ROLLING-BALL DATA ANALYSIS

A ball rolling down a ramp that was assumed to be straight, but in fact was curved slightly.

INTRODUCTION

It was suggested that a way to analyze the rolling-ball data would be to find the acceleration for each distance, and then in some manner compare these estimates to test the hypothesis that the acceleration was constant across distance down the ramp. This in fact had already been done in some prior experimentation, and the conclusion was that this approach was a relatively weak (in the statistical sense) test for a constant acceleration. It also would have required some new uncertainty analysis. Thus it was decided to stick with the (Galileo) ratio tests, for which the uncertainty analysis had already been developed.

However, calculation of the acceleration at each distance, and its uncertainty, is straightforward, and so this capability was added to the computer data analysis program. Graphing the acceleration estimates against distance down the ramp, that is, $a(s)$ vs. s, should have revealed random scatter around some average value, if the acceleration was constant down the ramp. That is, there should be no apparent trend.

ACCELERATION PLOTS

What these plots revealed, for nearly all data sets, was very clear, unambiguous evidence that *the acceleration was not constant.* In most cases there was an obvious downward trend with distance, so that the acceleration was slightly greater at the top end (elevated end) of the ramp; this corresponded to shorter distances. In one case there was the reverse situation, with the larger acceleration at the longer distances. In one other case there was no evident trend in the accelerations; for this student group the accelerations appeared to be constant.

NONCONSTANT-ACCELERATION DIFFERENTIAL EQUATION

To attempt to account for this nonconstant acceleration, it was assumed that there is a linear variation in acceleration with distance down the ramp. This would lead to the following second-order ordinary differential equation (ODE):

$$\frac{d^2 s}{dt^2} = a_0 - k\,s \tag{1}$$

where a_0 is the initial acceleration, that is, when the distance s of the ball down the ramp is zero, and k is a constant rate of change of the acceleration with s. For the case where the acceleration appears to increase with s, we would use the positive sign for the ks term. It is understood that this is undoubtedly too simple a model, and the acceleration would vary in a more complicated manner with s. But as is usual in engineering practice, a linear approximation can be an appropriate starting point for analysis.

DIFFERENTIAL EQUATION SOLUTION: A NEW MOTION MODEL

To obtain a unique solution for Eq(1), we need two initial conditions. These are the initial position $s(0)$, which we can always define to be zero, and the initial velocity of the ball, which is also zero. Using Laplace transforms to develop the solution, we find the time-dependent position of the ball to be

$$s(t) = \frac{a_0}{k}\left[1 - \cos\left(\sqrt{k}\,t\right)\right] \tag{2}$$

when the ks term in Eq(1) is negative, and

$$s(t) = \frac{a_0}{k}\left[\cosh\left(\sqrt{k}\,t\right) - 1\right] \tag{3}$$

when that term is positive. Needless to say, these look nothing like the usual kinematic equations that would apply if the acceleration was constant. Graphing them, however, reveals that they do have the general shape of the datasets.

MODEL WHEN ACCELERATION CHANGE IS SMALL

We would expect, if these new models for the motion are correct, that when the parameter k is small we should recover the usual kinematic equation for this situation, since if k is small Eq(1) reduces to the usual constant-acceleration ODE. To check this, consider that

$$\lim_{k \to 0} \left\{ \frac{a_0}{k} \left[1 - \cos\left(\sqrt{k}\, t \right) \right] \right\} = \frac{1}{2} a_0\, t^2$$

and similarly for Eq(3). This is, to say the least, not obvious. To see how this happens, expand the cosine and hyperbolic cosine in a Taylor series, and then take the limit; all terms after the first contain k, and thus vanish.

NONLINEAR ESTIMATION OF PARAMETERS IN MODEL

At this point we appear to have two new models for the motion of the ball down the ramp. These will be of little use unless we can estimate the parameters a_0 and k, using the observed data. These models are nonlinear, in the statistical sense, and so this estimation requires specialized analytical tools. These models were implemented in the computer data analysis program, and the results were excellent.

The datasets that showed significant lack of fit using the simple kinematics model now can be "fitted" very well using Eq(2), in most cases, and Eq(3) in a couple of other cases. The student dataset that did not show an apparent change in the acceleration can either be estimated using Eq(2), in which case the parameter k will test to be zero, or with the simple "Galileo" quadratic:

$$s(t) = \frac{1}{2} a_0\, t^2 \tag{4}$$

As a further verification of the model Eq(1), a simulation was developed that can generate data corresponding to Eq(2) or Eq(3), and this synthetic data does have the same characteristics as the observed datasets.

CONCLUSIONS

The conclusions of this analysis are as follows:

> The acceleration is not constant for the apparatus used in this experiment.
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> Since the acceleration is not constant then the ratio tests used in the analysis will usually fail, and we will be unable to verify Galileo's assertion about the relation between the time and distances.
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> A linear model for the change in acceleration, while undoubtedly oversimplified, still leads to estimation models, Eq(2) and Eq(3), that capture the structure in the datasets far better than does the constant-acceleration model Eq(4).
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> The nonconstant accelerations are most probably caused by a slight bending of the metal ramps; this would cause a somewhat steeper angle at the top (raised) end, and a shallower angle at the lower end. Then the acceleration would be a bit larger at the top end (shorter distances s), and decreasing as the distance s increases. This situation would be reversed for the group that used the *lower* end of the ramp as the endpoint for all time measurements, and this idea is consistent with their data.
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> The effects noted here are subtle, and would not be noticed without careful data analysis.