Cycloidal motion in an electromagnetic field

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Building the motion equations

A charged particle is introduced in a region where there is a uniform electric field (intensity ***E***) and a uniform magnetic field (intensity ***B***).
We'll call q and m the charge and mass of the particle and we'll assume that the electric field is directed as the y axis and that the magnetic field is directed as the z axis.

We'll also assume that the particle starts at the origin of the reference frame with an initial speed whose components are $v_x(0)=v_{0x}$, $v_y(0)=v_{0y}$ and $v_z(0)=v_{0z}$. Using the standard notation we'll indicate $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ the unit vectors (versors) directed respectively as the x, y, and z axes.
The forces acting on the particle will then be the electric force F_E and the Lorentz force F_L:

$$\mathbf{F_E} = qE \cdot \mathbf{j}$$

$$\mathbf{F_L} = qB\left(v_y \cdot \mathbf{i} - v_x \cdot \mathbf{j}\right)$$

There's no force directed along the z axis so we'll just notice that, if $v_{0z} \neq 0$, there will be a vertical motion with equation $z(t) = v_{0z}t$ adding to the motion in the xy plane. We'll make no further reference to this vertical motion in the following lines, mainly focused on the xy motion.

Considering separately the resulting motion in the x and y direction we get the differential equations

$$\dot{v}_x = \frac{qB}{m} v_y \qquad (1)$$

$$\dot{v}_y = \frac{q}{m}\left(E - Bv_x\right) \qquad (2)$$

Assuming[1] $B \neq 0$ we can extract v_y from eq. (1) and differentiate

$$v_y = \frac{m}{qB}\dot{v}_x$$

$$\dot{v}_y = \frac{m}{qB}\ddot{v}_x$$

Putting this in eq. (2) we get a single second order *ODE* in $v_x(t)$

$$\frac{m}{qB}\ddot{v}_x = \frac{q}{m}\left(E - Bv_x\right)$$

that is

[1] If $B=0$ the motion is a simple parabolic motion with $\begin{cases} \dot{v}_x = 0 \\ \dot{v}_y = \frac{q}{m}E \end{cases} \rightarrow \begin{cases} x(t) = v_{0x}t + x(0) \\ y(t) = \frac{q}{m}E \cdot t^2 + v_{0y}t + y(0) \end{cases}$

$$\ddot{v}_x + \left(\frac{qB}{m}\right)^2 v_x = \left(\frac{q}{m}\right)^2 BE \qquad (3)$$

Above equation is a *second order non homogeneous linear differential equation* with constant coefficients that can be solved exactly.

If we set $\omega = \frac{Bq}{m}$ the solution for $v_x(t)$ is

$$v_x(t) = \left(v_{0x} - \frac{E}{B}\right)\cos(\omega t) + v_{0y}\sin(\omega t) + \frac{E}{B} \qquad (4)$$

From $v_y = \frac{m}{qB}\dot{v}_x$ (eq. (2)) we also have

$$v_y(t) = -\left(v_{0x} - \frac{E}{B}\right)\sin(\omega t) + v_{0y}\cos(\omega t) \qquad (5)$$

The parametric equations of the trajectory can be derived by integrating eq. (4) and eq. (5)

$$x(t) = x(0) + \frac{m}{Bq}\left(\frac{q}{m}Et + \left(v_{0x} - \frac{E}{B}\right)\sin(\omega t) - v_{0y}\cos(\omega t) + v_{0y}\right)$$

$$y(t) = y(0) + \frac{m}{Bq}\left(\left(v_{0x} - \frac{E}{B}\right)\cos(\omega t) + v_{0y}\sin(\omega t) - v_{0x} + \frac{E}{B}\right)$$

Recalling that we have assumed $x(0)=0$, $y(0)=0$ and rearranging the terms we get the following parametric equations:

$$\begin{cases} x(t) = \frac{m}{Bq}\left(\left(v_{0x} - \frac{E}{B}\right)\sin(\omega t) - v_{0y}\cos(\omega t)\right) + \frac{E}{B}t + \frac{m}{Bq}v_{0y} \\ y(t) = \frac{m}{Bq}\left(\left(v_{0x} - \frac{E}{B}\right)\cos(\omega t) + v_{0y}\sin(\omega t)\right) + \frac{m}{Bq}\left(\frac{E}{B} - v_{0x}\right) \end{cases} \qquad (6)$$

