in which $h^2+r^2-2rfc+c^2$ is the squared radius vector, and $h^2+(r-fc)^2$ is the squared perpendicular on the tangent of the development; this makes

$$\Phi = \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma - s} \frac{ds}{\sqrt{S}}, \tag{110}$$

or

$$\Phi = \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s_2 - \sigma} \int \frac{s - s_2}{s - \sigma} \frac{ds}{\sqrt{S}}, \tag{111}$$

exactly as for the determination of ψ in the preceding geodesics.]

On a Regular Rectangular Configuration of Ten Lines. By F. MORLEY. Received May 28th, 1898. Read June 9th, 1898.

1. *The Construction.*—I shall say that a straight line is normal to another when they intersect and are perpendicular. Three lines in space form three pairs, and each pair has a common normal. The three lines, with the three normals, form a rectangular hexagon. The three pairs of opposite sides of this hexagon give three more lines—their common normals. It will be shown that *these last three have one common normal.* Thus, if, starting with three lines, we keep on constructing all possible common normals (excluding the common normals of intersecting lines) we get only ten lines in all, forming a regular configuration in the sense that each line has three normals.

2. *The Points at Infinity.*—Taking five points a, b, c, d, e in space and cutting the ten lines such as ab and the ten planes such as abc by an arbitrary plane, we get a well-known configuration (Fig. 1). [*Cf.* Cayley's *Math. Papers*, Vol. I., p. 318.]

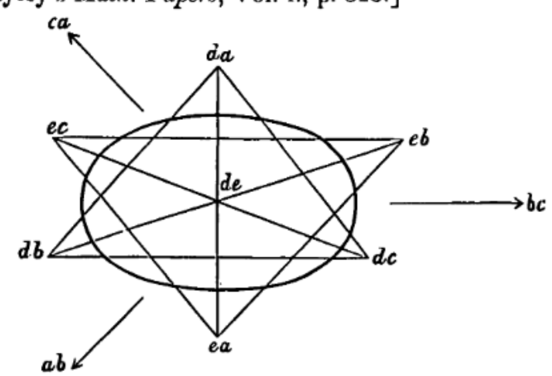

Fig. 1.

Each line *abc* in the plane is the polar of the point *de* with regard to a conic, as appears most easily from the consideration of the space-cubics through a, b, c, d, e (Reye, *Geometrie der Lage*, Vol. II., p. 273 of the second edition). Three ways of regarding the configuration are to be noticed:—

(i.) Removing a point *de* and its polar *abc*, we have two reciprocal triangles as to the conic, namely, *da*, *db*, *dc* and *ea*, *eb*, *ec*.

(ii.) The complete four-point *ea*, *eb*, *ec*, *ed* contains four of the points and six of the lines; the pairs of lines are conjugate as to the conic. The reciprocal of the four-point as to the conic is the other four lines and the other six points.

(iii.) To a pentagon, such as *ab*, *bc*, *cd*, *de*, *ea*, another *ac*, *ce*, *eb*, *bd*, *da* is both inscribed and circumscribed. The point *ac* is taken arbitrarily on the line *abc*, the second pentagon closing of its own accord.

We are here concerned with the case when the conic is the absolute circle at ∞; the five points a, b, c, d, e are then *orthocentric*, that is, the line through any two is perpendicular to the plane through the other three.

3. *The Proof.*—Returning to the three lines in space, we suppose them to be (Fig. 2)

$$y = \beta,\ z = -\gamma; \quad z = \gamma,\ x = -a; \quad x = a,\ y = -\beta;$$

and the points at ∞ on them to be

$$ea, \quad eb, \quad ec.$$

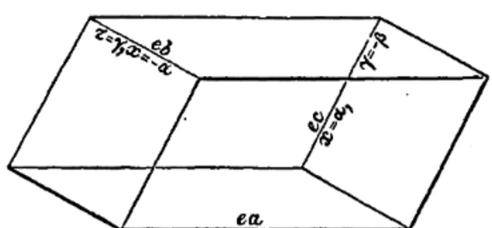

Fig. 2.

