the curve has a turning point at $x = 4$.

i. From this information write down three simultaneous equations in a, b and c.

ii. Hence show that $a = 1$, $b = -6$ and $c = 16$.

3 + 2 = 5 marks

b. Find, giving exact values

i. the coordinates of M and P

ii. the length of the tunnel

iii. the maximum depth of the valley below the train track.

The driver sees a large rock on the track at a point Q, 6.2 km from P. The driver puts on the brakes at the instant that the front of the train comes out of the tunnel at P.
From its initial speed of w km/h, the train slows down from point P so that its speed v km/h is given by

$$v = k\log_e\left(\frac{(d+1)}{7}\right),$$

where d km is the distance of the front of the train from P and k is a real constant.

c. Find the value of k in terms of w.

1 mark

d. If $v = \frac{120\log_e(2)}{\log_e(7)}$ when $d = 2.5$, find the value of w.

2 marks

e. Find the exact distance from the front of the train to the large rock when the train finally stops.

2010 Exam 1

Question 1

a. Differentiate x^3e^{2x} with respect to x.

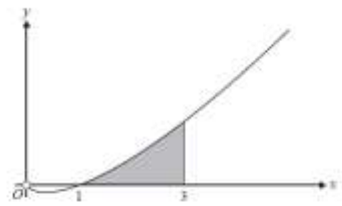
2 marks

b. For $f(x) = \log_e(x^2+1)$, find $f'(2)$.

2 marks

Question 9

Part of the graph of $f: R^+ \to R, f(x) = x\log_e(x)$ is shown below.



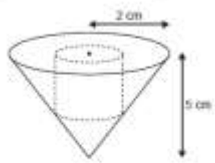
a. Find the derivative of $x^2\log_e(x)$.

1 mark

b. Use your answer to part **a.** to find the area of the shaded region in the form $a\log_e(b) + c$ where a, b and c are non-zero real constants.

Question 11

A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane as shown in the diagram below. The height of the cone is 5 cm and the radius of the cone is 2 cm. The radius of the cylinder is r cm and the height of the cylinder is h cm.



For the cylinder inscribed in the cone as shown above

a. find h in terms of r

2 marks

The total surface area, S cm^2, of a cylinder of height h cm and radius r cm is given by the formula

$$S = 2\pi rh + 2\pi r^2$$

b. find S in terms of r

1 mark

c. find the value of r for which S is a maximum.

2 marks

2010 Exam 2

Question 6

A function g with domain R has the following properties.

- $g'(x) = x^2 - 2x$
- the graph of $g(x)$ passes through the point $(1, 0)$

$g(x)$ is equal to

A. $2x - 2$

B. $\frac{x^3}{3} - x^2$

C. $\frac{x^3}{3} - x^2 + \frac{2}{3}$

D. $x^2 - 2x + 2$

E. $3x^3 - x^2 - 1$

Question 16

The gradient of the function $f: R \to R, f(x) = \frac{5x}{x^2+3}$ is negative for

A. $-\sqrt{3} < x < \sqrt{3}$

B. $x > 3$

C. $x \in R$

D. $x < -\sqrt{3}$ and $x > \sqrt{3}$

E. $x < 0$

Question 17

The function f is differentiable for all $x \in R$ and satisfies the following conditions.

- $f'(x) < 0$ where $x < 2$
- $f'(x) = 0$ where $x = 2$
- $f'(x) = 0$ where $x = 4$
- $f'(x) > 0$ where $2 < x < 4$
- $f'(x) > 0$ where $x > 4$

Which one of the following is true?

A. The graph of f has a local maximum point where $x = 4$.

B. The graph of f has a stationary point of inflection where $x = 4$.

C. The graph of f has a local maximum point where $x = 2$.

D. The graph of f has a local minimum point where $x = 4$.

E. The graph of f has a stationary point of inflection where $x = 2$.

Question 4

Consider the function $f: R \to R, f(x) = \frac{1}{27}(2x-1)^3(6-3x)+1$.

a. Find the x-coordinate of each of the stationary points of f and state the nature of each of these stationary points.

4 marks

In the following, f is the function $f: R \to R, f(x) = \frac{1}{27}(ax-1)^3(b-3x)+1$ where a and b are real constants.

b. Write down, in terms of a and b, the possible values of x for which $(x, f(x))$ is a stationary point of f.

1 mark

c. For what value of a does f have no stationary points?

3 marks

d. Find a in terms of b if f has one stationary point.

2 marks

e. What is the maximum number of stationary points that f can have?

1 mark

f. Assume that there is a stationary point at (1, 1) and another stationary point (p, p) where $p \neq 1$.
Find the value of p.

3 marks

Total 14 marks

2011 Exam 1

Question 1

a. Differentiate $\sqrt{4-x}$ with respect to x.

1 mark

b. If $g(x) = x^2\sin(2x)$, find $g'\left(\frac{\pi}{6}\right)$.