The motion in the x direction is an oscillatory motion around a moving point having speed E/B. The moving oscillation center follows the motion equation $x_{ave}(t) = \frac{E}{B}t + \frac{m}{Bq}v_{0y}$

The motion in the y direction is also an oscillatory motion around the middle point with $y_{ave} = \frac{m}{Bq}\left(\frac{E}{B} - v_{0x}\right)$

Both motions have the same oscillation amplitude

$$A = \frac{m}{Bq}\sqrt{\left(v_{0x} - \frac{E}{B}\right)^2 + {v_{0y}}^2}$$

if $v_{0x} = 0$ and $v_{0y} = 0$ the trajectory will be a *cycloid* and the oscillation amplitude will be

$$A_0 = \frac{mE}{B^2 q}$$

The trajectory is a trochoid (general case of a cycloid)

The parametric equations of motion (6) can be rewritten in the form

$$\begin{cases} x(t) = A\sin(\omega t - \varphi) + \dfrac{1}{\omega}\dfrac{E}{B}\omega t + \dfrac{1}{\omega}v_{0y} \\ y(t) = A\cos(\omega t - \varphi) + \dfrac{1}{\omega}\left(\dfrac{E}{B} - v_{0x}\right) \end{cases} \qquad (7)$$

where

$$A = \frac{1}{\omega}\sqrt{\left(v_{0x} - \frac{E}{B}\right)^2 + {v_{0y}}^2}\,,\ \cos\varphi = \frac{v_{0x} - \dfrac{E}{B}}{\sqrt{\left(v_{0x} - \dfrac{E}{B}\right)^2 + {v_{0y}}^2}}\,,\ \sin\varphi = \frac{v_{0y}}{\sqrt{\left(v_{0x} - \dfrac{E}{B}\right)^2 + {v_{0y}}^2}}\,,\ \omega = \frac{Bq}{m}$$

The parametric equations (7) are those of the curve called ***trochoid***.
The trochoid is defined as the curve traced by a point P fixed at some distance r_t from the center of a circle that rolls without slipping along a line.

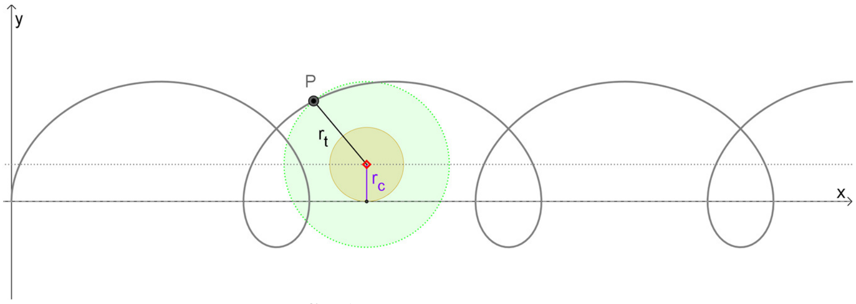

fig. 1: a prolated trochoid

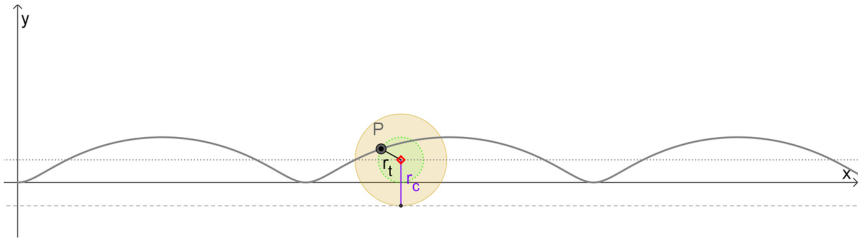

fig. 2: a curtated trochoid

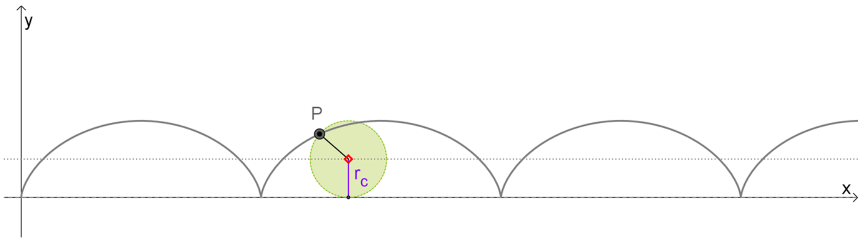

fig. 3: a cycloid (common trochoid)

