

A reference book has a formulation for the optimum launch angle θ at which we fire the projectile for a sloped terrain, with angle α measured positive clockwise from the horizontal (French, *Newtonian Mechanics*, p112). This angle is stated to be

$$\theta_{\text{opt}} = \frac{\pi}{4} - \frac{\alpha}{2} \quad (4)$$

and the downslope (along the hill) distance, for any θ , is claimed to be

$$L(\theta) = \frac{2 v_0^2}{g} \frac{\sin(\theta + \alpha) \cos(\theta)}{\cos^2(\alpha)} \quad (5)$$

Let's see if this is the same as Eq(2). We will need the fact that ϕ is $-\alpha$. Using this and taking a ratio,

$$\frac{\frac{2 v_0^2}{g} \frac{\sin(\theta - \phi) \cos(\theta)}{\cos^2(-\phi)}}{\frac{2 v_0^2}{g} \frac{\cos^2(\theta) (\tan(\theta) - \tan(\phi))}{\cos(\phi)}} = \frac{\sin(\theta - \phi)}{\cos(\theta) \cos(\phi)} \left(\frac{\sin(\theta)}{\cos(\theta)} - \frac{\sin(\phi)}{\cos(\phi)} \right) = \frac{\sin(\theta - \phi)}{\sin(\theta) \cos(\phi) - \sin(\phi) \cos(\theta)}$$

and the last expression is unity, so Eq(2) and Eq(5) are equivalent. Next we'd like to find the optimum angle. This can be done for both Eq(2) and (5) using trig, or calculus. First the trig. For Eq(5) we have the terms involving θ (all else is constant since α is given, as is v_0):

$$\sin(\theta + \alpha) \cos(\theta) = \frac{1}{2} (\sin(\theta + \alpha - \theta) + \sin(\theta + \alpha + \theta)) = \frac{1}{2} (\sin(\alpha) + \sin(2\theta + \alpha))$$

using the formula for the product of the sine and cosine of different angles. Now, to maximize this quantity we want the second sine to be unity, the largest it can be. For this to happen, it must be the case that

$$2\theta + \alpha = \frac{\pi}{2} \quad \text{so that} \quad \theta_{\text{opt}} = \frac{\pi}{4} - \frac{\alpha}{2} \quad \text{QED}$$

Similarly for Eq(2), the terms involving θ are (here, ϕ is given):

$$\cos^2(\theta) (\tan(\theta) - \tan(\phi)) = \cos^2(\theta) \left(\frac{\sin(\theta - \phi)}{\cos(\theta) \cos(\phi)} \right)$$

The second parentheses comes from a table of trig identities (Tuma, *Engineering Mathematics Handbook*, Third Edition, pp. 58-64 for this and other relations used here). Such a reference is indispensable for this kind of work. Here we used a formula for the difference of two tangents. Now we have just the θ -dependent factors

$$\cos(\theta) \sin(\theta - \phi) = \frac{1}{2} (\sin(\theta - \phi - \theta) + \sin(\theta - \phi + \theta))$$

using the product formula again. As before we end up with the sine argument

$$2\theta - \phi = \frac{\pi}{2} \quad \theta_{\text{opt}} = \frac{\pi}{4} + \frac{\phi}{2} \quad \text{QED}$$

Note that the form is the same, but the sign is reversed for ϕ as opposed to α , since they are measured in opposite senses.

We can also find this optimum angle using calculus, of course, by differentiating Eq(2) or Eq(5) with respect to θ and setting the derivative to zero. Solving for θ gives the critical value. This is tedious; it has been done by hand. It is an exercise in trig identities and there is no point in typing all that, since there is no new physics there. Suffice to say that we do get the same results as shown above for the optimum angle, for both formulations.

Finally we would like to have the maximum distance, when we use the optimum angle. From Eq(5),

$$L_{\max} = \frac{2 v_0^2}{g} \frac{\sin\left(\frac{\pi}{4} - \frac{\alpha}{2} + \alpha\right) \cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}{\cos^2(\alpha)}$$

which is, using the sine-cosine product again,

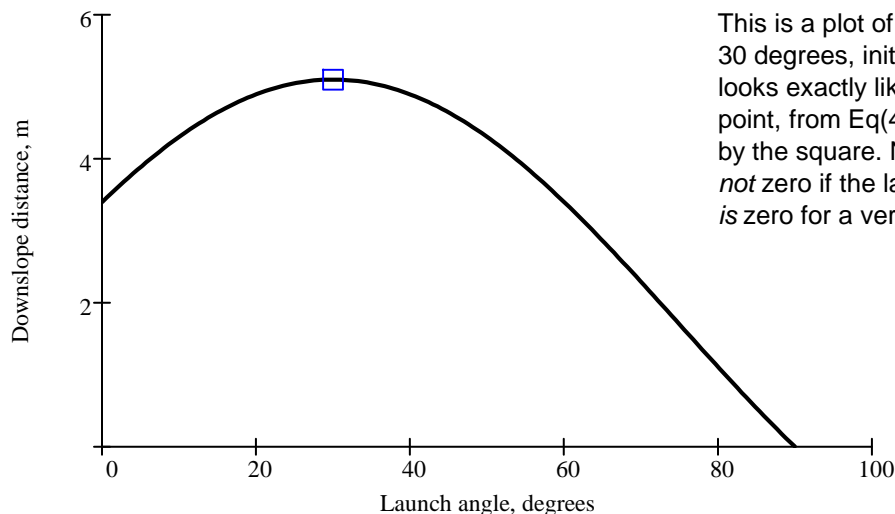
$$\frac{2 v_0^2}{g \cos^2(\alpha)} \left[\frac{1}{2} \left(\sin\left(\frac{\pi}{4} + \frac{\alpha}{2} - \frac{\pi}{4} + \frac{\alpha}{2}\right) + \sin\left(\frac{\pi}{4} + \frac{\alpha}{2} + \frac{\pi}{4} - \frac{\alpha}{2}\right) \right) \right] = \frac{v_0^2}{g} \frac{(1 + \sin(\alpha))}{\cos^2(\alpha)}$$

This is the result given in French, but it can be written more simply as

$$L_{\text{opt}} := \frac{v_0^2}{g} \frac{1}{1 - \sin(\alpha)} \quad (6)$$

Note that if the angle of the hill is zero, the maximum range is as we have for the $y(0) = 0$ case for a horizontal terrain, and the optimum angle from Eq(4) will just be 45 degrees. For Eq(2) we can go through a similar exercise, and we will find that

$$L_{\text{opt}} = \frac{v_0^2}{g} \frac{1}{1 + \sin(\phi)} \quad (7)$$



This is a plot of Eq(5) for a hill angle α of 30 degrees, initial velocity 5 m/s. Eq(2) looks exactly like this. The optimum point, from Eq(4) and Eq(6), is indicated by the square. Note that the distance is *not* zero if the launch angle is zero, but it *is* zero for a vertical launch.