15 Trigonometric Graphs

A very important feature of trigonometric functions, especially sine and cosine functions, is that they repeat themselves after a certain interval, i.e. they are periodic. This makes them suitable for modelling many situations in real life where quantities repeat over regular intervals. This feature of trigonometric functions is best seen through an analysis of trigonometric graphs.

One obvious example is the length of daylight throughout a year. Other examples abound in physics, engineering and even in economics. For example, in physics, alternating electric current is an example of a quantity that varies according to a sine or cosine function. The graphs of these functions are often referred to as sinusoidal waves.

One key example was the study of the transfer of heat across a metal plate in the early 1800s. A French mathematician, Jean-Baptiste Fourier, analysed this using trigonometric functions and came up with what are now known as Fourier series and Fourier transforms, which are of huge importance in electronics and other areas.

15.1 Graphs of Sine, Cosine and Tan

1. The Sine Graph

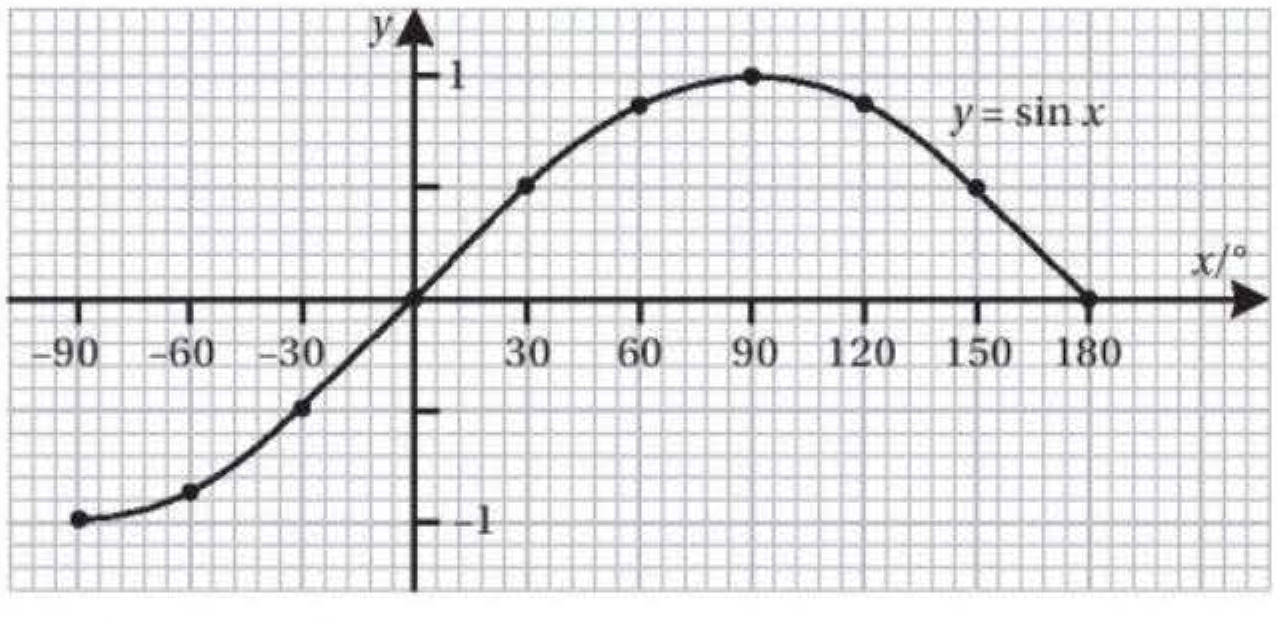
Consider the function $f(x) = \sin x$, which is defined for all $x \in \mathbb{R}$. This is valid whether x is measured in degrees or in radians.

We can construct a graph of the curve $y = \sin x$, representing this function by obtaining points on the curve from a table of values, plotting these points and joining them by a smooth curve. The curve $y = f(x)$ is also commonly referred to as the graph $y = f(x)$.

For example, suppose we are asked to construct the graph of $y = \sin x$ for the domain $-90° \le x \le 180°$. Then we can construct the following table.

$x/°$	−90	−60	−30	0	30	60	90	120	150	180
$y = \sin x$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

Bearing in mind that $\frac{\sqrt{3}}{2}$ is approximately 0·87, we can construct the following graph.