The general trochoid parametric equations are

$$\begin{cases} x(t) = a\omega t + b\sin(\omega t) \\ y(t) = a + b\cos(\omega t) \end{cases}$$

where a is the distance between P and the center of the rolling circle and b is the radius of the rolling circle.
If P lies inside the circle ($b < a$) the trochoid is called *curteted trochoid*.
If P lies on the circumference ($b = a$) the trochoid is called *common trochoid* or **cycloid**.
If P lies outside the circle ($b > a$) the trochoid is called *prolate trochoid*.

A prolate trochoid contains loops. A cycloid (common trochoid) has cusps at the points where *P* touches the line on which the circle rolls.

In our case the distance of the point *P* generating the trochoid from the center of the rolling circle is: $r_t = \frac{1}{\omega}\sqrt{\left(v_{0x} - \frac{E}{B}\right)^2 + v_{0y}{}^2}$

while the "rolling circle" has radius

$r_c = \frac{1}{\omega}\frac{E}{B}$

Its center follows the line $y = \frac{1}{\omega}\left(\frac{E}{B} - v_{0x}\right)$ with speed $\frac{E}{B}$

A cycloid is a special case of a trochoid occurring when $r_t = r_c$. So the condition to have a cycloid is:

$\frac{1}{\omega}\frac{E}{B} = \frac{1}{\omega}\sqrt{\left(v_{0x} - \frac{E}{B}\right)^2 + v_{0y}{}^2}$ that is $\sqrt{\left(v_{0x} - \frac{E}{B}\right)^2 + v_{0y}{}^2} = \frac{E}{B}$

The trochoidal/cycloidal motion and the *equivalence principle*

The motion described by equations (6) in the original frame of reference *S*

$$S:\begin{cases} x(t) = \frac{m}{Bq}\left(\left(v_{0x} - \frac{E}{B}\right)\sin(\omega t) - v_{0y}\cos(\omega t)\right) + \frac{E}{B}t + \frac{m}{Bq}v_{0y} \\ y(t) = \frac{m}{Bq}\left(\left(v_{0x} - \frac{E}{B}\right)\cos(\omega t) + v_{0y}\sin(\omega t)\right) + \frac{m}{Bq}\left(\frac{E}{B} - v_{0x}\right) \end{cases} \quad (6)$$

can be transformed to a different set of equations describing the motion as seen in an alternative frame of reference S' with a moving origin $O'\left(\frac{E}{B}t;0\right)$ (canceling the *x*-translation speed $\frac{E}{B}$ that is present in *S*).

In this new set of equations the initial speeds will be $v'_{0x} = v_{0x} - \frac{E}{B}$, $v'_{0y} = v_{0y}$ and in S' the parametric equations of motion will then be:

$$S':\begin{cases} x'(t) = \frac{m}{Bq}\left(v'_{0x}\sin(\omega t) - v_{0y}\cos(\omega t)\right) + \frac{m}{Bq}v_{0y} \\ y'(t) = \frac{m}{Bq}\left(v'_{0x}\cos(\omega t) + v_{0y}\sin(\omega t)\right) - \frac{m}{Bq}v'_{0x} \end{cases} \quad (8)$$

representing a circular motion around the fixed point $\left(\frac{m}{Bq}v'_{0y};\frac{m}{Bq}(-v'_{0x})\right)$

This same motion can be obtained in the original (static) frame of reference $S:(0;0)$ if we set $E = 0$ in equations (6). Thence we can define another system S'' identical to S but with the electric field removed:

$$S'':\begin{cases} x''(t) = \frac{m}{Bq}\left(v_{0x}\sin(\omega t) - v_{0y}\cos(\omega t)\right) + \frac{m}{Bq}v_{0y} \\ y''(t) = \frac{m}{Bq}\left(v_{0x}\cos(\omega t) + v_{0y}\sin(\omega t)\right) - \frac{m}{Bq}v_{0x} \end{cases} \quad (9)$$

The two motions, seen in the two different frames of reference S' and S'', are identical.

