[MAA 4.5-4.7] PROBABILITY I (VENN DIAGRAMS – TABLES)

SOLUTIONS

Compiled by: Christos Nikolaidis

O. Practice questions

1. (a)

$n(A)$	15	$n(B)$	25	$n(A \cap B)$	10
$n(A')$	35	$n(B')$	25	$n(A \cup B)$	30
$n(A' \cap B)$	15	$n(A \cap B')$	5	$n(A' \cap B')$	20
$n(A' \cup B)$	45	$n(A \cup B')$	35	$n(A' \cup B')$	40

(b)

$P(A)$	15/50	$P(A')$	35/50	$P(A \cup B)$	30/50
$P(A' \cap B)$	15/50	$P(A' \cup B)$	45/50	$P(B' \cup A)$	35/50

(c)

$P(A \mid B)$	10/25	$P(A' \mid B)$	15/25	$P(B' \mid A)$	5/15
$P(B \mid A)$	10/15	$P(A \mid B')$	5/25	$P(A' \mid B')$	20/25

2. (a)

$P(A)$	0.5	$P(A')$	50	$P(A \cap B)$	0.2
$P(A \cup B)$	0.9	$P(A' \cap B)$	0.4	$P(A' \cup B)$	0.7

(b)

$P(A \mid B)$	1/3	$P(A' \mid B)$	2/3	$P(B' \mid A)$	3/5
$P(B \mid A)$	2/5	$P(A \mid B')$	3/4	$P(A' \mid B')$	1/4

3. (a)

$P(Boy)$	30/80	$P(Group\, C)$	15/80
$P(Boy\ and\ Group\ C)$	10/80	$P(Boy\ or\ Group\ C)$	35/80

(b)

$P(Boy \mid Group\, C)$	10/15	$P(Group\, C \mid Boy)$	10/30
$P(Boy \mid NOT\, Group\, C)$	20/65	$P(NOT\, Group\, C \mid Boy)$	20/30

4. (a)

$P(A)$	$a+c$	$P(A' \cap B)$	b
$P(A')$	$b+d$	$P(A' \cup B)$	$b+c+d$
$P(A \cap B)$	c	$P(A' \cap B')$	d
$P(A \cup B)$	$a+b+c$	$P(A' \cup B')$	$a+b+d$

(b)

$P(A \mid B)$	$c / (b+c)$	$P(B' \mid A)$	$a / (a+c)$
$P(B \mid A)$	$c / (a+c)$	$P(B \mid A')$	$b / (b+d)$
$P(A' \mid B)$	$b / (b+c)$	$P(A' \mid B')$	$d / (a+d)$
$P(A \mid B')$	$a / (a+d)$	$P(B' \mid A')$	$d / (b+d)$

(c) $P(A \cap B) = P(A)P(B) \Rightarrow c = (a+c)(b+c) \Rightarrow a = (2a)(2a)$

$\Rightarrow a = 4a^2 \Rightarrow a = 0.25$, so $b = c = d = 0.25$

5.   (a) $P(A) = 0.4$ $P(A \cup B) = 0.7$

         (b) $x = 0$.

         (c) $x = 0.2$

         (d) $x = 0.3$

         (e) $x = 0.1$

A.   Exam style questions (SHORT)

6.   (a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

             $65 = 30 + 50 - n(A \cap B) \Rightarrow n(A \cap B) = 15$ (may be on the diagram)

             $n(B \cap A') = 50 - 15 = 35$

         (b) $P(B \cap A') = \dfrac{n(B \cap A')}{n(U)} = \dfrac{35}{100} = 0.35$

7.   (a)   (i) $n(A \cap B) = 2$

              (ii) $P(A \cap B) = \dfrac{2}{36}\left(\text{or } \dfrac{1}{18}\right)$

         (b) $n(A \cap B) \neq 0$ (or equivalent)

8.   (a)   (i) $n = 0.1$

              (ii) $m = 0.2, p = 0.3, q = 0.4$

         (b) $P(B') = 0.6$

9.   (a)   (i) $p = 0.2$

              (ii) $q = 0.4$

              (iii) $r = 0.1$

         (b) $P(A \mid B') = \dfrac{2}{3}$

         (c)   valid reason *e.g.* $\dfrac{2}{3} \neq 0.5, 0.35 \neq 0.3$ thus, A and B are not independent

