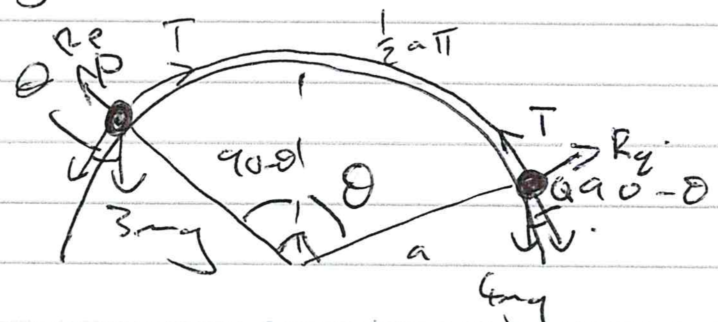
STEP III 2011
Mechanics Questions

9) Resolving tangentially works (rather than resolving towards the centre as in most M3 questions). See M3 book pages 137 to 139 for details.



~~Taking tangents at each particle~~ initially when system is in equilibrium.

$$Ta + 4mg\sin\theta = Ta + 3mg\cos\theta.$$

$$\Rightarrow 4\sin\theta = 3\cos\theta.$$

$$\Rightarrow \tan\theta = \frac{3}{4} \Rightarrow \cos\theta = \frac{4}{5} \quad \sin\theta = \frac{3}{5}.$$

Effectively $t = 0$ conditions.

Resolving tangential when system is moving. Tangential acceleration $= r\ddot{\theta}$

Note $r\ddot{\theta} = r\dfrac{d\frac{1}{2}\dot{\theta}^2}{d\theta}$ just like $\ddot{x} = \dfrac{d\frac{1}{2}\dot{x}^2}{dx}$.

For P $\Rightarrow 3ma\ddot{\theta} = T - 3mg\cos\theta$

For Q $\Rightarrow 4ma\ddot{\theta} = 4mg\sin\theta - T.$

Adding gives.

$7ma\ddot{\theta} = 4mg\sin\theta - 3mg\cos\theta$

$\Rightarrow 7a\ddot{\theta} = 4g\sin\theta - 3g\cos\theta$

$\Rightarrow 7a\dfrac{d\frac{1}{2}\dot{\theta}^2}{d\theta} = 4g\sin\theta - 3g\cos\theta.$

$\Rightarrow 7a\frac{1}{2}\dot{\theta}^2 = -4g\cos\theta - 3g\sin\theta + c.$

when $t=0$ $\dot{\theta}=0$ $\cos\theta = \frac{4}{5}$ $\sin\theta = \frac{3}{5}.$

$\Rightarrow 0 = -\frac{16g}{5} - \frac{9g}{5} + c.$

$\Rightarrow c = \frac{25g}{5} = 5g$

$\Rightarrow 7a\frac{1}{2}\dot{\theta}^2 = -4g\cos\theta - 3g\sin\theta + 5g.$

$\Rightarrow \frac{7}{2}a\dot{\theta}^2 + 4g\cos\theta + 3g\sin\theta = 5g$

$\Rightarrow 7a\dot{\theta}^2 + 8g\cos\theta + 6g\sin\theta = 10g$ — (1)

i) Resolving towards the centre for Q gives

$4mg\cos\theta - R_Q = 4ma\dot{\theta}^2.$

$R_Q = 0 \Rightarrow a\dot{\theta}^2 = g\cos\theta$

Inserting into (1) gives. Changing θ to β

$$7g\cos\beta + 8g\cos\beta + 6g\sin\beta = 10g$$

$$\Rightarrow 15\cos\beta + 6\sin\beta = 10$$

ii) Resolving to centre equations

$$3ma\ddot{\theta} = T - 3mg\cos\theta$$
$$4ma\ddot{\theta} = 4mg\sin\theta - T.$$

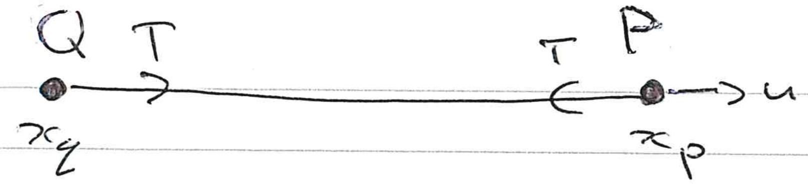
$$\Rightarrow \frac{T - 3mg\cos\theta}{3} = \frac{4mg\sin\theta - T}{4.}$$

$$\Rightarrow 4T - 12mg\cos\theta = 12mg\sin\theta - 3T.$$

$$\Rightarrow 7T = 12mg(\sin\theta + \cos\theta)$$

$$\Rightarrow T = \frac{12}{7}mg(\sin\theta + \cos\theta).$$

(0)



$l = a$

$\lambda = \frac{1}{2} m a \omega^2$

Applying Hooke's law.

$$T = \frac{\lambda (x_p - x_q - a)}{a} \Rightarrow T = \frac{1}{2} m \omega^2 (x_p - x_q - a)$$