Above arguments shows that the trochoidal/cycloidal motion can be reduced to a pure circular motion in a frame of reference with a moving origin having coordinates $O'\left(\frac{E}{B}t;0\right)$ or in a static physical system where there is only the magnetic field **B**.
This also means that we can set an alternative inertial frame of reference S'' in which the motion is just driven by the magnetic field B instead of one in which the motion is the result of the combined effects of both an electric and a magnetic field. The effect of the electric field E in S can be eliminated in S'' by setting $E''=0$ and $v''_{0x}=v_{0x}-\frac{E}{B}$
The following table summarize the different situations described above

S with $O(0;0)$	S' with $O'\left(\frac{E}{B}t;0\right)$	S'' with $O''(0;0)$
Both **E** and **B** are present	Both **E** and **B** are present. The observer moves with speed $\frac{E}{B}$	**E=0** and only **B** is present.
Initial speeds: v_{0x}, v_{0y}	Initial speeds: $v'_{0x}=v_{0x}-\frac{E}{B}$, $v'_{0y}=v_{0y}$	Initial speeds: v_{0x}, v_{0y}
Trochoidal motion	Circular motion around O'	Circular motion around O'

Since the *principle of relativity* says that the forces acting in nature must have the same form in every inertial frame of reference it follows that the magnetic force (Lorentz force) and the electric force are just different aspects of some more general force that combine both of them.
That's the ***electromagnetic force***. It will be a pure electric force within some frame of reference, a pure magnetic force within some other frame of reference and a combination of both of them in some other frame of reference.
The problem is that we should use a tensor (electromagnetic tensor) to represent the electromagnetic force (instead of separated vectors).

The energy

The general motion equation (6) shows that, whatever are the initial conditions, if both **E** and **B** are present, the charged particle will always end up translating in the *x*-direction (direction of the vector $\mathbf{E}\times\mathbf{B}$) with an average speed $v_{ave}=\frac{E}{B}$.

This motion can be very simple in the case of the *Wiener filter* (with initial conditions $v_{0x}=\frac{E}{B}$, $v_{0y}=0$) where the motion will be the simple linear uniform motion described by the equations

$$\begin{cases} x(t)=\frac{E}{B}t \\ y(t)=0 \end{cases}$$

With other initial conditions for the initial speed we'll have the trochoidal motions examined before but they can nonetheless be considered as a combination of linear motion with constant speed $v_{ave}=\frac{E}{B}$ in the *x*-direction and an oscillatory motion (with zero average speed).
In the case of a null initial speed ($v_{0x}=0$, $v_{0y}=0$) we get the cycloidal motion

$$\begin{cases} x(t) = -\dfrac{mE}{B^2 q}\sin(\omega t) + \dfrac{E}{B}t \\ y(t) = \dfrac{mE}{B^2 q}(1-\cos(\omega t)) \end{cases}$$

and this motion too has an average translational speed in the *x*-direction $v_{ave} = \frac{E}{B}$.

That's a little surprising. If we start with no kinetic energy how comes that we end up with the some positive kinetic energy?

The answer is in the potential energy.

There's no potential energy for the magnetic field (since the Lorentz' force doesn't do any work being always perpendicular to the infinitesimal displacement). But there is a potential energy for the electric field.

The gain in the average kinetic energy is balanced by a loss in the average potential energy of the electric field.

In fact we can see from equations (6) that the average displacement in the *y*-direction (with $v_{0x} = 0$) is

$$y_{ave} = \frac{mE}{B^2 q} - \frac{m}{Bq}v_{0x}$$

and the average electric potential energy will then be

$$\Delta U_{ave} = -qEy_{ave} = -m\frac{E}{B}\left(\frac{E}{B} - v_{0x}\right)$$

Let's now calculate the average kinetic energy. Even if the average translational speed in the *x*-direction is $v_{ave} = \frac{E}{B}$ this doesn't mean that the average kinetic energy is $\frac{1}{2}mv^2{}_{ave} = \frac{1}{2}\left(\frac{E}{B}\right)^2$. In fact the *wobbling* in the trochoidal motion contribute some extra kinetic energy.