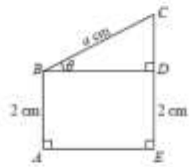
2 marks

Question 10

The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. The point D is on line segment EC with $AB = ED = 2$ cm and $BC = a$ cm, where a is a positive constant.

$$\angle BAE = \angle CEA = \frac{\pi}{2}$$

Let $\angle CBD = \theta$ where $0 < \theta < \frac{\pi}{2}$.



a. Find BD and CD in terms of a and θ.

2 marks

b. Find the length, L cm, of the wire in the frame, including length BD, in terms of a and θ.

c. Find $\frac{dL}{d\theta}$, and hence show that $\frac{dL}{d\theta} = 0$ when $BD = 2CD$.

2 marks

d. Find the maximum value of L if $a = 3\sqrt{5}$.

2011 Exam 2

Question 4

The derivative of $\log_e(2f(x))$ with respect to x is

A. $\frac{f'(x)}{f(x)}$

B. $2\frac{f'(x)}{f(x)}$

C. $\frac{f'(x)}{2f(x)}$

D. $\log_e(2f'(x))$

E. $2\log_e(2f'(x))$

Question 17

The normal to the curve with equation $y = x^{\frac{3}{2}} + x$ at the point (4, 12) is parallel to the straight line with equation

A. $4x = y$

B. $4y + x = 7$

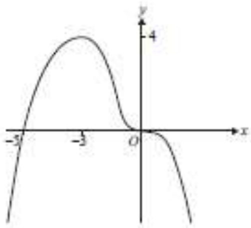
C. $y = \frac{x}{4} + 1$

D. $x - 4y = -5$

E. $4y + 4x = 20$

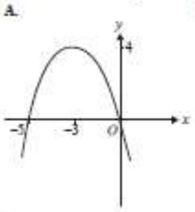
Question 9

The graph of the function $y = f(x)$ is shown below.

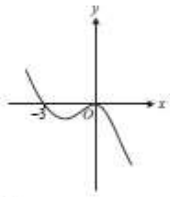


Which of the following could be the graph of the derivative function $y = f'(x)$?

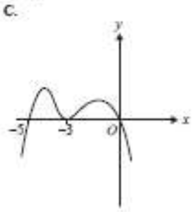
A.



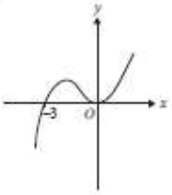
B.



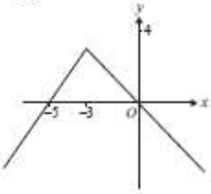
C.



D.



E.



Question 18

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A. $-7 < w < 25$

B. $w \leq -7$

C. $w \geq 25$

D. $w < -7$ or $w > 25$

E. $w > 1$

Question 3

a. Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9$.

i. Find $f'(x)$

ii. Explain why $f'(x) \geq 5$ for all x.

1 + 1 = 2 marks

b. The cubic function p is defined by $p: R \to R, p(x) = ax^3 + bx^2 + cx + k$, where a, b, c and k are real numbers.

i. If p has m stationary points, what possible values can m have?

ii. If p has an inverse function, what possible values can m have?

1 + 1 = 2 marks

c. The cubic function q is defined by $q: R \to R, q(x) = 3 - 2x^3$.

i. Write down an expression for $q^{-1}(x)$.

ii. Determine the coordinates of the point(s) of intersection of the graphs of $y = q(x)$ and $y = q^{-1}(x)$.

2 + 2 = 4 marks

d. The cubic function g is defined by $g: R \to R, g(x) = x^3 + 2x^2 + cx + k$, where c and k are real numbers.

i. If g has exactly one stationary point, find the value of c.

ii. If this stationary point occurs at a point of intersection of $y = g(x)$ and $y = g^{-1}(x)$, find the value of k.

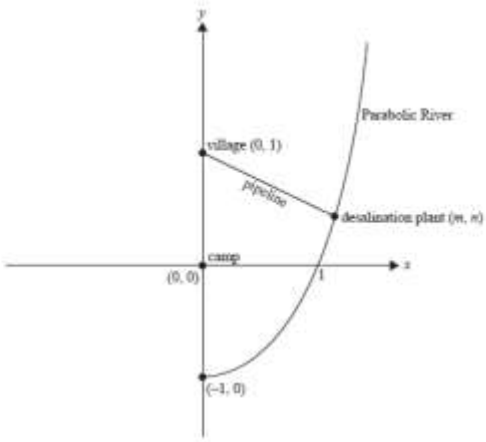
3 + 3 = 6 marks

Total 14 marks

Question 4

Deep in the South American jungle, Tasmania Jones has been working to help the Quetzacotl tribe to get drinking water from the very salty water of the Parabolic River. The river follows the curve with equation $y = x^2 - 1, x \geq 0$ as shown below. All lengths are measured in kilometres.