Applying F=ma for both particles.

$$m\ddot{x}_p = -T = \frac{1}{2} m \omega^2 (a - (x_p - x_q))$$

$$m\ddot{x}_q = T = \frac{1}{2} m \omega^2 ((x_p - x_q) - a)$$

Adding gives $m\ddot{x}_p + m\ddot{x}_q = 0$.

$$\Rightarrow \ddot{x}_p + \ddot{x}_q = 0.$$

let $y = x_p + x_q \Rightarrow \ddot{y} = \ddot{x}_p + \ddot{x}_q = 0.$

$\Rightarrow y = At + B$

when $t = 0$ $y = a \Rightarrow B = a.$

when $t = 0$ $\dot{y} = u \Rightarrow A = u.$

$\Rightarrow y = ut + a$

substracting $z = x_p - x_q$

$$\Rightarrow \ddot{z} = \ddot{x}_p - \ddot{x}_q$$

$$m\ddot{x}_p - m\ddot{x}_q = m\omega^2 (a - (x_p - x_q))$$

$$\Rightarrow \ddot{x}_p - \ddot{x}_q = \omega^2 (a - (x_p - x_q)).$$

In terms of z.

$$\ddot{z} = \omega^2 (a - z).$$

$$\Rightarrow \ddot{z} + \omega^2 z = \omega^2 a.$$

$$\Rightarrow z = A\sin(\omega t + B)$$ complementary.

P.I. $z = \text{const} \Rightarrow z = a$.

$\therefore$ solution $z = A\sin(\omega t + B) + a$

$t = 0 \Rightarrow z = a$

$$\Rightarrow A\sin B + a = a \Rightarrow \sin B = 0$$
$$\Rightarrow B = 0.$$

$$\Rightarrow z = A\sin(\omega t) + a$$

$$\frac{dz}{dt} = A\omega\cos\omega t$$

$t = 0 \quad \dot{z} = u. \Rightarrow A\omega = u$
$$A = \frac{u}{\omega}.$$

$$z = \frac{u}{\omega}\sin(\omega t) + a.$$

$y = x_p + x_q \qquad z = x_p - x_q.$

$$\Rightarrow x_p = \frac{1}{2}(y + z) = \frac{1}{2}\left(ut + a + \frac{u}{\omega}\sin(\omega t) + a\right)$$
$$= \frac{1}{2}ut + \frac{1}{2}\frac{u}{\omega}\sin(\omega t) + a.$$

$$x_q = \frac{1}{2}(y - z) = \frac{1}{2}\left(ut + a - \left(\frac{u}{\omega}\sin(\omega t) + a\right)\right)$$

$$= \frac{1}{2}ut - \frac{1}{2}\frac{u}{\omega}\sin(\omega t).$$

clearly particles with return to a separation of 'a' when $t = \frac{\pi}{\omega}$

$t = 0 \qquad x_p = a \qquad x_q = 0.$

$t = \frac{\pi}{\omega} \qquad x_p = \frac{1}{2}u\frac{\pi}{\omega} + a \qquad x_q = \frac{1}{2}u\frac{\pi}{\omega}$

So both particles have travelled a distance of

$$\underline{\underline{\frac{1}{2}\pi\frac{u}{\omega}}}$$

When string is no longer taut $z \leq a$ particles will move with the ~~sa~~ speed they had at the instant the string become slack $(z = a)$

$t = \frac{\pi}{\omega} \qquad \dot{x}_p = \frac{1}{2}u + \frac{1}{2}u\cos(\omega t).$

$$= \frac{1}{2}u + \frac{1}{2}u\cos(\pi).$$

$$= 0.$$

$t = \frac{\pi}{u}$ $\quad \dot{x}_q = \frac{1}{2}u - \frac{1}{2}u\cos(ut)$.

$= \frac{1}{2}u - \frac{1}{2}u\cos(\pi)$.

$= u$.

So P is station & Q moves with no forces acting on it with speed u.

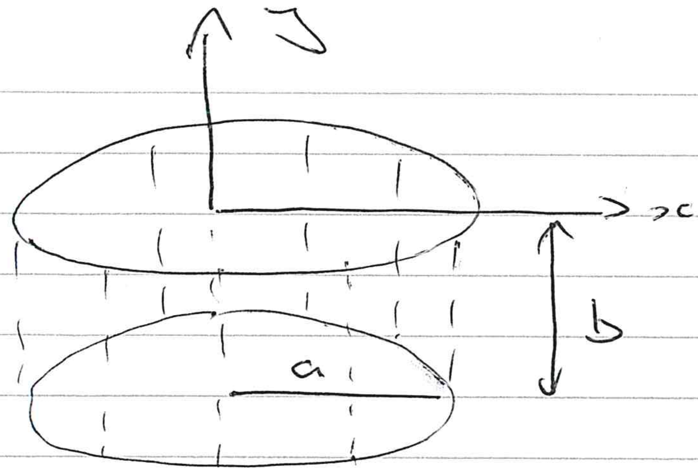
$$\text{speed} = \frac{\text{dist}}{\text{time}} \Rightarrow u = \frac{a}{t}.$$

$$\Rightarrow t = \frac{a}{u}.$$

So total time from impulse to collision is.