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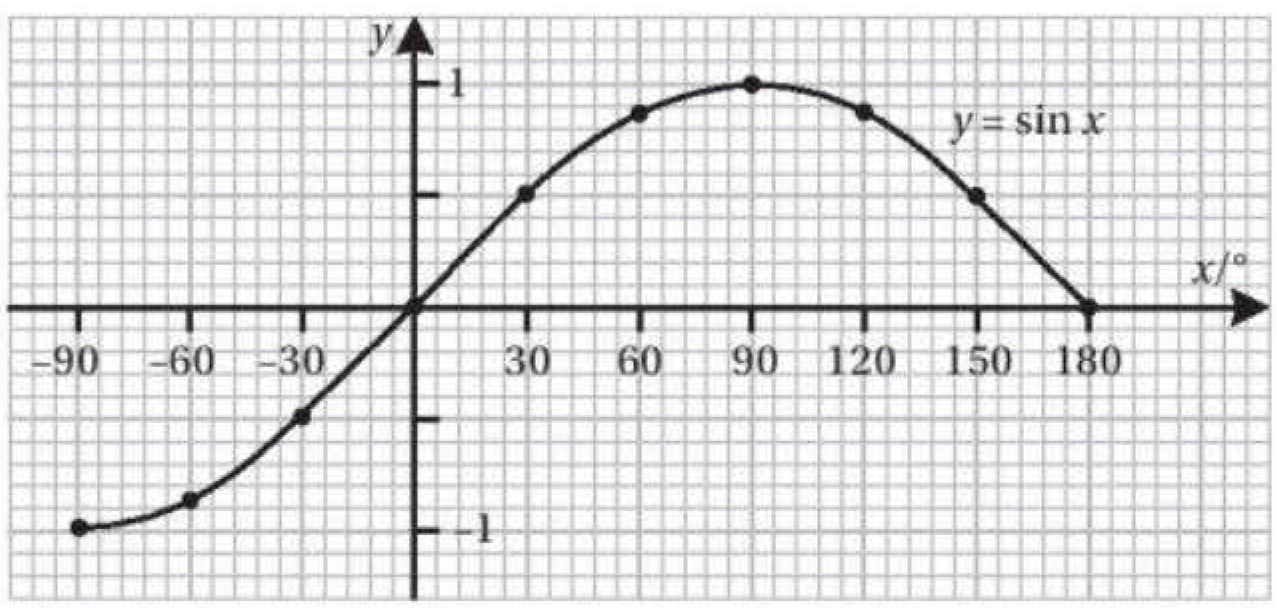
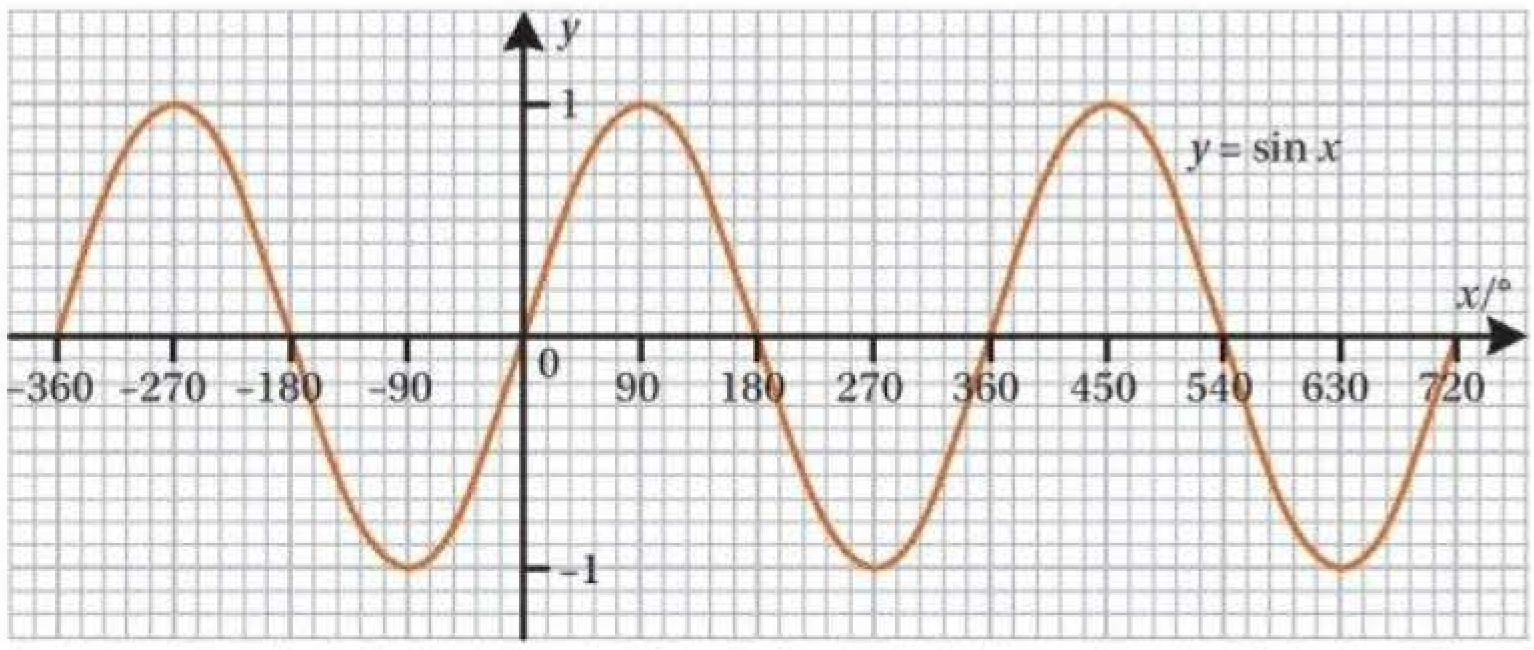
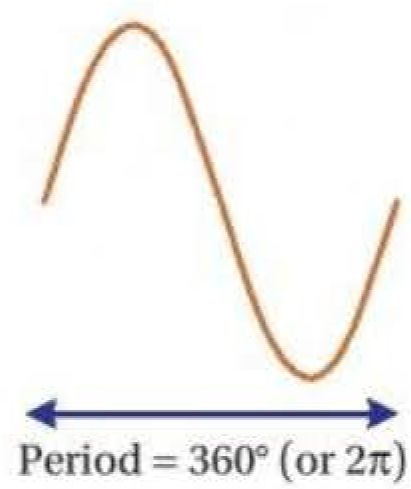


If we extend the domain, say to $-360° \le x \le 720°$, we can see that the graph of $y = \sin x$ repeats itself periodically, both left and right.

x/°	–360	–270	–180	–90	0	90	180	270	360	450	540	630	720
y = sin x	0	1	0	–1	0	1	0	–1	0	1	0	–1	0