The coordinates are oblique; we suppose that the absolute cone is

$$a_1x^2 + b_1y^2 + c_1z^2 + 2fyz + 2gzx + 2hxy = 0,$$

and its reciprocal, the absolute circle, is

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta = 0.$$

The coordinates of the ten points at ∞ in Fig. 1 are known from the theory of a conic; thus

$$da \text{ is } A, H, G;$$
$$bc \text{ is } 0, g, -h;$$
$$dc \text{ is } 1/F, 1/G, 1/H.$$

The problem of writing the common normals is hereby much simplified. Naming each line in space by its point at ∞, we have for the equations of da, which is the common normal of eb and ec,

$$\frac{z-\gamma}{G} = \frac{x+\alpha}{A}, \quad \frac{x-\alpha}{A} = \frac{y+\beta}{H};$$

for we have merely to pick out from the pencils determining eb and ec the planes through the point

$$x/A = y/H = z/G = \infty.$$

Hence, using the same process, the common normal bc of da and ea, which passes through the point bc, has the equations

$$\frac{A(z-\gamma)-G(x+\alpha)}{-h} = \frac{H(x-\alpha)-A(y+\beta)}{-g},$$

$$\frac{y-\beta}{g} = -\frac{z+\gamma}{h}.$$

Therefore the plane through bc and the point de at ∞ is

$$\frac{(hH+gG)x+\alpha(gG-hH)-gA(z-\gamma)-hA(y+\beta)}{(hH+gG)(GH-AF)} = \frac{h(y-\beta)+g(z+\gamma)}{(hH+gG)F},$$

or, since $$GH-AF = \Delta . f,$$

$$Fx(gG+hH)-Gy.hH-Hz.gG+F\alpha(gG-hH)-G\beta.hH + H\gamma.gG+2\Delta(hf\beta-fg\gamma) = 0.$$

It is clear that the three such equations, obtained by permuting cyclically x, y, z; f, g, h; F, G, H; and α, β, γ, are consistent, since the left sides have the sum zero. Thus the three planes which should determine the third set of common normals belong to a pencil; which proves the point.

4. *Restatements.*—As in the plane configuration of § 2, the configuration of ten lines in space can be broken up in different ways. Thus we can state our theorem as follows:—

(i.) The common normals of opposite sides of a rectangular hexagon have a common normal.

(ii.) Four lines in space can be regarded as two pairs in three ways. If the common normals of the two pairs are themselves normal in two of the ways, they are so in the third. This statement of the case was suggested by Mr. Richmond.

(iii.) Two rectangular pentagons can be normal to each other; that is, each side of the one can be normal to a side of the other.

There are in the configuration ten rectangular hexagons, five systems of four lines of the kind just mentioned, and six pairs of mutually normal rectangular pentagons.

Point-Groups in a Plane, and their effect in determining Algebraic Curves. By F. S. MACAULAY, D.Sc. Read and received June 9th, 1898.

I. INTRODUCTION.

The following is a continuation of my former paper on "Point-Groups in relation to Curves" in Vol. XXVI. of the *Proceedings*, p. 519. It deals especially with the reduction of point-groups which supply a known number of conditions for an algebraic curve of any order.

The effect of a group of N points in determining an algebraic curve of order n (called hereafter a C_n) need not depend on N and n alone. It may, and often does, happen that the N points do not supply N independent conditions for a C_n, but only a smaller number $N-r_n$. In any case, if the point-group N is given, the number r_n has a definite positive* (integral or zero) value. The extreme case is that in which all the N points lie on a straight line; and we then have $r_n = N-(n+1)$ if $n \leqslant N-1$, and $r_n = 0$ if $n \geqslant N-1$.

For the case in which the N points form the complete intersection of two curves, the values of r_n for all values of n have long been known. Thus, if N consists of the complete intersection of a C_l and C_m, and if n is less than $l+m$, but not less than l or m, then

$$r_n = \tfrac{1}{2}(l+m-n-1)(l+m-n-2),$$

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* In this paper the curves are subject to no other conditions than those of passing through points, *i.e.*, they are general algebraic curves through given point-groups in which the parameters or coefficients enter linearly. For linear systems of curves no value of r could be negative; this would not be true for non-linear systems.