Tasmania has his camp site at (0, 0) and the Quetzacotl tribe's village is at (0, 1). Tasmania builds a desalination plant, which is connected to the village by a straight pipeline.



a. If the desalination plant is at the point (m, n) show that the length, L kilometres, of the straight pipeline that carries the water from the desalination plant to the village is given by

$$L = \sqrt{m^4 - 3m^2 + 4}.$$

b. If the desalination plant is built at the point on the river that is closest to the village

i. find $\frac{dL}{dm}$ and hence find the coordinates of the desalination plant

ii. find the length, in kilometres, of the pipeline from the desalination plant to the village.

3 + 2 = 5 marks

The desalination plant is actually built at $\left(\frac{\sqrt{7}}{2}, \frac{3}{4}\right)$.

If the desalination plant stops working, Tasmania needs to get to the plant in the minimum time.

Tasmania runs in a straight line from his camp to a point (x, y) on the river bank where $x \le \frac{\sqrt{7}}{2}$. He then swims up the river to the desalination plant.

Tasmania runs from his camp to the river at 2 km per hour. The time that he takes to swim to the desalination plant is proportional to the difference between the y-coordinates of the desalination plant and the point where he enters the river.

c. Show that the total time taken to get to the desalination plant is given by

$$T = \frac{1}{2}\sqrt{x^4 - x^2 + 1} + \frac{1}{4}k\left(7 - 4x^2\right) \text{ hours where } k \text{ is a positive constant of proportionality.}$$

3 marks

The value of k varies from day to day depending on the weather conditions.

d. If $k = \frac{1}{2\sqrt{13}}$

i. find $\frac{dT}{dx}$

ii. hence find the coordinates of the point where Tasmania should reach the river if he is to get to the desalination plant in the minimum time.

1 + 2 = 3 marks

e. On one particular day, the value of k is such that Tasmania should run directly from his camp to the point $(1, 0)$ on the river to get to the desalination plant in the minimum time. Find the value of k on that particular day.

2 marks

f. Find the values of k for which Tasmania should run directly from his camp towards the desalination plant to reach it in the minimum time.

2012 Exam 1

Question 1

a. If $y = (x^2 - 5x)^4$, find $\frac{dy}{dx}$.

1 mark

b. If $f(x) = \frac{x}{\sin(x)}$, find $f'\left(\frac{\pi}{2}\right)$.

2 marks

Question 10

Let $f: R \to R$, $f(x) = e^{-mx} + 3x$, where m is a positive rational number.

a. i. Find, in terms of m, the x-coordinate of the stationary point of the graph of $y = f(x)$.

ii. State the values of m such that the x-coordinate of this stationary point is a positive number.

2 + 1 = 3 marks

b. For a particular value of m, the tangent to the graph of $y = f(x)$ at $x = -6$ passes through the origin. Find this value of m.

2012 Exam 2

Question 2

For the function with rule $f(x) = x^3 - 4x$, the average rate of change of $f(x)$ with respect to x on the interval $[1, 3]$ is

A. 1

B. 3

C. 5

D. 6

E. 9

Question 4

Given that g is a differentiable function and k is a real number, the derivative of the composite function $g(e^{kx})$ is

A. $kg'(e^{kx})\, e^{kx}$

B. $kg(e^{kx})$

C. $ke^{kx}\, g(e^{kx})$

D. $ke^{kx}\, g'(e^{x})$

E. $\frac{1}{k} e^{kx} g'(e^{kx})$

Question 8

The function $f: R \to R$, $f(x) = ax^3 + bx^2 + cx$, where a is a negative real number and b and c are real numbers.

For the real numbers $p < m < 0 < n < q$, we have $f(p) = f(q) = 0$ and $f'(m) = f'(n) = 0$.

The gradient of the graph of $y = f(x)$ is negative for

A. $(-\infty, m) \cup (n, \infty)$

B. (m, n)

C. $(p, 0) \cup (q, \infty)$

D. $(p, m) \cup (0, q)$

E. (p, q)

Question 9

The normal to the graph of $y = \sqrt{b - x^2}$ has a gradient of 3 when $x = 1$.

The value of b is

A. $-\frac{10}{9}$

B. $\frac{10}{9}$

C. 4

D. 10

E. 11

Question 16

The graph of a cubic function f has a local maximum at $(a, -3)$ and a local minimum at $(b, -8)$.

The values of c, such that the equation $f(x) + c = 0$ has exactly one solution, are

A. $3 < c < 8$

B. $c > -3$ or $c < -8$

C. $-8 < c < -3$

D. $c < 3$ or $c > 8$

E. $c < -8$

Question 18

The tangent to the graph of $y = \log_e(x)$ at the point $(a, \log_e(a))$ crosses the x-axis at the point $(b, 0)$, where $b < 0$.

Which of the following is **false**?