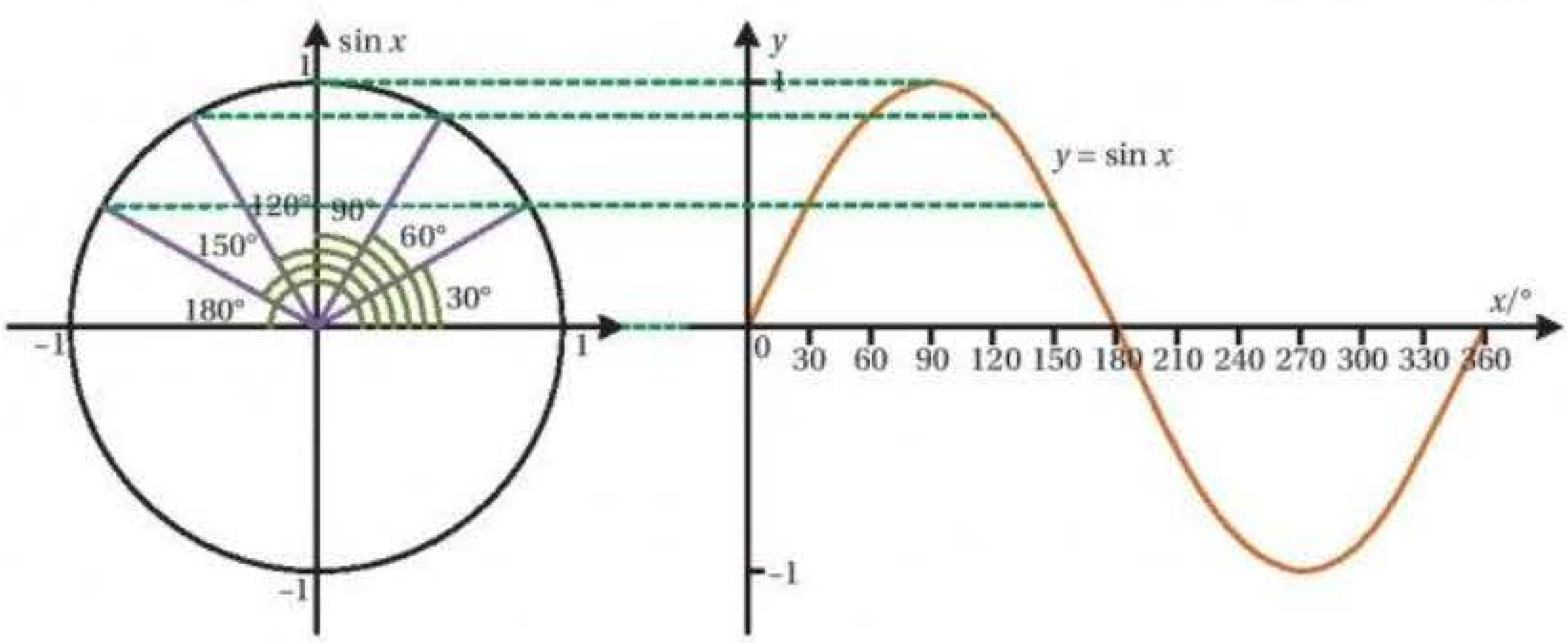


A function such as $f(x) = \sin x$ which behaves in this way is called a **periodic function** and its graph $y = \sin x$ is a **periodic graph**. We may consider a periodic graph to be composed of a basic building block, repeated to infinity in both directions. The basic building block is not unique, but always has the same width, called the **period**. One possible view of the basic building block of the sine graph is shown opposite. The period of the sine graph is 360°, or 2π if working in radians.



One other feature we consider is the **range**. This is the interval on the y-axis from the lowest y value on the graph to the highest. From the graph above we can see that the lowest y value is –1 and the highest y value is 1. Thus, the range of the sine graph is [–1,1].

If we are asked to construct the sine graph by **using only a ruler, a compass and a protractor,** i.e. without using a calculator or tables, we can draw a unit circle and the graph side by side as shown below.

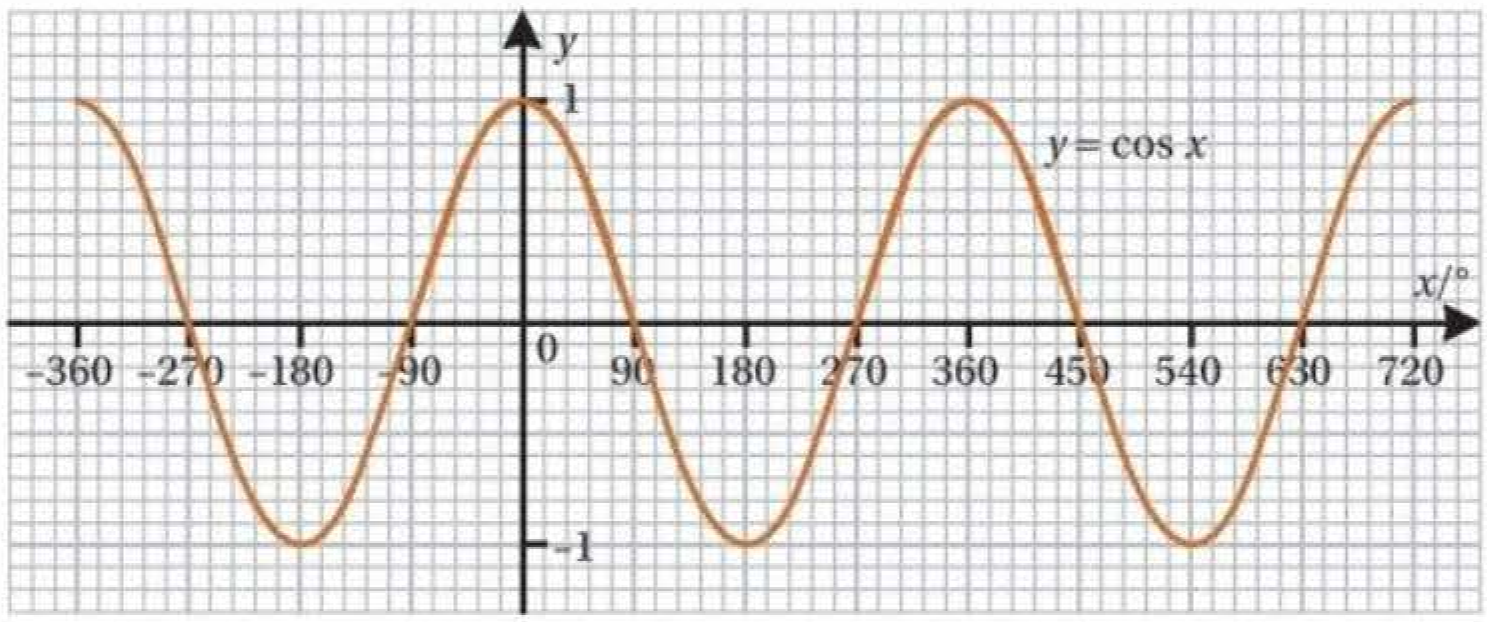


Using a protractor, we can indicate the point on the unit circle which corresponds to an angle of 30°. Then, drawing a horizontal line from this point, we can mark a point at the same level above '30' on the graph. By repeating this process for many different angles, we obtain a number of points on the graph. Joining up these points, we obtain the standard sine graph.

2. The Cosine Graph

The graph of the function $f(x) = \cos x$ is $y = \cos x$, and it may be constructed in the same way as the sine graph. If required, we can make a table of values of x in the given domain, and calculate the corresponding values of y.

The shape of the cosine graph is identical to that of the sine graph. The only difference is in the location. The cosine graph cuts the y-axis at (0,1), unlike the sine graph, which cuts the y-axis at (0,0). The diagram below shows the cosine graph in the domain $-360° \leq x \leq 720°$.



The period of the cosine graph is 360°, or 2π if working in radians. The basic building block of which the graph is constructed may be considered to be the U shape shown opposite. However, like the sine graph, this is not unique.

The range of the cosine graph is [–1,1], as can be seen from the diagram above.

To construct the cosine graph **using only a ruler, a compass and a protractor**, i.e. without a calculator or tables, we can repeat what we did for the sine graph, except that for the unit circle, we show the x-axis as vertical and the y-axis as horizontal. The details are left as an exercise.

We can summarise some of the key features of the graphs $y = \sin x$ and $y = \cos x$.

(i) **Midway Line**
The midway line is the x-axis. This is the horizontal line that lies half way between the highest points and the lowest points on the graphs.

(ii) **Maximum and Minimum**
The maximum (y value) is the greatest y value that the graph reaches. For these curves, this is 1.
The minimum (y value) is the least y value that the graph reaches. For these curves, this is –1.

(iii) **Range**
The range of both graphs is [–1,1], the interval from the minimum to the maximum.

(iv) **Amplitude**
The amplitude is the distance from the midway line to the maximum (or the minimum). In both cases, this is 1.

(v) **Period**
The period of both graphs is 360° or 2π radians, as discussed above.