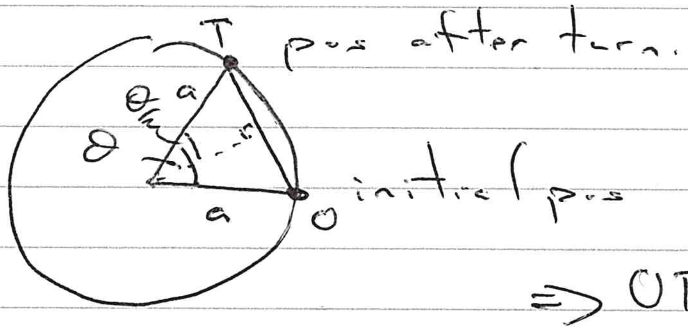
$$\underline{\underline{\frac{\pi}{u} + \frac{a}{u}}}$$

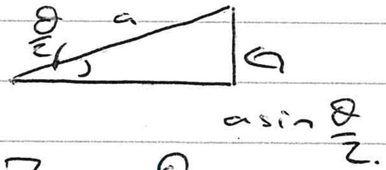
11)



$b > 2a$.

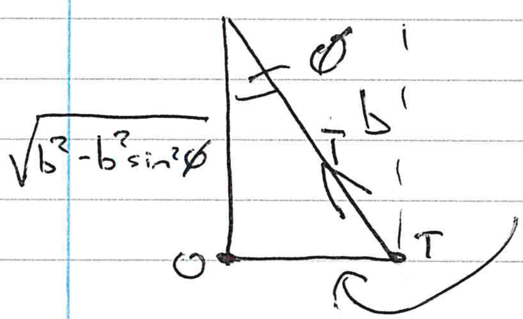
disc turned through angle θ.





$\Rightarrow OT = 2a\sin\frac{\theta}{2}$

Side view.



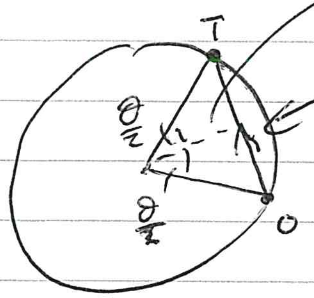
$b\sin\phi$

Equating expressions for OT gives.

$b\sin\phi = 2a\sin\frac{\theta}{2}$.

Resolving vertically $\Rightarrow mg = nT\cos\phi$. $\Rightarrow T = mg/n\cos\phi$

Taking moments moment for couple.



T sin ϕ

perpendicular distance $= a\cos\frac{\theta}{2}$.

$\therefore$ moment for a string acting at T is

$$\underbrace{\frac{mg}{n\cos\phi}\sin\phi}_{\text{force}} \times a\cos\frac{\theta}{2}.$$

from side view triangle. $= 1/\cos\phi$

$$= \frac{mg}{n}\left(\frac{b}{\sqrt{b^2-b^2\sin^2\phi}}\right)\times \sin\phi \times a\cos\frac{\theta}{2}.$$

$b\sin\phi$

$$= \frac{mg}{n}\,\frac{2a\sin\frac{\theta}{2}}{\sqrt{b^2-b^2\sin^2\phi}} \times a\cos\frac{\theta}{2}.$$

$$= \frac{mg}{n}\,\frac{a\sin\theta}{\sqrt{b^2-b^2\sin^2\phi}}.$$ Given n strings.

$$\Rightarrow \frac{mga\sin\theta}{\sqrt{b^2-b^2\sin^2\phi}} = \frac{mga\sin\theta}{\sqrt{b^2-4a^2\sin^2\frac{\theta}{2}}}.$$

When released the disc will drop & potential energy will be transformed into kinetic & rotational energy. At the lowest point there will be ~~[illegible]~~ no kinetic energy and all the P.E. will be converted into rotational energy

Rotational energy for disc $= \frac{1}{2} I\omega^2$.

Where $I =$ inertia $= \frac{1}{2} Mr^2$.

loss of P.E. $= mg(b - b\cos\phi)$

$$= \frac{1}{2}\left(\frac{1}{2} ma^2\right)\omega^2.$$

$$\Rightarrow \cancel{m}g(b - b\cos\phi) = \frac{1}{4}\cancel{m}a^2\omega^2.$$

$$g(b - b\cos\phi) = \frac{1}{4}a^2\omega^2.$$

$$b - \sqrt{b^2 - b^2\sin^2\phi} = \frac{a^2\omega^2}{4g}.$$

$$\Rightarrow b - \sqrt{b^2 - 4a^2\sin^2\frac{\theta}{2}} = \frac{a^2\omega^2}{4g}$$