10.   (a) $p(A \cap B) = 0.6 + 0.8 - 1 = 0.4$

         (b) $p(A' \cup B') = p((A \cap B)') = 1 - 0.4 = 0.6$

11.   (a) $\dfrac{19}{120} \left(= 0.158\right)$

         (b) $35 - (8 + 5 + 7)(= 15)$

             Probability $= \dfrac{15}{120} \left(= \dfrac{3}{24} = \dfrac{1}{8} = 0.125\right)$

         (c)   Number studying $= 76$

             Number not studying $= 120 -$ number studying $= 44$

             Probability $= \dfrac{44}{120} \left(= \dfrac{11}{30} = 0.367\right)$

12.   (a)   (i) $P(P \mid C) = \dfrac{20}{20 + 40} = \dfrac{1}{3}$

              (ii) $P(P \mid C') = \dfrac{30}{30 + 60} = \dfrac{1}{3}$

         (b)   Investigating conditions, or some relevant calculations

P is independent of *C*, **with** valid reason

13. (a)

	Boy	Girl	Total
TV	**13**	**25**	**38**
Sport	33	29	**62**
Total	46	**54**	100

P(TV) = $\dfrac{38}{100}$

(b) P(TV | Boy) = $\dfrac{13}{46}$

14. (a)

	Males	Females	Totals
Unemployed	**20**	**40**	**60**
Employed	**90**	**50**	**140**
Totals	**110**	**90**	200

(b) (i) $P(\text{unemployed female}) = \dfrac{40}{200} = \dfrac{1}{5}$

(ii) $P(\text{male I employed person}) = \dfrac{90}{140} = \dfrac{9}{14}$

15. (a) 46/97

(b) 13/51

(c) 59/97

16. (a) (i) P(male or tennis) = $\dfrac{12}{20}$ $\left(=\dfrac{3}{5}\right)$

(ii) P(not football | female) = $\dfrac{6}{11}$

(b) P(first not football) = $\dfrac{11}{20}$, P(second not football) = $\dfrac{10}{19}$

P(neither football) = $\dfrac{11}{20} \times \dfrac{10}{19}$

P(neither football) = $\dfrac{110}{380}$ $\left(=\dfrac{11}{38}\right)$

17. (a) Independent (I)
(b) Mutually exclusive (M)
(c) Neither (N)

18. Using a table,

	ARGENTINE	NOT ARGENTINE	TOTAL
SPANISH	12	3	15
ENGLISH	3	3	6
TOTAL	15	6	21

$p(S| A) = \dfrac{12}{15} = \dfrac{4}{5}$
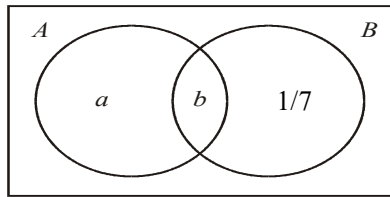
19. (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = \dfrac{1}{2} + \dfrac{3}{4} - \dfrac{7}{8} = \dfrac{3}{8}$

(b) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \left(= \dfrac{\frac{3}{8}}{\frac{3}{4}} \right) = \dfrac{1}{2}$

(c) Yes, the events are independent

EITHER $P(A|B) = P(A)$ **OR** $P(A \cap B) = P(A)P(B)$

20.

$P(A'|B) = \dfrac{1}{3} \Rightarrow \dfrac{\frac{1}{7}}{b + \frac{1}{7}} = \dfrac{1}{3} \Rightarrow \dfrac{1}{7} = \dfrac{1}{3}b + \dfrac{1}{21} \Rightarrow \dfrac{1}{3}b = \dfrac{1}{21} - \dfrac{1}{7} \Rightarrow b = \dfrac{2}{7}$

$P(B) = \dfrac{2}{7} + \dfrac{1}{7} = \dfrac{3}{7}$

21. (a) $P(A) = \dfrac{1}{11}$

(b) $P(B|A) = \dfrac{2}{10}$

(c) $P(A \cap B) = \dfrac{1}{11} \times \dfrac{2}{10} = \dfrac{2}{110}$

22. (a) Independent $\Rightarrow P(A \cap B) = P(A) \times P(B) = 0.3 \times 0.8 = 0.24$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.8 - 0.24 = 0.86$