From the graphs $y = \sin x$ and $y = \cos x$, we can see that the functions $f(x) = \sin x$ and $f(x) = \cos x$, defined from $\mathbb{R}$ to $\mathbb{R}$, are neither injective nor surjective and, hence, are not bijective. Later, we will see that by restricting the domain and the codomain, we can construct bijective functions for these.

3. The Tan Graph

The graph, $y = \tan x$, of the function $f(x) = \tan x$ is very different from the sine and cosine graphs. Some of the key features of the tan graph are as follows.

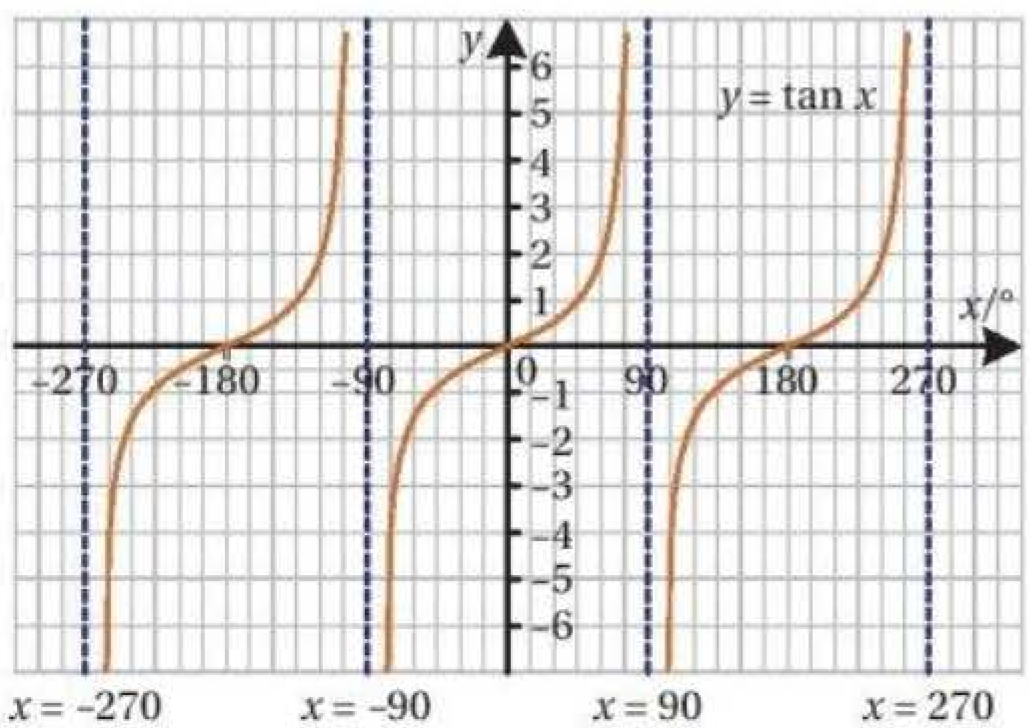
(i) As $\tan x = \frac{\sin x}{\cos x}$, $\tan x$ is not defined when $\cos x = 0$. $\cos x = 0$ when $x = \pm 90°, \pm 270°, \ldots$ This means that there is a break in the graph at each of these values.

(ii) $\tan 0° = 0$, $\tan 30° = \frac{1}{\sqrt{3}} \approx 0{\cdot}58$, $\tan 45° = 1$, $\tan 60° = \sqrt{3} \approx 1{\cdot}73$. As x rises towards $90°$, the value of $\tan x$ rises very steeply. For example, $\tan 85° = 11{\cdot}4$.

(iii) As x goes from $0°$ to $-90°$, the same thing happens in reverse. The y values become more and more negative, and the graph goes steeply downwards.

(iv) As x rises towards $90°$, the tan graph gets closer and closer to the vertical straight line $x = 90°$. This line is called an **asymptote** of the graph. Other asymptotes are $x = 270°$, $x = -90°$ and $x = -270°$, etc.

The graph $y = \tan x$ in the domain $-270° < x < 270°$, $x \neq \pm 90°$ is shown below.



The tan graph is periodic. The basic building block for the graph is shown opposite. The width of this basic building block is 180°, or π if working in radians. Thus, the period of the tan graph is 180° or π.

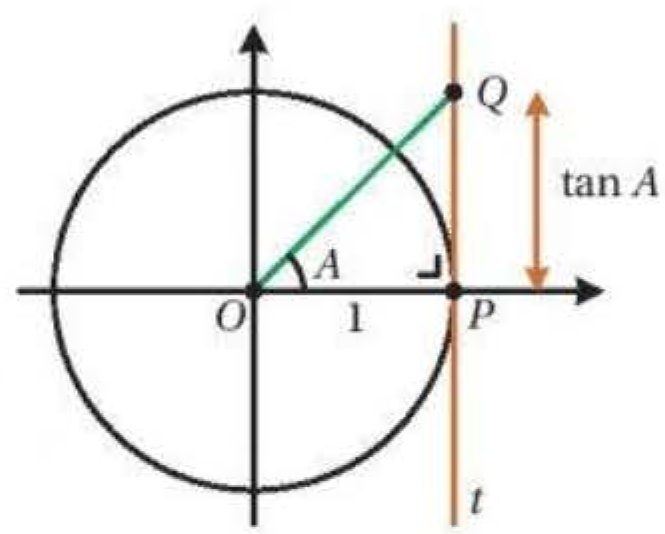
Period = 180° (or π)

The range of the tan graph is the whole set of real numbers, as the graph goes both down and up forever. The range of the tan graph is $\mathbb{R}$.

Thus, $f(x) = \tan x$ is a surjective function, although it is not injective. Later, we will restrict the domain to make a bijective function.

