[MAA 3.4] ARCS AND SECTORS

SOLUTIONS

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O. Practice questions

1. (a) (i) $\frac{\pi}{9}$ (ii) $\frac{\pi}{10}$ (iii) 3π

(b) (i) 10° (ii) 36° (iii) 450°.

2. (a)

	A	B	C	D
in degrees	30°	150°	210°	330°
in radians	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$

(b)

	A	B	C	D
in degrees	40°	140°	220°	320°
in radians	$\frac{2\pi}{9}$	$\frac{7\pi}{9}$	$\frac{11\pi}{9}$	$\frac{16\pi}{9}$

3.

	In degrees	
2nd period backwards	$-720° \le \theta < -360°$	-690°
1st period backwards	$-360° \le \theta < 0°$	-330°
1st period	$0° \le \theta < 360°$	30°
2nd period	$360° \le \theta < 720°$	390°
3rd period	$720° \le \theta < 1080°$	750°

in radians	
$-4\pi \le \theta < 2\pi$	$-\frac{23\pi}{6}$
$-2\pi \le \theta < 0$	$-\frac{11\pi}{6}$
$0 \le \theta < 2\pi$	$\frac{\pi}{6}$
$2\pi \le \theta < 4\pi$	$\frac{13\pi}{6}$
$4\pi \le \theta < 6\pi$	$\frac{25\pi}{6}$

4. **(a)**

Length of	the **minor** arc AB	$10 \times 1.5 = 15$
	the **major** arc AB	$10 \times (2\pi - 1.5) = 20\pi - 15$
Area of	the **minor** sector	$\frac{1}{2} \times 10^2 \times 1.5 = 75$
	the **major** sector.	$\frac{1}{2} \times 10^2 \times (2\pi - 1.5) = 100\pi - 75$
Perimeter of	the **minor** sector	$15 + 20 = 35$
	the **major** sector	$(20\pi - 15) + 20 = 20\pi + 5$

(b)

Length of	the **minor** arc AB	$10\times\frac{\pi}{2}=5\pi$
	the **major** arc AB	$10\times(2\pi-\frac{\pi}{2})=10\times\frac{3\pi}{2}=15\pi$
Area of	the **minor** sector	$\frac{1}{2}\times10^2\times\frac{\pi}{2}=25\pi$
	the **major** sector.	$\frac{1}{2}\times10^2\times(2\pi-\frac{\pi}{2})=50\times\frac{3\pi}{2}=75\pi$
Perimeter of	the **minor** sector	$5\pi+20$
	the **major** sector	$15\pi+20$

5. (a) $l = r\theta = (8)(1.3) = 10.4$

(b) $A_{OACB} = \frac{1}{2}r^2\theta = \frac{1}{2}(8)^2(1.3) = 41.6$

(c) $AB^2 = 8^2 + 8^2 - 2(8)(8)\cos 1.3 \Rightarrow AB = 9.6829\ldots \cong 9.68$

(d) $A_{OAB} = \frac{1}{2}r^2\sin\theta = \frac{1}{2}(8)^2\sin(1.3) = 30.8338\ldots \cong 30.8$

(e) $A_{shaded} = A_{OACB} - A_{OAB} = 41.6 - 30.8338\ldots \cong 10.8$

$A_{shaded} = \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2}(8)^2[1.3 - \sin(1.3)] = 10.766\ldots \cong 10.8$

6. (a) **METHOD A**

$A_{triangle} = A_{shaded} \Rightarrow \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta-\sin\theta) \Rightarrow \sin\theta = \theta - \sin\theta \Rightarrow 2\sin\theta = \theta$

METHOD B

$A_{triangle} = \frac{1}{2}A_{sector} \Rightarrow \frac{1}{2}r^2\sin\theta = \frac{1}{2}\frac{1}{2}r^2\theta \Rightarrow \sin\theta = \frac{1}{2}\theta \Rightarrow 2\sin\theta = \theta$

Then, by GDC, $\theta = 1.89549\ldots \cong 1.90$

(b) **METHOD A**

$\frac{l_{minor}}{l_{major}} = \frac{1}{5} \Rightarrow \frac{r\theta}{r(2\pi-\theta)} = \frac{1}{5} \Rightarrow \frac{\theta}{2\pi-\theta} = \frac{1}{5} \Rightarrow 5\theta = 2\pi - \theta \Rightarrow 6\theta = 2\pi \Rightarrow \theta = \frac{\pi}{3}$