(c) No, **since** $P(A \cap B) \neq 0$ or $P(A \cup B) \neq P(A) + P(B)$

23. (a) $P(F) = \dfrac{P(E \cap F)}{P(E)}$, $P(F) = \dfrac{\frac{1}{3}}{\frac{2}{3}} = \dfrac{1}{2}$

(b) $P(E \cup F) = P(E) + P(F) - (P(E \cap F)) = \dfrac{2}{3} + \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{5}{6}(=0.833)$

24. (a) $\dfrac{3}{4}$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \dfrac{2}{5} + \dfrac{3}{4} - \dfrac{7}{8} = \dfrac{11}{40} (0.275)$

(c) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ $\left(= \dfrac{\frac{11}{40}}{\frac{3}{4}} \right) = \dfrac{11}{30}$ (0.367)

25.   (a) $P(A \cap B) = P(A) \times P(B) = 0.6x$

       (b)   (i) $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

                  $\Leftrightarrow 0.80 = 0.6 + x - 0.6x \Leftrightarrow 0.2 = 0.4x \Leftrightarrow x = 0.5$

           (ii) $P(A \cap B) = 0.3$

       (c) $P(A \cap B) \neq 0$

26.   Let $P(A) = x$ then $P(B) = 3x$ and $P(A \cap B) = P(A) \times 3P(A) = 3x^2$

       $P(A \cup B) = P(A) + P(B) - P(A \cap B) \Leftrightarrow 0.68 = x + 3x - 3x^2$

       $3x^2 - 4x + 0.68 = 0 \Leftrightarrow x = 0.2$ ($x = 1.133$, not possible)

       $P(B) = 3x = 0.6$

27.   (a) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = \dfrac{3}{11} + \dfrac{4}{11} - \dfrac{6}{11} = \dfrac{1}{11}$ (0.0909)

       (b)   For independent events, $P(A \cap B) = P(A) \times P(B) = \dfrac{3}{11} \times \dfrac{4}{11} = \dfrac{12}{121}$ (0.0992)

28.

       (a)   (i)   Use of $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

                      $0.6 = 0.5 + 0.3 - P(A \cap B)$

                        $P(A \cap B) = 0.2$

           (ii) $P(A)P(B) = 0.15 \neq P(A \cap B)$

                Hence not independent

       (b)   Use of $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$

                      $= \dfrac{0.2}{0.5}$

                      $= 0.4$

29.   As $P(A \mid B) = P(A)$ then A and B are independent events

       Using $P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$

       to obtain $0.8 = 0.6 + P(B) - 0.6 \times P(B)$

       $0.8 = 0.6 + 0.4P(B) \implies P(B) = 0.5$ **OR** by using Venn diagram

30.

       (a) $P(A \cap B) = P(A) \times P(B \mid A)$

                  $= \dfrac{1}{5} \times \dfrac{1}{4}$ $\left(= \dfrac{1}{20} \right)$

       (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

                  $\Rightarrow P(B) = \dfrac{7}{10} - \dfrac{1}{5} + \dfrac{1}{20}$

                        $= \dfrac{11}{20}$

(c)   **METHOD 1**

$P(B) = \dfrac{11}{20}$ and $P(B|A) = \dfrac{1}{4}$

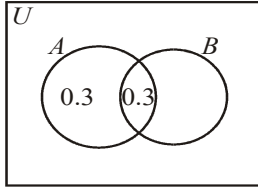$P(B) \neq P(B|A)$

$\Rightarrow A$ and B are not independent

METHOD 2

$P(A) \times P(B) = \dfrac{1}{5} \times \dfrac{11}{20} = \dfrac{11}{100}$ and $P(A \cap B) = \dfrac{1}{20}$

$P(A \cap B) \neq P(A) \times P(B)$

$\Rightarrow A$ and B are not independent

31. (Venn diagram)