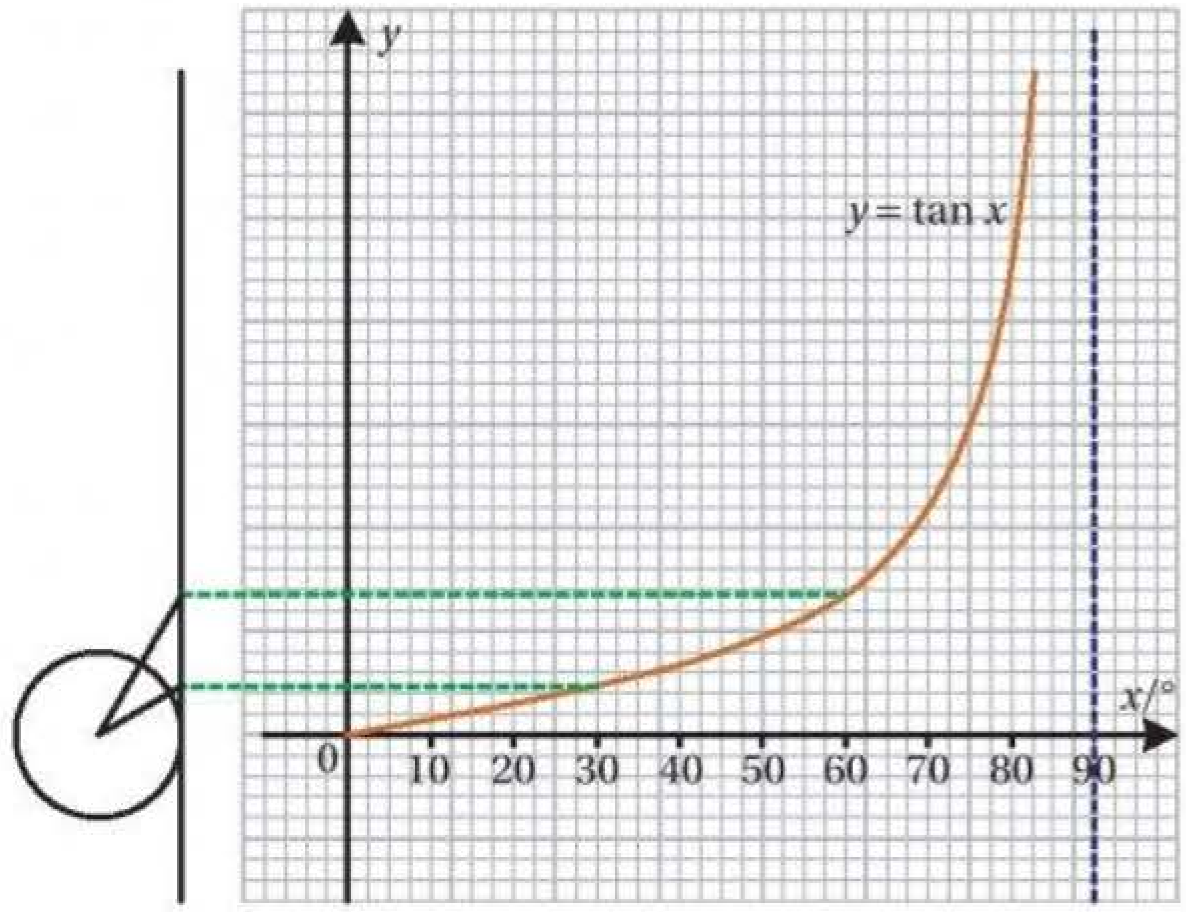
In order to construct the tan graph **using only a ruler, a compass and a protractor**, we need to determine how tan can be represented by a height.

Consider the tangent t to the unit circle at the point $P(1,0)$. This tangent is vertical. If we now take an angle A at the centre and produce the radius to meet t at the point Q, then OPQ is a right angled triangle, where O is the origin.



Then $\tan A = \frac{|PQ|}{1} = |PQ|$.

Thus, the tan of each angle A is given by the position of the point Q on the tangent t. The construction of the tan graph using only a ruler, a compass and a protractor is shown below.



Here is a summary of the main features of all three of the basic trig functions and their graphs.

Graph	Domain	Range	Period
$y = \sin x$	$\mathbb{R}$	$[-1,1]$	360° (2π)
$y = \cos x$	$\mathbb{R}$	$[-1,1]$	360° (2π)
$y = \tan x$	$\mathbb{R}\setminus\{\pm 90°, \pm 270°, ...\}$	$\mathbb{R}$	180° (π)

Exercises 15.1

1. The function $f(x) = \sin x$ is defined for all $x \in \mathbb{R}$. We want to draw the graph $y = f(x)$ in the domain $-360° \leq x \leq 360°$.
   - **(i)** Make a table of values, taking all multiples of 30° between −360° and 360°.
   - **(ii)** Plot the points you obtain and construct the graph $y = f(x)$.
   - **(iii)** State the period and the range of the function.
   - **(iv)** By drawing the horizontal line $y = \frac{1}{2}$, find the values of x in the domain for which $\sin x = \frac{1}{2}$.

2. **(i)** Make a table of values for $y = \sin x$, for $0 \leq x \leq 4\pi$, taking all multiples of $\frac{\pi}{4}$ in this domain.
   - **(ii)** Construct this graph in the given domain.
   - **(iii)** State the maximum and minimum values of y and give the amplitude.
   - **(iv)** By drawing the horizontal line $y = -\frac{1}{\sqrt{2}}$, find the values of x in the domain for which $\sin x = -\frac{1}{\sqrt{2}}$.

3. We want to plot the graph $y = \cos x$ in the domain $0° \leq x \leq 540°$.
   - **(i)** Make a table of values, taking all multiples of 30° in the domain.
   - **(ii)** Plot the points you obtain and construct the graph.
   - **(iii)** State the period and the range.
   - **(iv)** By drawing the horizontal line $y = -\frac{1}{2}$, find the values of x in the domain for which $\cos x = -\frac{1}{2}$.

4. **(i)** Make a table of values for $y = \cos x$, for $-2\pi \leq x \leq 2\pi$, taking all multiples of $\frac{\pi}{4}$ in this domain.
   - **(ii)** Construct the graph $y = \cos x$ in the given domain.
   - **(iii)** State the period and the range.
   - **(iv)** State the values of x in the domain for which the graph is increasing, i.e. going up as the graph goes from left to right.

15.2 Other Sine and Cosine Graphs

From Section 1, we are aware of the shape of the graphs $y = \sin x$ and $y = \cos x$, and their important properties. It is also possible to construct the graphs of other sine and cosine functions, either by constructing a table of values or by analysing their properties. In general, sine and cosine curves are referred to as **sinusoidal** curves.