A. $1 < a < e$

B. The gradient of the tangent is positive

C. $a > e$

D. The gradient of the tangent is $\frac{1}{a}$

E. $a > 0$

Question 22

The graph of a differentiable function f has a local maximum at (a, b), where $a < 0$ and $b > 0$, and a local minimum at (c, d), where $c > 0$ and $d < 0$.

The graph of $y = -f(x - 2)$ has

A. a local minimum at $(a - 2, -b)$ and a local maximum at $(c - 2, d)$

B. local minima at $(a + 2, -b)$ and $(c + 2, d)$

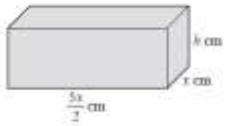
C. local maxima at $(a + 2, b)$ and $(c + 2, -d)$

D. a local minimum at $(a - 2, -b)$ and a local maximum at $(a - 2, -d)$

E. local minima at $(c + 2, -d)$ and $(a + 2, -b)$

Question 1

A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 sq cm.

a. Show that if the height of the block is h cm, $h = \frac{6480 - 5x^2}{7x}$.

b. The volume, V cm^3, of the block is given by $V(x) = \frac{5x(6480 - 5x^2)}{14}$.

Given that $V(x) > 0$ and $x > 0$, find the possible values of x.

2 marks

c. Find $\frac{dV}{dx}$, expressing your answer in the form $\frac{dV}{dx} = ax^2 + b$, where a and b are real numbers.

3 marks

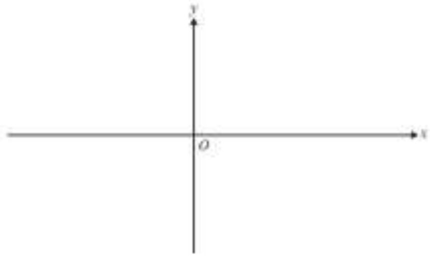
d. Find the exact values of x and h if the block is to have maximum volume.

2 marks

Question 2

Let $f: R \setminus \{2\} \to R$, $f(x) = \frac{1}{2x - 4} + 3$.

a. Sketch the graph of $y = f(x)$ on the set of axes below. Label the axes intercepts with their coordinates and label each of the asymptotes with its equation.



3 marks

b. **i.** Find $f'(x)$.

ii. State the range of f'.

iii. Using the result of **part ii.** explain why f has no stationary points.

1 + 1 + 1 = 3 marks

c. If (p, q) is any point on the graph of $y = f(x)$, show that the equation of the tangent to $y = f(x)$ at this point can be written as $(2p-4)^2(y-3) = -2x + 4p - 4$.

d. Find the coordinates of the points on the graph of $y = f(x)$ such that the tangents to the graph at these points intersect at $\left(-1, \frac{7}{2}\right)$.

e. A transformation $T: R^2 \to R^2$ that maps the graph of f to the graph of the function $g: R \setminus \{0\} \to R, g(x) = \frac{1}{x}$ has rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$, where a, c and d are non-zero real numbers.

Find the values of a, c and d.

2013 Exam 1

Question 1 (5 marks)

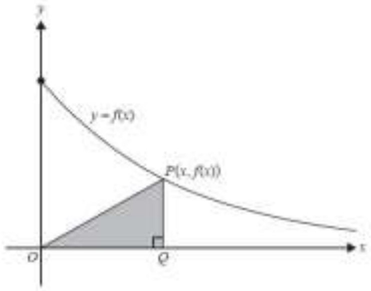
a. If $y = x^2\log_e(x)$, find $\frac{dy}{dx}$. 2 marks

b. Let $f(x) = e^{x^2}$.
Find $f'(3)$. 3 marks

Question 10 (7 marks)

Let $f: [0, \infty) \to R, f(x) = 2e^{-\frac{x}{5}}$.

A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.



a. Find the area, A, of the triangle OQP in terms of x. 1 mark

b. Find the maximum area of triangle OQP and the value of x for which the maximum occurs. 3 marks

2013 Exam 2

Question 6

For the function $f(x) = \sin(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\frac{1}{4}, 5\right]$ is

A. 0

B. $\frac{34}{19}$

C. $\frac{7}{2}$

D. $\frac{2\pi + 10}{4}$

E. $\frac{23}{4}$

Question 11

If the tangent to the graph of $y = e^{ax}$, $a \neq 0$, at $x = c$ passes through the origin, then c is equal to

A. 0

B. $\frac{1}{a}$

C. 1

D. a

E. $-\frac{1}{a}$

Question 12

Let $y = 4\cos(x)$ and x be a function of t such that $\frac{dx}{dt} = 3e^{2t}$ and $x = \frac{3}{2}$ when $t = 0$.