$P(A \cap B) = P(A)P(B)$ $0.3 = 0.6 \times P(B)$

$P(B) = 0.5$

Therefore, $P(A \cup B) = 0.8$

32. (a) $0.88 = 0.4 + P(B) - 0.4P(B)$

        $0.6P(B) = 0.48 => P(B) = 0.8$

   (b) $P(A \cup B) - P(A \cap B) = 0.88 - 0.32 = 0.56$

33. for independence $P(A \cap B) = P(A) \times P(B)$

$P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$

$0.52 = P(A) + 2P(A) - 2P(A)P(A)$

$P(B) = 0.4$

34. Total number of possible outcomes $= 36$

(i) $P(E) = P(1,1) + P(2,2) + P(3,3) + P(4,4) + P(5,5) + P(6,6) \; = \dfrac{6}{36}$

(ii) $P(F) = P(6,4) + P(5,5) + P(4,6) = \dfrac{3}{36}$

(iii) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

       $P(E \cap F) = \dfrac{1}{36}$

       $P(E \cup F) = \dfrac{6}{36} + \dfrac{3}{36} - \dfrac{1}{36} \; \left(= \dfrac{8}{36} = \dfrac{2}{9}, 0.222 \right)$

35. Sample space $= \{(1,1), (1,2) \dots (6,5), (6,6)\}$

(This may be indicated in other ways, e.g, a grid or a tree diagram, partly or fully completed)
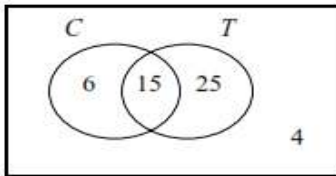
(a) $P(S < 8) = 21/36 = 7/12$

(b) $P(\text{at least one } 3) = \dfrac{11}{36}$

(c) $P(\text{at least one } 3 \,|\, S < 8) = 7/21 \; = \dfrac{1}{3}$

36.

(a)   **METHOD 1**

$P(T \cap C) = 0.3$

METHOD 2

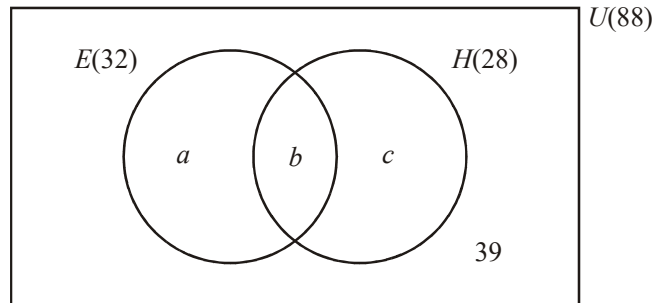$n(T \cup C) = n(T) + n(C) - n(T \cap C)$
$\qquad 46 = 40 + 21 - n(T \cap C)$
$n(T \cap C) = 15$
$P(T \cap C) = 0.3$

(b) $P(T \mid C) = \dfrac{P(T \cap C)}{P(C)}$

$\qquad = \dfrac{0.3}{0.42} \quad (= \dfrac{5}{7}, \; 0.714 \text{ to 3 s.f.})$

37.   (a)

$n(E \cup H) = a + b + c = 88 - 39 = 49$

$n(E \cup H) = 32 + 28 - b = 49 \Leftrightarrow b = 11$

$a = 32 - 11 = 21$
$c = 28 - 11 = 17$

(b)   (i) $P(E \cap H) = \dfrac{11}{88} = \dfrac{1}{8}$

     (ii) $P(H' \mid E) = \dfrac{P(H' \cap E)}{P(E)} = \dfrac{\frac{21}{88}}{\frac{32}{88}} = \dfrac{21}{32} \; (= 0.656)$ Or directly $= \dfrac{21}{32}$

(c)   (i)   P(none in economics) $= \dfrac{56 \times 55 \times 54}{88 \times 87 \times 86} = 0.253$

     (ii)   P(at least one) $= 1 - 0.253 = 0.747$
        OR

$3\left(\dfrac{32}{88} \times \dfrac{56}{87} \times \dfrac{55}{86}\right) + 3\left(\dfrac{32}{88} \times \dfrac{31}{87} \times \dfrac{56}{86}\right) + \dfrac{32}{88} \times \dfrac{31}{87} \times \dfrac{30}{86} = 0.747$

38. (a) $P(F \cup S) = 1 - 0.14 \ (= 0.86)$

$P(F \cap S) = 0.93 - 0.86 = 0.07$

Note: You can use *Venn Diagram*

(b) $P(F \mid S) \left(= \dfrac{P(F \cap S)}{P(S)} \right) = \dfrac{0.07}{0.62} = 0.113$