On our course, we have to consider graphs of the form

$$y = a + b\sin cx$$

and $y = a + b\cos cx$,

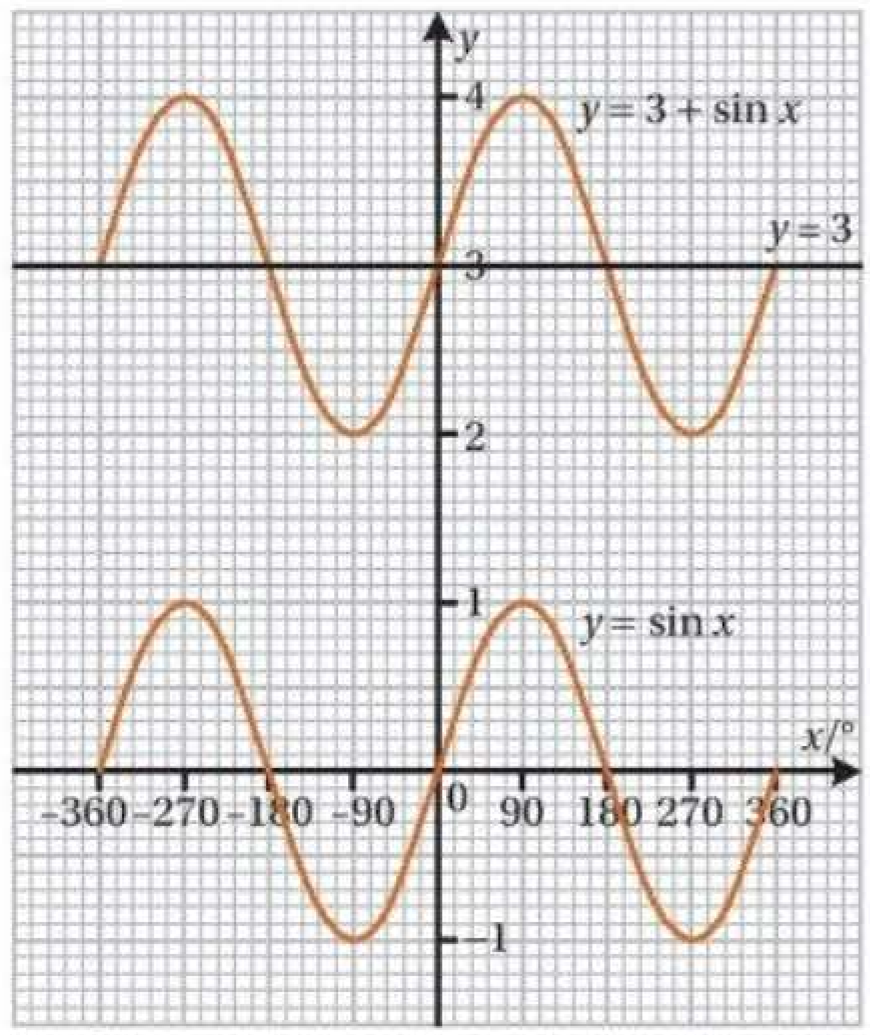
where $a, b, c \in \mathbb{R}$.

Each of the constants, a, b and c will have an effect on the shape and location of the graph, and on all the features outlined in Section 1. We will examine each of these in turn and refer to the transformations that produce these changes.

1. **Midway Line (value of *a*)**

Rule: The line $y = a$ is the midway line of $y = a + b\sin cx$ and $y = a + b\cos cx$.

Transformation: For the graph $y = a + \sin x$ or $y = a + \cos x$ the value of a gives the **vertical translation** of the standard sine or cosine graph.



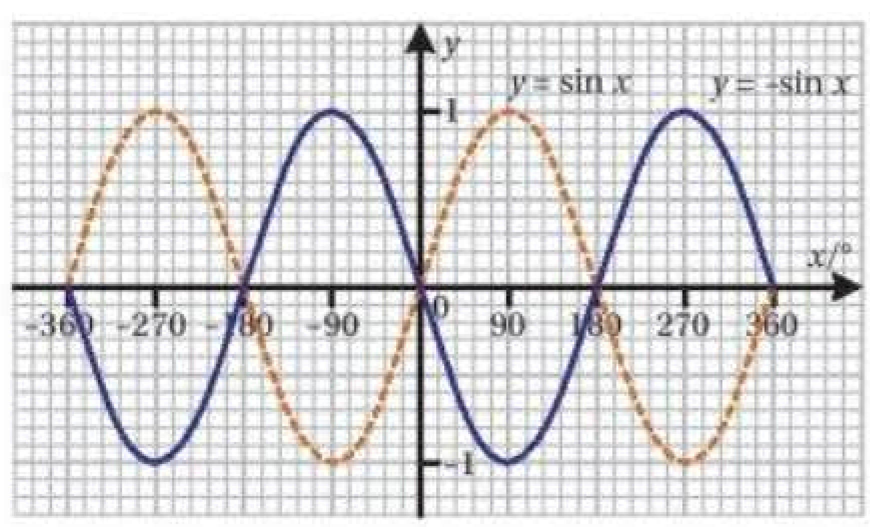
The diagram opposite shows the curves $y = \sin x$ and $y = 3 + \sin x$ on the same graph.

Notice that the curve $y = 3 + \sin x$ has exactly the same shape as that of $y = \sin x$. It has just been shifted vertically a distance of 3.

2. **Orientation: Normal or Inverted (sign of *b*)**

Rule: If b is positive then the graph is normal, i.e. it is orientated the same as the standard sine and cosine graphs. If b is negative, then the curve is inverted about the midway line.

Transformation: If b is negative, then the standard sine or cosine curve is reflected in the midway line.



The graph shows both the standard curve $y = \sin x$ and its reflection in its midway line (the x-axis), $y = -\sin x$.

3. **Amplitude (modulus of *b*)**

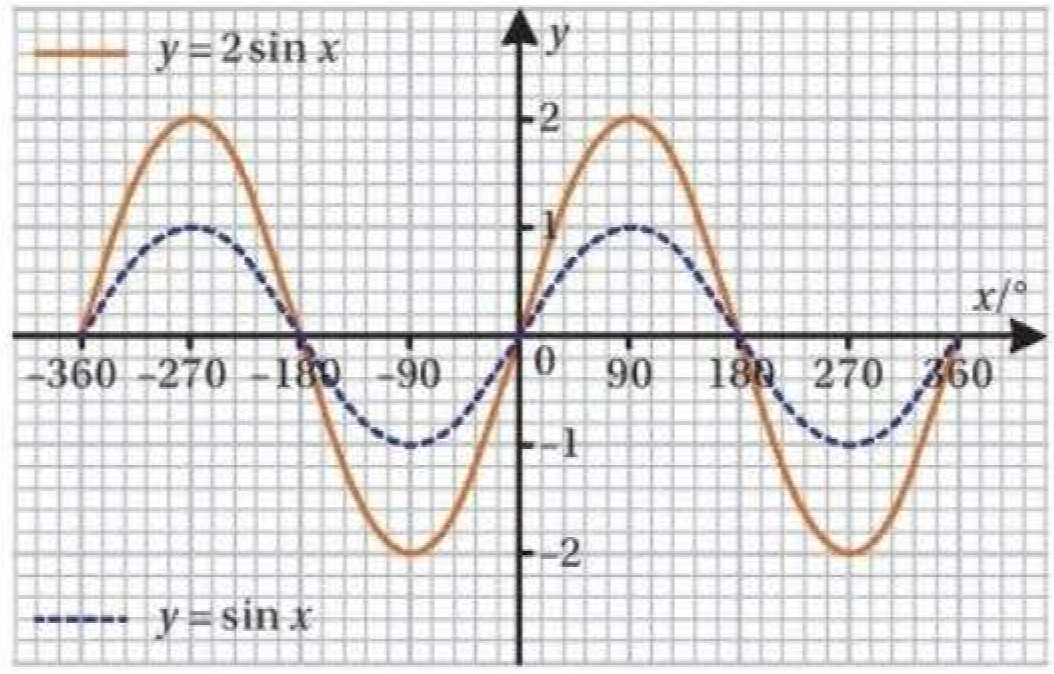
Rule: The amplitude of a sine or cosine graph is the modulus of b, i.e. $|b|$. For example, the amplitude of

$$y = -3\cos x$$

is $|-3| = 3$.