METHOD B

$l_{minor} = \frac{1}{6}l_{circumference} \Rightarrow r\theta = \frac{1}{6}(2\pi r) \Rightarrow \theta = \frac{\pi}{3}$

(c) $A_{shaded} = \frac{1}{6}A_{circle} \Rightarrow \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{6}(\pi r^2) \Rightarrow \theta - \sin\theta = \frac{\pi}{3}$

Then, by GDC, $\theta = 1.9689\ldots \cong 1.97$

A. Exam style questions (SHORT)

7. (a) $l = r\theta$ or ACB = 2 × OA= 30 cm

(b) $A\hat{O}B$ (obtuse) $= 2\pi - 2$

Area $= \frac{1}{2}r^2\theta = \frac{1}{2}(15)^2(2\pi - 2) = 482$ cm^2 (3 sf)

8. Perimeter = 5(2π – 1) + 10 = (10π + 5) cm (= 36.4, to 3 sf)

9. (a) arc length $l = r\theta = 10 \times \frac{\pi}{3} = \frac{10\pi}{3}$

(b) area of large sector = $\frac{1}{2} \times 10^2 \times \frac{\pi}{3}\left(= \frac{100\pi}{6}\right)$

area of small sector = $\frac{1}{2} \times 8^2 \times \frac{\pi}{3}\left(= \frac{64\pi}{6}\right)$

area shaded = $\frac{36\pi}{6} = 6\pi$

10. (a) **METHOD 1**

cosine rule AB = $\sqrt{3.9^2 + 3.9^2 - 2(3.9)(3.9)\cos 1.8}$ = AB = 6.11(cm)

METHOD 2

using right-angled triangles

If $x = \frac{1}{2}$AB

$\sin 0.9 = \frac{x}{3.9} \Rightarrow x = 3.9 \sin 0.9$

AB = 2x = 6.11 (cm)

(b) For major sector: $A\hat{O}B = 2\pi - 1.8$ (= 4.4832)

$A = \frac{1}{2}(3.9)^2(4.4832...) = 34.1$ (cm^2)

11. $\frac{1}{2} \times (5.4)^2\theta = 21.6 \Rightarrow \theta = \frac{4}{2.7}$ (= 1.481 radians)

$AB = r\theta = 5.4 \times \frac{4}{2.7} = 8$ cm

12. (a) $A = \frac{1}{2}r^2\theta \Leftrightarrow 27 = \frac{1}{2}(1.5)r^2 \Leftrightarrow r^2 = 36 \Rightarrow r = 6$ cm

(b) Arc length $= r\theta = 1.5 \times 6 = 9$ cm

13. (a) $3\pi = r\frac{2\pi}{9} \Leftrightarrow r = 13.5$ (cm)

(b) perimeter = 27+3π (cm)(= 36.4)

(c) area = $\frac{1}{2} \times 13.5^2 \times \frac{2\pi}{9} = 20.25\pi$ (cm^2) (= 63.6)

14. $A = \frac{1}{2}r^2\theta \Leftrightarrow \frac{1}{2}r^2\theta = \frac{4}{3}\pi$

$l = r\theta \Leftrightarrow r\theta = \frac{2}{3}\pi$

Solving the system: $r = 4$, $\theta = \frac{\pi}{6}$

15 $A = \frac{1}{2}r^2\theta \Leftrightarrow \frac{1}{2}r^2\theta = 180$

$l = r\theta \Leftrightarrow r\theta = 24$

Solving the system: $r = 15$, $\theta = 1.6$

16. (a) Area = $\frac{1}{2}r^2\theta = \frac{1}{2}(15^2)(2) = 225$ (cm^2)

(b) Area ΔOAB = $\frac{1}{2}15^2 \sin 2 = 102.3$

Area = 225 – 102.3 = 122.7 (cm^2) = 123 (3 sf)

17. (a) perimeter = $r + r$ + arc length ⇔ 20 = 2r + $r\theta$ ⇔ $\theta = \frac{20-2r}{r}$

(b) $A = \frac{1}{2}r^2\left(\frac{20-2r}{r}\right)$ ⇔ $10r - r^2 = 25$ ⇔ $r = 5$ cm