The value of $\frac{dy}{dt}$ when $x = \frac{\pi}{2}$ is

A. 0

B. $3\pi \log_e\left(\frac{\pi}{2}\right)$

C. -4π

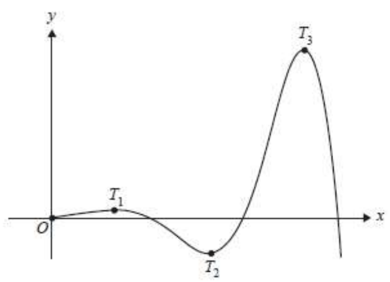
D. -2π

E. $-12e$

Question 19

Part of the graph of a function $f: [0, \infty) \to R, f(x) = e^{x\sqrt{3}}\sin(x)$ is shown below.

The first three turning points are labelled T_1, T_2 and T_3.



The x-coordinate of T_3 is

A. $\frac{8\pi}{3}$

B. $\frac{16\pi}{3}$

C. $\frac{13\pi}{6}$

D. $\frac{17\pi}{6}$

E. $\frac{29\pi}{6}$

Question 21

The cubic function $f: R \to R, f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, points when

A. $c > \frac{b^2}{4a}$

B. $c < \frac{b^2}{4a}$

C. $c < 4b^2a$

D. $c > \frac{b^2}{3a}$

E. $c < \frac{b^2}{3a}$

Question 1 (12 marks)

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature (T °C) is given by $T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right)$, $0 \le t \le 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a. State the maximum temperature in the greenhouse and the values of t when this occurs. 2 marks

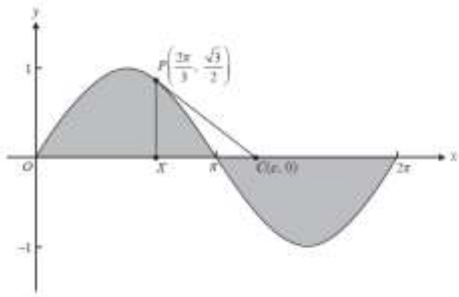
b. State the period of the function T. 1 mark

c. Find the smallest value of t for which $T = 26$. 2 marks

d. For how many hours during the 24-hour time interval is $T \ge 26$? 2 marks

Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \le x \le 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram below. He includes a garden path, which is shown as line segment PC.

The line through points $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P.



e. i. Find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$. 1 mark

ii. Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$. 1 mark

In further planning for the garden, Trigg uses a transformation of the plane defined as a dilation of factor k from the x-axis and a dilation of factor m from the y-axis, where k and m are positive real numbers.

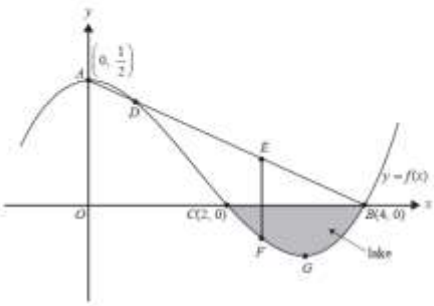
f. Let X', P' and C' be the image, under this transformation, of the points X, P and C respectively.

i. Find the values of k and m if $X'P' = 10$ and $X'C' = 30$. 2 marks

ii. Find the coordinates of the point P'. 1 mark

Question 3 (19 marks)

Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains. He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram below.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}$$

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2, 0)$ and $B(4, 0)$ are the points at the edge of the lake, situated in the valley. All distances are measured in kilometres.

a. Find the coordinates of G, the deepest point in the lake. 3 marks

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A, to the edge of the lake, B. The section of the railway line from A to D passes through a tunnel in the mountain.

b. Write down the equation of the line that passes through A and B. 2 marks

c. i. Show that the x-coordinate of D, the end point of the tunnel, is $\frac{2}{3}$. 1 mark

ii. Find the length of the tunnel AD. 2 marks

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V: [0, 4] \to R, V(x) = k\sqrt{x} - mx^2,$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line.

The train then slows to a stop at $B(4, 0)$, that is $V(4) = 0$.

e. Find k in terms of m. 1 mark

f. Find the value of x for which the speed, V, is a maximum. 2 marks

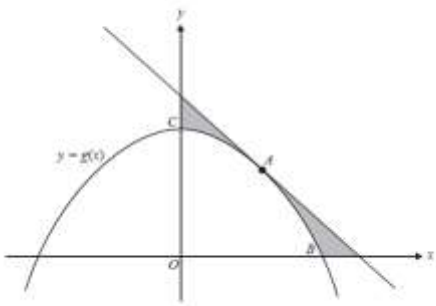
Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m remains the same.

g. If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place. 2 marks

h. If, on another day, the maximum value of V is 120, find the value of m. 2 marks

Question 4 (16 marks)

Part of the graph of a function $g: R \to R$, $g(x) = \frac{16 - x^2}{4}$ is shown below.



a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram above. The tangent to the graph of g at the point A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A. 2 marks

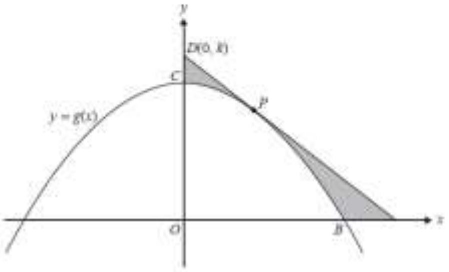
ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the point A, and the x-axis and y-axis.