Transformation: The modulus of b, $|b|$, gives the factor of the **vertical dilation** (stretch or shrink parallel to the y-axis). If $|b| > 1$, the graph is stretched, while if $|b| < 1$, the graph is shrunk.

The diagram opposite shows the curves $y = \sin x$ and $y = 2 \sin x$ on the same graph. Note that the curve $y = 2 \sin x$ is the curve $y = \sin x$ stretched by a factor of 2 parallel to the y-axis.



Like the other examples given above, we can construct the graph $y = 2 \sin x$ by forming a table of values, if required. A table of values for $y = 2 \sin x$ is given below, for $-360° \leq x \leq 360°$.

x/°	−360	−270	−180	−90	0	90	180	270	360
sin x	0	1	0	−1	0	1	0	−1	0
$y = 2 \sin x$	0	2	0	−2	0	2	0	−2	0

4. Period (value of c)

Note that we only need to consider values of $c > 0$. This is because of the negative angle results from Chapter 14:

$\cos(-A) = \cos A$ and $\sin(-A) = -\sin A$.

Thus, for example, we can write

$y = 2 + 3 \sin(-4x)$

as $y = 2 - 3 \sin 4x$.

Rule: The period of

$y = a + b \sin cx$

or $y = a + b \cos cx$

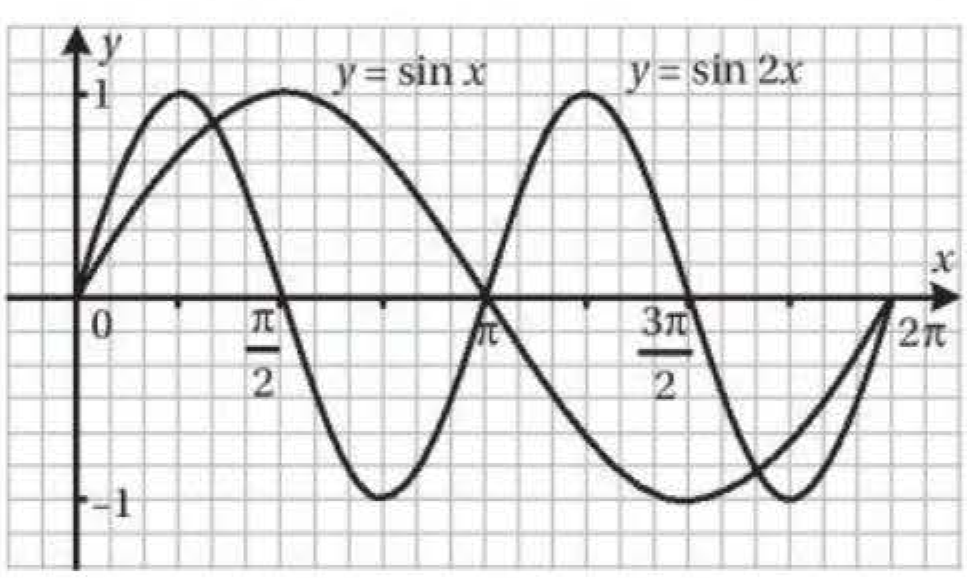
where $c > 0$, is

$\frac{360°}{c}$ (in degrees)

or $\frac{2\pi}{c}$ (in radians).

Transformation: The value of $c > 0$ determines the **horizontal dilation** (stretch or shrink about the y-axis). The curves $y = \sin x$ and $y = \cos x$ are shrunk by a factor of c $\left(\text{stretched by a factor of } \frac{1}{c}\right)$ when forming the curves $y = a + b \sin cx$ and $y = a + b \cos cx$ respectively.

The diagram below shows the curves $y = \sin x$ and $y = \sin 2x$, for $0 \leq x \leq 2\pi$. This can easily by verified by making a table of values. From the graph, it can be seen that the period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$ (or 180°).



To explain why the period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$, notice that the curve begins to repeat itself once the angle reaches 2π. If the angle is $2x$, then this will be when

$$2x = 2\pi$$

i.e. $x = \pi$.

In general, the graph of

$$y = a + b \sin cx$$

or $y = a + b \cos cx$

will begin to repeat itself when the angle is 2π. In each case, this will be when

$$cx = 2\pi$$

i.e. $x = \frac{2\pi}{c}$.

This is the period of each function.

In summary, we state the following results.

Sine and Cosine Graphs (Sinusoidal Graphs)

Consider the sine graph $y = a + b \cos cx$
and the cosine graph $y = a + b \cos cx$,
where a, b and c are constants, and $c > 0$.
These graphs have the following properties.

1. **Midway Line:** $y = a$
2. **Orientation:** normal if $b > 0$, inverted if $b < 0$
3. **Amplitude:** $|b|$
4. **Range:** $[a - |b|, a + |b|]$
   minimum is $a - |b|$, maximum is $a + |b|$
5. **Period:** $\frac{360°}{c}$ or $\frac{2\pi}{c}$.

Example 1

The function f is defined by

$$f: x \rightarrow 2 - 2 \cos 4x,$$

for $-\frac{\pi}{2} \leq x \leq \pi$, and is represented by the graph $y = 2 - 2 \cos 4x$.

(i) Identify the horizontal midway line, the amplitude, the range and the period of this graph.

(ii) Is the orientation of the curve about the midway line normal or inverted? Give a reason.

(iii) Using this information, construct the graph of $y = f(x)$ for $-\frac{\pi}{2} \leq x \leq \pi$.

(iv) Starting with the standard cosine graph, $y = \cos x$, outline a sequence of transformations that will map this graph to the graph $y = f(x)$.

(v) Use the graph of $y = f(x)$ to find all the solutions of the equation

$$f(x) = 3$$

for $-\frac{\pi}{2} \leq x \leq \pi$.

