PROJECTILE MOTION *Range*

The range is the maximum x-displacement of the projectile. This occurs at the time of flight, so that, starting with the kinematic equation for the x-motion,

$$x(t) = x_0 + v_0\, t \cos(\theta) + \frac{1}{2} a_x t^2$$

and with the initial x defined to be zero (we can always shift the coordinate system to make this so), and having zero acceleration in the x direction, we have

$$x(T) = R = v_0\, T \cos(\theta)$$

The TOF has been developed elsewhere, and is

$$T = \frac{1}{g}\left[v_0 \sin(\theta) + \sqrt{\left(v_0 \sin(\theta)\right)^2 + 2\, g\, y_0}\right]$$

Thus we can write a general expression for the range:

$$R = \frac{v_0}{g} \cos(\theta)\left[v_0 \sin(\theta) + \sqrt{\left(v_0 \sin(\theta)\right)^2 + 2\, g\, y_0}\right] \qquad (1)$$

In the special case of a zero initial height we have the usual textbook result

$$R = \frac{2\, {v_0}^2 \sin(\theta) \cos(\theta)}{g} = \frac{{v_0}^2}{g} \sin(2\,\theta) \qquad y_0 = 0$$

This is the range for a given angle; to get a given range X, the angle needed is found by inverting, so that

$$\theta_X = \frac{1}{2} \operatorname{asin}\left(\frac{X\, g}{{v_0}^2}\right)$$

If the initial angle is zero, then

$$R = v_0 \sqrt{\frac{2\, y_0}{g}} \qquad \theta = 0$$

which is just the initial velocity times the TOF for this case.

We can plot the range as a function of its parameters but first we develop expressions for the maximum range, so that these can be indicated on the graphs. We seek the angle that will maximize the range, and then the range that results from a trajectory at that angle. To do this, we find the derivative of R as a function of the angle, set it to zero, and solve for the angle. This tedious work results in

$$\theta_{maxR} = \operatorname{acos}\left[\frac{\sqrt{2}}{2}\left(\sqrt{\frac{1 + \frac{2\, g\, y_0}{{v_0}^2}}{1 + \frac{g\, y_0}{{v_0}^2}}}\right)\right] \qquad \theta_{maxR} = \frac{\pi}{2} - \operatorname{atan}\left(\sqrt{1 + \frac{2\, g\, y_0}{{v_0}^2}}\right) \qquad (2)$$

Different versions of a symbolic math processor developed these two solutions; they can be shown (numerically) to be equal. We will use the arctan form, since it is a bit simpler.

Observe that for a zero initial height, Eq(2) is 45 degrees. We can also see this using the range expression for this case

$$R = \frac{v_0^2}{g} \sin(2\,\theta) \qquad\qquad y_0 = 0 \tag{3}$$

and recognizing that this will be a maximum when 2θ is 90 degrees, so θ must be 45 degrees. This can of course also be done by the usual calculus procedure, with the same result.

Next we seek the range that will result if we use this optimum angle. Substituting Eq(2) into Eq(1) and doing a lot of algebra (or letting a symbolic math program do it), we get a remarkably simple result

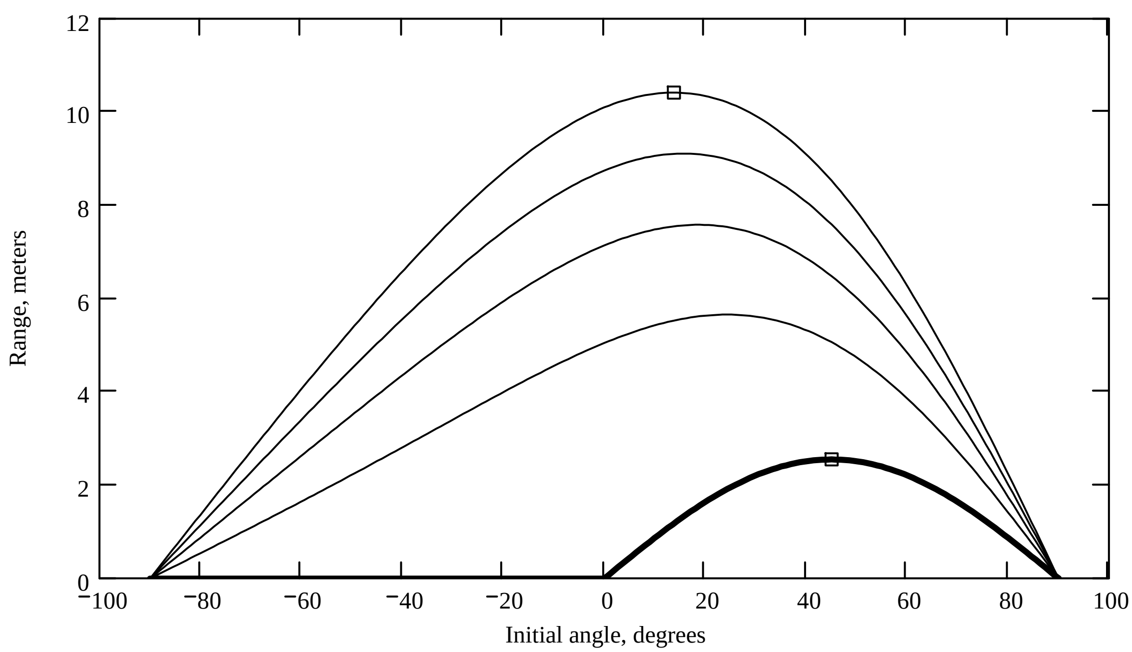
$$R_{max} = \frac{v_0}{g}\sqrt{v_0^2 + 2\,g\,y_0} = \frac{v_0\,v_f}{g} \tag{4}$$

This says that the maximum range is the product of the initial and final velocities, divided by g. When the initial height is zero,

$$R_{max} = \frac{v_0^2}{g}$$

which we can immediately see from Eq(3); recall that for this case the initial and final velocities are equal.

Finally, we should plot the range as a function of its parameters. More detailed plots are in the graphics portion of this packet; for now here is one version.



The initial velocity is 5 m/s; the initial heights are 0 (thick line), 5, 10, 15, 20 m. The calculated maximum points from Eqs(2) and (4) are indicated by the squares for the first and last case. Note that only the zero initial height case is symmetric- this relates to the "Galileo angles" discussed on another sheet.