The average kinetic energy can be calculated as

$$KE_{ave} = \frac{1}{T}\int_0^T \frac{1}{2}m\left(v_x(t)^2 + v_y(t)^2\right)\mathrm{d}t$$

where $T = \frac{2\pi}{\omega} = \frac{m}{Bq}2\pi$, $v_x(t) = \left(v_{0x} - \frac{E}{B}\right)\cos(\omega t) + v_{0y}\sin(\omega t) + \frac{E}{B}$, $v_y(t) = -\left(v_{0x} - \frac{E}{B}\right)\sin(\omega t) + v_{0y}\cos(\omega t)$

Doing the calcs yields

$$KE_{ave} = \frac{1}{2}m\frac{\left(2E^2 - 2BEv_{0x} + B^2\left({v_{0x}}^2 + {v_{0y}}^2\right)\right)}{B^2}$$

Subtracting the initial kinetic energy $KE_0 = \frac{1}{2}m\left({v_{0x}}^2 + {v_{0y}}^2\right)$ we have

$$\Delta K_{ave} = KE_{ave} - KE_0 = \frac{1}{2}m\frac{\left(2E^2 - 2BEv_{0x} + B^2\left({v_{0x}}^2 + {v_{0y}}^2\right)\right)}{B^2} - \frac{1}{2}m\left({v_{0x}}^2 + {v_{0y}}^2\right)$$

$$\Delta K_{ave} = KE_{ave} - KE_0 = m\frac{\left(E^2 - BEv_{0x}\right)}{B^2} = m\frac{E}{B}\left(\frac{E}{B} - v_{0x}\right)$$

then we have $\Delta K_{ave} + \Delta U_{ave} = 0$, respecting the conservation of the total energy

The same result can be obtained more precisely (with calculations more complicated) considering the punctual values of the kinetic energy $\Delta K(t)$ and of the electric potential energy $\Delta U(t)$ and showing that, for every *t* is:

$$\Delta K(t) + \Delta U(t) = 0$$

In fact

$$\Delta K(t) = \frac{1}{2} m\left(v_x^2(t) + v_y^2(t)\right) - \frac{1}{2} m\left(v_{0x}^2 + v_{0y}^2\right)$$

$$\Delta K(t) = \frac{1}{2} m\left[\left(\left(v_{0x} - \frac{E}{B}\right)\cos(\omega t) + v_{0y}\sin(\omega t) + \frac{E}{B}\right)^2 + \left(-\left(v_{0x} - \frac{E}{B}\right)\sin(\omega t) + v_{0y}\cos(\omega t)\right)^2 - \left(v_{0x}^2 + v_{0y}^2\right)\right]$$

that simplify to

$$\Delta K(t) = m\frac{E}{B}\left(\left(\frac{E}{B} - v_{0x}\right)\left(1 - \cos(\omega t)\right) + v_{0y}\sin(\omega t)\right)$$

The electric potential energy is $\Delta U(t) = -q\,E\,y(t)$. We need rearrange the expression of $y(t)$ in equations (6) from

$$y(t) = \frac{m}{Bq}\left(\left(v_{0x} - \frac{E}{B}\right)\cos(\omega t) + v_{0y}\sin(\omega t)\right) + \frac{m}{Bq}\left(\frac{E}{B} - v_{0x}\right)$$

to

$$y(t) = \frac{m}{Bq}\left(\left(\frac{E}{B} - v_{0x}\right)\left(1 - \cos(\omega t)\right) + v_{0y}\sin(\omega t)\right)$$

Then we have

$$\Delta U(t) = -q\,E\,y(t) = -m\frac{E}{B}\left(\left(\frac{E}{B} - v_{0x}\right)\left(1 - \cos(\omega t)\right) + v_{0y}\sin(\omega t)\right)$$

It is clearly $\Delta K(t) + \Delta U(t) = 0$

Conclusions

In the setting proposed, with a uniform electric field and a perpendicular uniform magnetic fields acting on a charged particle, these fields produce, in the plane perpendicular to the magnetic field, a trochoidal trajectory (general case of a cycloidal motion). We get a cycloid in the special case of zero initial speed.
Choosing an appropriate different inertial reference frame this motion can be accounted of by the only magnetic field. In this specific frame of reference the resulting motion is a circular motion with constant speed.
Then the different effects of the magnetic and electric field depends on the choice of the inertial frame of reference.
There is a great lecture by Richard Feynman that investigate about these themes "The relativity of magnetic and electric fields".