18. Area sector OAB $= \frac{1}{2}\left(\frac{3\pi}{4}\right)(5)^2 = \frac{75}{8}\pi$

Area of ΔOAB = $\frac{1}{2}(5)(5)\sin\frac{3\pi}{4} = \frac{25\sqrt{2}}{4}$

⇒ Shaded area = area of sector OAB – area of ΔOAB

= 20.6 (cm)

19. $r\theta = 2\tau$

$\theta = 2$

Area $= \frac{\theta}{2}r^2 - \frac{1}{2}r^2\sin\theta$

$\theta = 2 \Rightarrow \text{area} = r^2 - \frac{1}{2}r^2(\sin 2)\ \left(= r^2(1 - \frac{1}{2}\sin 2)\right)$

$k = 1 - \frac{1}{2}\sin 2 \quad (= 0.545)$

20. (a) Area of sector = 3 (Area of segment)

Area of sector $= \frac{\theta}{2}$

Area of triangle $= \frac{\sin\theta}{2}$

$\frac{\sin\theta}{2} = 3\left(\frac{\theta}{2} - \frac{\sin\theta}{2}\right)$

$3\theta = 4\sin\theta$

(b) θ = 1.28 radians (accept 73.1°)

21. **METHOD A**

$$\frac{A_{shaded}}{A_{triangle}} = \frac{2}{5} \Leftrightarrow \frac{\frac{1}{2}r^2(\theta - \sin\theta)}{\frac{1}{2}r^2\sin\theta} = \frac{2}{5} \Leftrightarrow \frac{\theta - \sin\theta}{\sin\theta} = \frac{2}{5} \Leftrightarrow 5\theta - 5\sin\theta = 2\sin\theta$$

METHOD B

$A_{shaded} : A_{triangle}$ = 2:5 (so the whole part, i.e. sector, is 7). Therefore,

$A_{triangle} : A_{sector}$ = 5:7

$$\frac{A_{triangle}}{A_{sector}} = \frac{5}{7} \Leftrightarrow \frac{\frac{1}{2}r^2\sin\theta}{\frac{1}{2}r^2\theta} = \frac{5}{7} \Leftrightarrow \frac{\sin\theta}{\theta} = \frac{5}{7} \Leftrightarrow 5\theta = 7\sin\theta$$

22. $T\hat{O}A = 60°$

Area of $\Delta = \frac{1}{2} \times 6 \times 12 \times \sin 60 = 18\sqrt{3}$ Area of sector $= \frac{1}{2} \times 6 \times 6 \times \frac{\pi}{3} = 6\pi$

Shaded area = $18\sqrt{3} - 6\pi$ [= 12.3 cm^2 (3 sf)]

OR

$O\hat{T}A = 90°$

AT = $\sqrt{12^2 - 6^2} = 6\sqrt{3}$ $T\hat{O}A = 60° = \frac{\pi}{3}$

Area = area of triangle – area of sector = $\frac{1}{2} \times 6 \times 6\sqrt{3} - \frac{1}{2} \times 6 \times 6 \times \frac{\pi}{3} = 18\sqrt{3} - 6\pi$ [= 12.3 cm^2]

23. Area sector OAB $= \frac{1}{2}(5)^2(0.8) = 10$

$\cos 0.8 = ON/5 \Rightarrow ON = 5\cos 0.8 \ (= 3.483...)$

Area of $\Delta AON = \frac{1}{2} ON \times 5 \times \sin 0.8 = 6.249...$

Shaded area $= 10 - 6.249.. = 3.75$

24. $h = r$ so $2r^2 = 100 \Rightarrow r^2 = 50 \Rightarrow r = 5\sqrt{2}$

Hence circumference = $2\pi r = 10\pi\sqrt{2}$

$l = 10\theta = 10\pi\sqrt{2}$

$\Rightarrow \theta = \pi\sqrt{2}$

25. (a) area of sector ABDC = $\frac{1}{4}\pi(2)^2 = \pi$

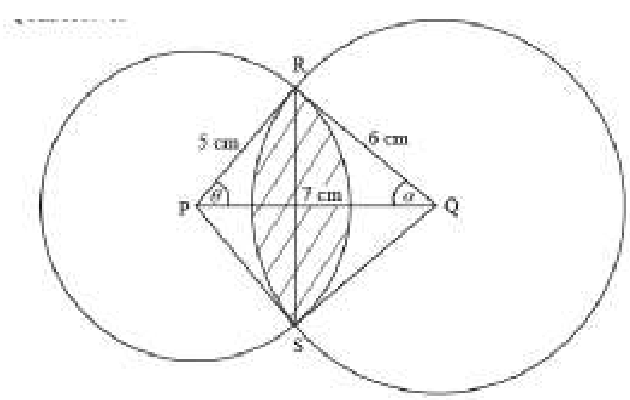
area of segment BDCP = π – area of $\Delta ABC = \pi - 2$