Evaluate the area of this shaded region. 3 marks

b. Let Q be a point on the graph of $y = g(x)$.

Find the positive value of the x-coordinate of Q, for which the distance OQ is a minimum and find the minimum distance. 3 marks

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y-axis at point $D(0, k)$, where $5 \le k \le 8$.



c. Find the gradient of the tangent in terms of k. 2 marks

d. **i.** Find the rule $A(k)$ for the function of k that gives the area of the shaded region. 2 marks

ii. Find the **maximum** area of the shaded region and the value of k for which this occurs. 2 marks

iii. Find the **minimum** area of the shaded region and the value of k for which this occurs. 2 marks

2014 Exam 1

Question 1 (5 marks)

a. If $y = x^2 \sin(x)$, find $\frac{dy}{dx}$. 2 marks

b. If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$. 3 marks

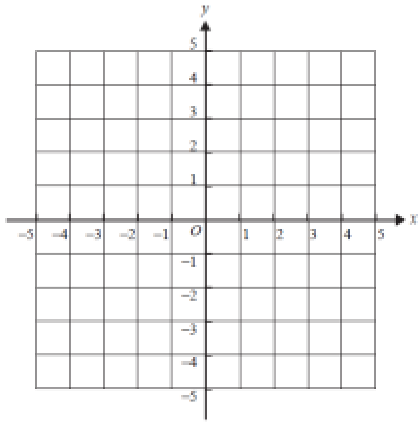
Question 5 (7 marks)

Consider the function $f : [-1, 3] \to R,\ f(x) = 3x^2 - x^3$.

a. Find the coordinates of the stationary points of the function. 2 m

b. On the axes below, sketch the graph of f.

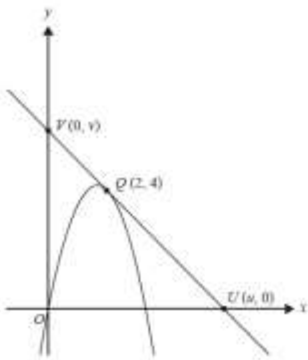
Label any end points with their coordinates. 2 marks



Question 10 (7 marks)

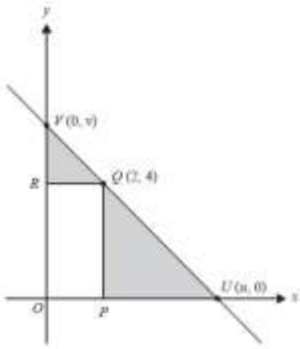
A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \le u \le 6$.

a. When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates $(2, 4)$, as shown.



If a and b are non-zero real numbers, find the values of a and b. 3 marks

b. The rectangle $OPQR$ has a vertex at Q on the line. The coordinates of Q are $(2, 4)$, as shown.



i. Find an expression for v in terms of u. 1 mark

ii. Find the **minimum** total shaded area and the value of u for which the area is a minimum. 2 marks

iii. Find the **maximum** total shaded area and the value of u for which the area is a maximum. 1 mark

2014 Exam 2

Question 4

Let f be a function with domain R such that $f'(5) = 0$ and $f'(x) < 0$ when $x \neq 5$.

At $x = 5$, the graph of f has a

A. local minimum.

B. local maximum.

C. gradient of 5.

D. gradient of -5.

E. stationary point of inflection.

Question 6

The function $f : D \to R$ with rule $f(x) = 2x^3 - 9x^2 - 168x$ will have an inverse function for

A. $D = R$

B. $D = (7, \infty)$

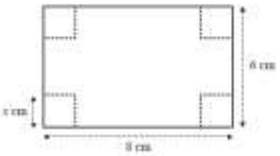
C. $D = (-4, 8)$

D. $D = (-\infty, 0)$

E. $D = \left[-\frac{1}{2}, \infty\right)$

Question 15

Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.



Zoe turns up the sides to form an open box.

The value of x for which the volume of the box is a maximum is closest to

A. 0.8

B. 1.1

C. 1.6

D. 2.0

E. 3.6

Question 21

The trapezium $ABCD$ is shown below. The sides AB, BC and DA are of equal length, p. The size of the acute angle BCD is x radians.