(c) F and S are **not** independent

EITHER

If independent $P(F \mid S) = P(F), \ 0.113 \neq 0.31$

OR

If independent $P(F \cap S) = P(F) \, P(S), \ 0.07 \neq 0.31 \times 0.62 \ (= 0.1922)$

(d) Let $P(F) = x$

$P(S) = 2P(F) = 2x$

$P(F \cup S) = P(F)P(S) - P(F)P(S) \Leftrightarrow 0.86 = x + 2x - 2x^2 \Leftrightarrow 2x^2 - 3x + 0.86 = 0$

$x = 0.386, \ x = 1.11$

$P(F) = 0.386$

39. (a) (i) $P(A) = \dfrac{80}{210} = \left(\dfrac{8}{21} = 0.381 \right)$

(ii) $P(\text{year 2 art}) = \dfrac{35}{210} = \left(\dfrac{1}{6} = 0.167 \right)$

(iii) No (the events are not independent

EITHER $P(A \cap B) = P(A) \times P(B)$ (to be independent)

$P(B) = \dfrac{100}{210} \left(= \dfrac{10}{21} = 0.476 \right)$ but $\dfrac{1}{6} \neq \dfrac{8}{21} \times \dfrac{10}{21}$

OR $P(A) = P(A \mid B)$ (to be independent)

$P(A \mid B) = \dfrac{35}{100}$ but $\dfrac{8}{21} \neq \dfrac{35}{100}$

OR $P(B) = P(B \mid A)$ (to be independent)

$P(B) = \dfrac{100}{210} \left(= \dfrac{10}{21} = 0.476 \right), P(B \mid A) = \dfrac{35}{80}$ but $\dfrac{35}{80} \neq \dfrac{100}{210}$

(b) $n(\text{history}) = 85$

$P(\text{year 1} \mid \text{history}) = \dfrac{50}{85} = \left(\dfrac{10}{17} = 0.588 \right)$

(c) $\left(\dfrac{110}{210} \times \dfrac{100}{209} \right) + \left(\dfrac{100}{210} \times \dfrac{110}{209} \right) \left(= 2 \times \dfrac{110}{210} \times \dfrac{100}{209} \right) = \dfrac{200}{399} (= 0.501)$

40. (a) (i) Venn diagram, 30

      (ii) 45

   (b) (i) $\dfrac{70}{100}\left(=\dfrac{7}{10}\right)$

      (ii) $\dfrac{45}{70}\left(=\dfrac{9}{14}\right)$

   (c) $P(A \cap B) = 0.3 \neq 0$

   (d) $P(A \cap B) \neq P(A) \times P(B)$, $\dfrac{30}{100} \neq \dfrac{75}{100} \times \dfrac{55}{100}$, $\dfrac{30}{55} \neq \dfrac{75}{100}$

      OR $P(B \mid A) \neq P(B)$, $\dfrac{30}{55} \neq \dfrac{75}{100}$

41. (a) (i) $s = 1$

      (ii) $q = 5$

      (iii) $p = 7, r = 3$

   (b) (i) P(art|music) $= \dfrac{5}{8}$

      (ii) **METHOD 1**

            $P(\text{art}) = \dfrac{12}{16}\left(=\dfrac{3}{4}\right)$ $\dfrac{3}{4} \neq \dfrac{5}{8}$ the events are not independent

          METHOD 2

            $P(\text{art}) \times P(\text{music}) = \dfrac{96}{256}\left(=\dfrac{3}{8}\right)$ $\dfrac{12}{16} \times \dfrac{8}{16} \neq \dfrac{5}{16}$ the events are not independent

   (c) P(first takes only music) $= \dfrac{3}{16}$

      P(second takes only art) $= \dfrac{7}{15}$

      P(music and art) $= \dfrac{3}{16} \times \dfrac{7}{15} = \dfrac{21}{240}\left(=\dfrac{7}{80}\right)$

42. (a) (i) 250   (ii) 166   (since 1000/6=166.67)   (iii) 83

   (b) 250/1000 = 1/4

   (c) 166/1000

   (d) 83/1000

   (e) 250-83 = 167   so P=167/1000.

   (f) 250 + 166 – 83 = 333   so P=333/1000

   (g) 250 + 166 – 2×83 = 250   so P=250/1000.