(b) BP = $\sqrt{2}$

area of semicircle of radius BP = $\frac{1}{2}\pi(\sqrt{2})^2 = \pi$

area of shaded region = $\pi - (\pi - 2) = 2$

26.



$\cos\theta = \frac{5^2 + 7^2 - 6^2}{2 \times 5 \times 7} = \frac{25 + 49 - 36}{70} = \frac{38}{70} \Rightarrow \theta = 0.997$

$\Rightarrow 2\theta = 1.99...$

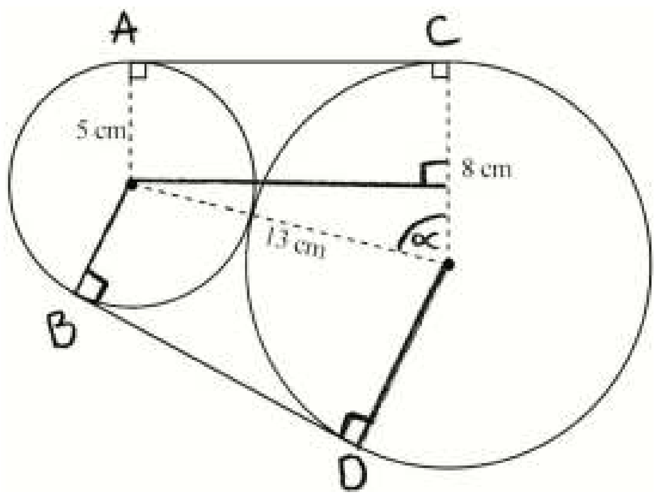
$\cos\alpha = \frac{7^2 + 6^2 - 5^2}{2 \times 7 \times 6} = \frac{49 + 36 - 25}{84} = \frac{60}{84} = 0.775$

$\Rightarrow 2\alpha = 1.55...$

Required area $= \frac{1}{2}5^2(1.99 - \sin 1.99) + \frac{1}{2}6^2(1.55 - \sin 1.55)$

$= 23.4 \text{ cm}^2$

27.



Let AC = BD = x

$x^2+3^2=13^2 \Rightarrow x=\sqrt{160}=4\sqrt{10}$ (=12.64…)

$\cos\alpha=\dfrac{3}{13}\Rightarrow a=1.337...$ rad

Arc length CD = $8(2\pi-2a)=28.85...$

Arc length AB = $5(2a)=13.37...$

Total length = 13.37… + 28.85… + 2×(12.64…) ≅ 67.5

28. (a) Let $A_1=\dfrac{1}{2}r^2\theta_1$, $A_2=\dfrac{1}{2}r^2\theta_2$, $A_3=\dfrac{1}{2}r^2\theta_3$, ...

Given that $A_2-A_1=A_3-A_2=...$ is constant

$\dfrac{1}{2}r^2(\theta_2-\theta_1)=\dfrac{1}{2}r^2(\theta_3-\theta_2)=...$ is constant

So $\theta_2-\theta_1=\theta_3-\theta_2=...$ is constant

(b) **METHOD 1 (considering the AS of the angles)**

Let $u_1=\theta$, $u_{12}=2\theta$ be the first and the last angles

$S_{12}=2\pi \Rightarrow \dfrac{12}{2}(\theta+2\theta)=2\pi \Rightarrow 18\theta=2\pi$

$\theta=\dfrac{\pi}{9}$

METHOD 2 (considering the AS of the sector areas)

Let $u_1=A$, $u_{12}=2A$ be the first and the last sector areas

$S_{12}=\pi r^2 \Rightarrow \dfrac{12}{2}(A+2A)=\pi r^2 \Rightarrow 18A=\pi r^2 \Rightarrow A=\dfrac{\pi r^2}{18}$