The area of the trapezium is a maximum when the value of x is

A. $\frac{\pi}{12}$

B. $\frac{\pi}{6}$

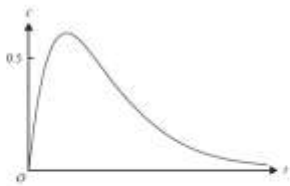
C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

E. $\frac{5\pi}{12}$

Question 3 (11 marks)

In a controlled experiment, Juan took some medicine at 8 pm. The concentration of medicine in his blood was then measured at regular intervals. The concentration of medicine in Juan's blood is modelled by the function $c(t) = \frac{5}{2}te^{-\frac{3t}{2}}$, $t \geq 0$, where c is the concentration of medicine in his blood, in milligrams per litre, t hours after 8 pm. Part of the graph of the function c is shown below.



a. What was the maximum value of the concentration of medicine in Juan's blood, in milligrams per litre, correct to two decimal places? 1 mark

b. **i.** Find the value of t, in hours, correct to two decimal places, when the concentration of medicine in Juan's blood first reached 0.5 milligrams per litre. 1 mark

ii. Find the length of time that the concentration of medicine in Juan's blood was above 0.5 milligrams per litre. Express the answer in hours, correct to two decimal places. 2 marks

c. **i.** What was the value of the average rate of change of the concentration of medicine in Juan's blood over the interval $\left[\frac{2}{3}, 3\right]$? Express the answer in milligrams per litre per hour, correct to two decimal places. 2 marks

ii. At times t_1 and t_2, the instantaneous rate of change of the concentration of medicine in Juan's blood was equal to the average rate of change over the interval $\left[\frac{2}{3}, 3\right]$.

Find the values of t_1 and t_2, in hours, correct to two decimal places. 2 marks

Alicia took part in a similar controlled experiment. However, she used a different medicine. The concentration of this different medicine was modelled by the function $n(t) = Ate^{-kt}$, $t \geq 0$, where A and $k \in R^+$.

d. If the **maximum** concentration of medicine in Alicia's blood was 0.74 milligrams per litre at $t = 0.5$ hours, find the value of A, correct to the nearest integer. 3 marks

2015 Exam 1

Question 1 (4 marks)

a. Let $y = (5x + 1)^7$.

Find $\frac{dy}{dx}$. 1 mark

b. Let $f(x) = \frac{\log_e(x)}{x^2}$.

i. Find $f'(x)$. 2 marks

ii. Evaluate $f'(1)$. 1 mark

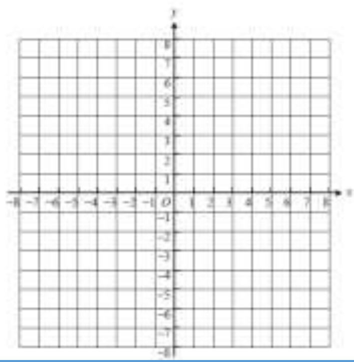
Question 4 (6 marks)

Consider the function $f : [-3, 2] \to R$, $f(x) = \frac{1}{2}\left(x^3 + 3x^2 - 4\right)$.

a. Find the coordinates of the stationary points of the function. 2 marks

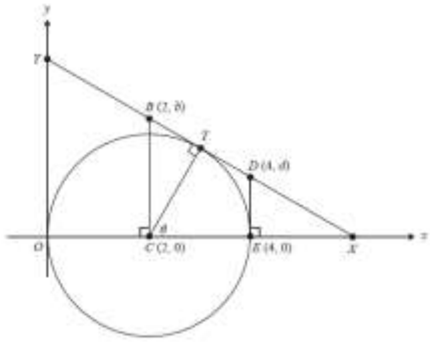
The rule for f can also be expressed as $f(x) = \frac{1}{2}(x-1)(x+2)^2$.

b. On the axes below, sketch the graph of f, clearly indicating axis intercepts and turning points. Label the end points with their coordinates. 2 marks



Question 10 (7 marks)

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates $(2, 0)$. The angle ECT is θ, where $0 < \theta \le \frac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Z.

a. Find the coordinates of T in terms of θ. 1 mark

b. Find the gradient of the tangent to the circle at T in terms of θ. 1 mark

c. The equation of the tangent to the circle at T can be expressed as

$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XZ.

Find b in terms of θ. 1 mark

ii. Point D, with coordinates $(4, d)$, is on the line segment XZ.

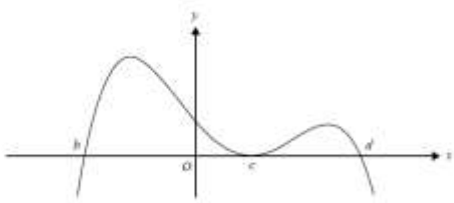
Find d in terms of θ. 1 mark

d. Consider the trapezium $CEDB$ with parallel sides of length b and d.

Find the value of θ for which the area of the trapezium $CEDB$ is a minimum. Also find the minimum value of the area. 3 marks

2015 Exam 2

Question 3



The rule for a function with the graph above could be

A. $y = -2(x + b)(x - c)^2(x - d)$

B. $y = 2(x + b)(x - c)^2(x - d)$

C. $y = -2(x - b)(x - c)^2(x - d)$

D. $y = 2(x - b)(x - c)(x - d)$

E. $y = -2(x - b)(x + c)^2(x + d)$

Question 4

Consider the tangent to the graph of $y = x^2$ at the point $(2, 4)$.