Hence, $\dfrac{1}{2}r^2\theta=\dfrac{\pi r^2}{18}\Rightarrow \theta=\dfrac{\pi}{9}$

29. (a) Arc AB = θ

$OB_1 = \cos\theta$

Arc $A_1B_1 = \theta\cos\theta$

Similarly,

Arc $A_2B_2 = \theta\cos^2\theta$

Sum = $\theta + \theta\cos\theta + \theta\cos^2\theta + \ldots = \dfrac{\theta}{1-\cos\theta}$

(b) $\dfrac{\theta}{1-\cos\theta} = \dfrac{2\pi}{3} \Rightarrow \theta = \dfrac{\pi}{3} \cong 1.05$

B. Exam style questions (LONG)

30. (a) $\dfrac{AD}{\sin 0.8} = \dfrac{4}{\sin 0.3} \Rightarrow AD = 9.71$ (cm)

(b) OAD = $\pi - 1.1 = (2.04)$

EITHER $OD^2 = 9.71^2 + 4^2 - 2 \times 9.71 \times 4 \times \cos(\pi - 1.1) \Rightarrow OD = 12.1$

OR. $\dfrac{OD}{\sin(\pi - 1.1)} = \dfrac{9.71}{\sin 0.8} = \dfrac{4}{\sin 0.3} \Rightarrow OD = 12.1$

(c) area = $0.5 \times 4^2 \times 0.8 = 6.4$

(d) area of triangle OAD: . $A = \dfrac{1}{2} \times 4 \times 12.1 \times \sin 0.8 = 17.3067$

(**OR** $A = \dfrac{1}{2} \times 4 \times 9.71 \times \sin 2.04 = 17.3067$

OR $A = \dfrac{1}{2} \times 12.1 \times 9.71 \times \sin 0.3 = 17.3067$

area ABCD = 17.3067 – 6.4 = 10.9 (cm^2)

31. (a) $AB^2 = 12^2 + 12^2 - 2 \times 12 \times 12 \times \cos 75° = 12^2(2 - 2\cos 75°) = 12^2 \times 2(1 - \cos 75°)$

$\Rightarrow AB = 12\sqrt{2(1-\cos 75°)}$

(b) $P\hat{O}B = 37.5°$, $\tan 37.5 = \dfrac{BP}{12} \Rightarrow BP = 12\tan 37.5° = 9.21$ cm

OR

$B\hat{P}A = 105°$ $\quad B\hat{A}P = 37.5°$

$\dfrac{AB}{\sin 105°} = \dfrac{BP}{\sin 37.5°} \Rightarrow BP = \dfrac{AB\sin 37.5°}{\sin 105°} = 9.21$(cm)

(c) (i) Area $\Delta OBP = \dfrac{1}{2} \times 12 \times 9.21 = 55.3$ (cm^2)

(ii) Area $\Delta ABP = \dfrac{1}{2}(9.21)^2 \sin 105° = 41.0$ (cm^2)

(d) Area of sector = $\dfrac{1}{2} \times 12^2 \times 75 \times \dfrac{\pi}{180} = 94.2$ (cm^2) (accept 30π)

(e) Shaded area = 2 × area ΔOPB – area sector = 16.4 (cm^2)

32. **(a)** Area of sector OAB $\frac{1}{2}r^2\theta$

Area of triangle OAB $=\frac{1}{2}r^2\sin\theta$

Shaded area = Area of sector CAB – Area of triangle OAB

$=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta$

$=\frac{1}{2}r^2(\theta-\sin\theta)$

(b) Area of the major segment = area of circle – shaded area

$=\pi r^2-\frac{1}{2}r^2(\theta-\sin\theta)\quad\left(=r^2\left(\pi-\frac{\theta}{2}+\frac{\sin\theta}{2}\right)\right)$

(c) Given ratio of segments is $3:2$

METHOD 1

$\frac{3}{2}r^2(\theta-\sin\theta)=2r^2\left(\pi-\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$