Which of the following points lies on this tangent?

A. $(1, -4)$

B. $(3, 8)$

C. $(-2, 6)$

D. $(1, 8)$

E. $(4, -4)$

Question 17

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts.

The set of all possible values of c is

A. R

B. R^+

C. $\{0, 4\}$

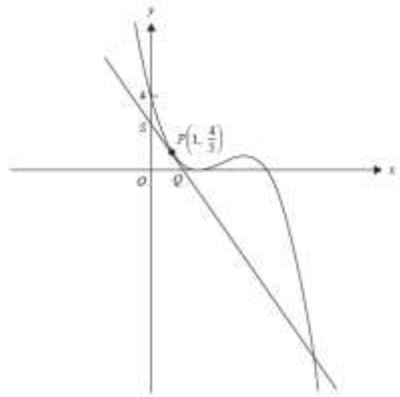
D. $(0, 4)$

E. $(-\infty, 4)$

Question 1 (9 marks)

Let $f: R \to R$, $f(x) = \frac{1}{5}(x-2)^2(5-x)$. The point $P\left(1, \frac{4}{5}\right)$ is on the graph of f, as shown below.

The tangent at P cuts the y-axis at S and the x-axis at Q.



a. Write down the derivative $f'(x)$ of $f(x)$. 1 mark

b. i. Find the equation of the tangent to the graph of f at the point $P\left(1, \frac{4}{5}\right)$. 1 mark

ii. Find the coordinates of points Q and S. 2 marks

c. Find the distance PS and express it in the form $\frac{\sqrt{b}}{c}$, where b and c are positive integers. 2 marks

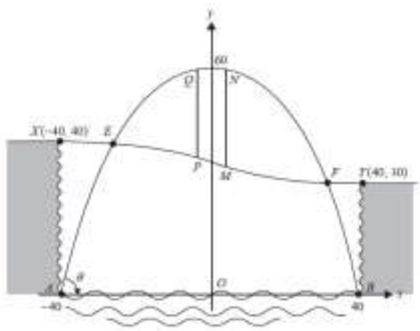
Question 2 (14 marks)

A city is located on a river that runs through a gorge.

The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.

A bridge is to be built that crosses the river and the gorge.

A diagram for the design of the bridge is shown below.



The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by $y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at $B(40, 0)$ and $A(-40, 0)$.

The road across the gorge is modelled by a cubic polynomial function.

a. Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree. 2 marks

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$.

b. Find the maximum downwards slope of the road. Give your answer in the form $-\frac{m}{n}$ where m and n are positive integers. 2 marks

Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame.
The supporting column, MN, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c. Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places. 3 marks

The second supporting column, PQ, has its lowest point at $P(-u, w)$.

d. Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ. 3 marks

e. Find the x-coordinates, correct to two decimal places, of E and F, the points at which the road meets the parabolic frame of the bridge. 3 marks

f. Find the area of the banner (shaded region), giving your answer to the nearest square metre. 1 mark

Question 5 (15 marks)

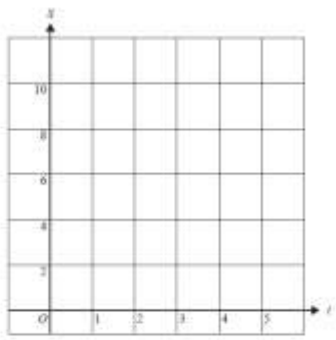
a. Let $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, where $0 \le t \le 5$.

i. Find $S(0)$ and $S(5)$. 1 mark

ii. The minimum value of S occurs when $t = \log_e(c)$.

State the value of c and the minimum value of S. 2 marks

iii. On the axes below, sketch the graph of S against t for $0 \le t \le 5$. Label the end points and the minimum point with their coordinates. 2 marks



iv. Find the value of the average rate of change of the function S over the interval $[0, \log_e(c)]$. 2 marks

Let $F: [0, 5] \to R$, $F(t) = de^{\frac{t}{3}} + (10 - d)e^{-\frac{2t}{3}}$, where d is a real number and $d \in (0, 10)$.

b. If the minimum value of the function occurs when $t = \log_e(9)$, find the value of d. 2 marks

c. i. Find the set of possible values of d such that the minimum value of the function occurs when $t = 0$. 2 marks

ii. Find the set of possible values of d such that the minimum value of the function occurs when $t = 5$. 2 marks

d. If the function F has a local minimum (a, w), where $0 \le a \le 5$, it can be shown that $w = \frac{k}{2}d^{\frac{2}{3}}(10 - d)^{\frac{1}{3}}$.

Find the value of k. 2 marks