$\Rightarrow 3\theta-3\sin\theta=4\pi-2\theta+2\sin\theta$

$\Rightarrow 5\theta-5\sin\theta=4\pi$

$\Rightarrow 5\sin\theta=5\theta-4\pi$

$\Rightarrow \sin\theta=\theta-\frac{4\pi}{5}$

METHOD 2

area of shaded region $=\frac{2}{5}\pi r^2$

$\Rightarrow \frac{1}{2}r^2(\theta-\sin\theta)=\frac{2}{5}\pi r^2$

$\Rightarrow 5(\theta-\sin\theta)=4\pi$

$\Rightarrow 5\theta-5\sin\theta=4\pi$

$\Rightarrow \sin\theta=\theta-\frac{4}{5}\pi$

(d) $\theta=2.82$ radians

33. (a) (i) OP = PQ (= 3cm) So Δ OPQ is isosceles

(ii) $\cos\ O\hat{P}Q=\frac{3^2+3^2-4^2}{2\times3\times3}=\frac{9+9-16}{18}\left(=\frac{2}{18}\right)=\frac{1}{9}$

(iii) $\sin^2 A+\cos^2 A=1\Rightarrow \sin\ O\hat{P}Q=\sqrt{1-\frac{1}{81}}\left(=\sqrt{\frac{80}{81}}\right)=\frac{\sqrt{80}}{9}$

(iv) Area triangle OPQ $=\frac{1}{2}\times OP\times PQ\ \sin P=\frac{\sqrt{80}}{2}\left(=\sqrt{20}\right)(=4.47)$

(b) (i) $O\hat{P}Q=1.4594...=1.46$,

(ii) Area sector OPQ $=\frac{1}{2}\times3^2\times1.4594\ldots=6.57$

(c) $Q\hat{O}P=\frac{\pi-1.4594\ldots}{2}(=0.841)$, Area sector QOS $=\frac{1}{2}\times4^2\times0.841=6.73$

(d) Area of small semi-circle is 4.5π (= 14.137...)

Area = area of semi-circle – area sector OPQ – area sector QOS + area triangle POQ

= 4.5π – 6.5675... – 6.7285... + 4.472... = 5.31

34. (a) cosine rule; $4^2 = 3^2 + 2^2 - 2\times3\times2\cos A\hat{O}P \Rightarrow A\hat{O}P = 1.82$ (radians)

(b) $A\hat{O}B = 2\pi - 2(1.82) = 2\pi - 3.64 = 2.64$ (radians)

(c) First find $A\hat{P}O$

Either using cosine rule $3^2 = 2^2 + 4^2 - 2\times2\times4\cos A\hat{P}O \Rightarrow A\hat{P}O = 0.8127\ldots$

Or using sine rule $\dfrac{4}{\sin 1.82} = \dfrac{3}{\sin A\hat{P}O} \Rightarrow A\hat{P}O \cong 0.813$

$A\hat{P}B = 2 \times 0.813 \cong 1.63$

(d) (i) Area of sector PAEB = $\dfrac{1}{2}\times4^2\times1.63 = 13.04$ (cm^2)

(ii) Area of sector OADB = $\dfrac{1}{2}\times3^2\times2.64 = 11.9$ (cm^2)

(e) Area of triangle OAP = $\dfrac{1}{2}\times3\times2\sin 1.82 = 2.907...$

AOBP = 2(2.907…) ≅ 5.81 cm^2.

(f) (i) Area AOBE = Area PAEB – Area AOBP (= 13.0 – 5.81) = 7.23

(ii) Area shaded = Area OADB – Area AOBE = 11.9 – 7.23 = 4.67

(accept answers between 4.63 and 4.72)

35. (a) $6 = 8\theta \Leftrightarrow A\hat{O}C = 0.75$

(b) area of triangle = $\dfrac{1}{2} \times 8 \times 8 \times \sin(0.75) = 21.8\ldots$

area of sector = $\dfrac{1}{2} \times 64 \times 0.75 = 24$

area of shaded region = area of sector – area of triangle = 2.19 cm^2

(or directly area of segment = = $\dfrac{1}{2} \times 8^2$ (0.75 – sin0.75= 2.19 cm^2).

(c) $45 = \dfrac{1}{2} \times 8^2 \times \theta \Rightarrow C\hat{O}E = 1.40625$ (1.41 to 3 sf)

(d) $E\hat{O}F = \pi - 0.75 - 1.41 = 0.985$

EF = $\sqrt{8^2 + 8^2 - 2\times8\times8\times\cos 0.985} = 7.57$ cm (other methods are also possible)