Solution

(i) Graph: $y = 2 - 2 \cos 4x$

$(a = 2, b = -2, c = 4)$

Midway line: $y = a$

$y = 2$

Amplitude: amplitude $= |b|$

amplitude $= |-2|$

amplitude $= 2$

Range: range $= [a - |b|, a + |b|]$

range $= [2 - 2, 2 + 2]$

range $= [0,4]$

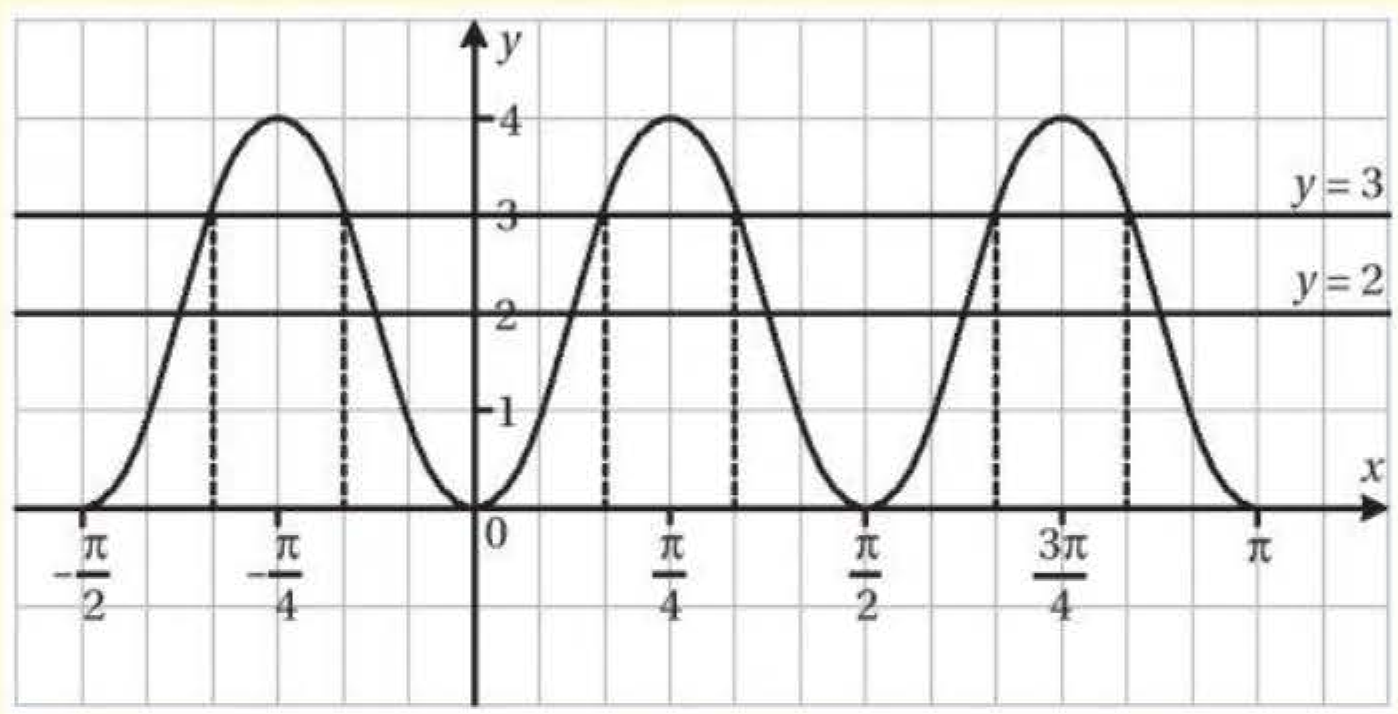
Period: period $= \frac{2\pi}{c}$

period $= \frac{2\pi}{4}$

period $= \frac{\pi}{2}$

(ii) The orientation of the curve about the midway line is inverted, as $b < 0$.

(iii) The graph is shown below.



(iv) Starting with the graph $y = \cos x$:

[1] Translate the graph a distance of 2 parallel to the y-axis, to get the midway line $y = 2$.

[2] Invert the graph about the midway line.

[3] Stretch the graph vertically by a factor of 2 about the midway line.

[4] Shrink the graph horizontally by a factor of 4 about the y-axis.

This gives the graph $y = 2 - 2\cos 4x$.

(v) To solve the equation

$f(x) = 3$,

we draw the line $y = 3$ and take the x co-ordinates of the points where the line cuts the graph. Noticing that the vertical lines on the grid above are separated by 15°, we can estimate the solutions of

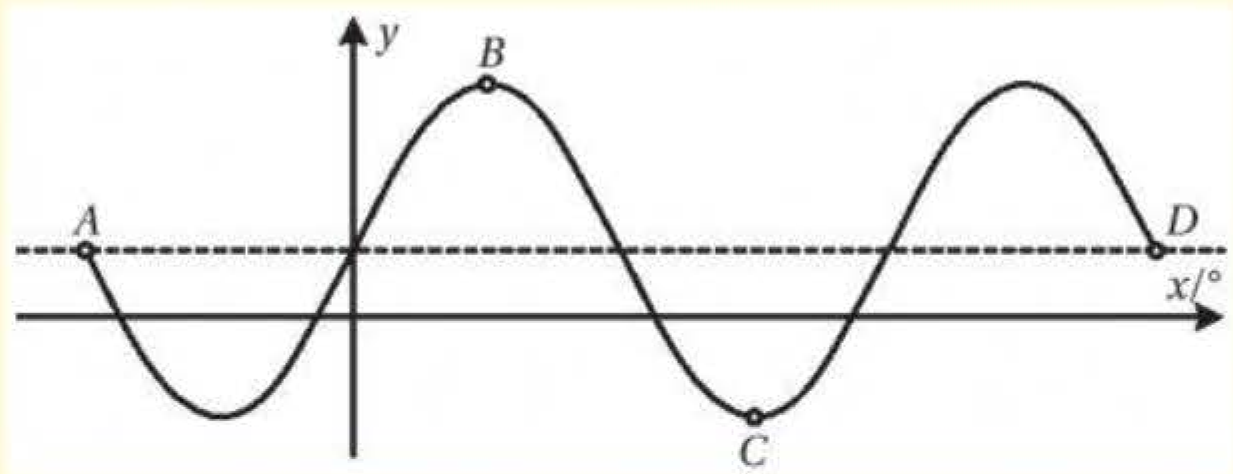
$f(x) = 3$

as $x = -60°\left(-\frac{\pi}{3}\right)$, $x = -30°\left(-\frac{\pi}{6}\right)$, $x = 30°\left(\frac{\pi}{6}\right)$, $x = 60°\left(\frac{\pi}{3}\right)$, $x = 120°\left(\frac{2\pi}{3}\right)$ and $x = 150°\left(\frac{5\pi}{6}\right)$.

When dealing with abstract questions on sine and cosine graphs, where we are not given the equation of the graph, it is important to recognise that each period (basic building block) can be broken down into four sections of equal width, which will be one quarter of the period. This can help us obtain co-ordinates of named points and, indeed, find the equation of the graph.

Example 2

The graph of a trigonometric function $y = f(x)$, where x is in degrees, is shown below. The co-ordinates of the points A and B are $(-30,2)$ and $(15,7)$ respectively.



(i) State whether $f(x)$ is a sine function or a cosine function. Give a reason.

(ii) Write the equation of the graph.

(iii) Find the co-ordinates of the points C and D.

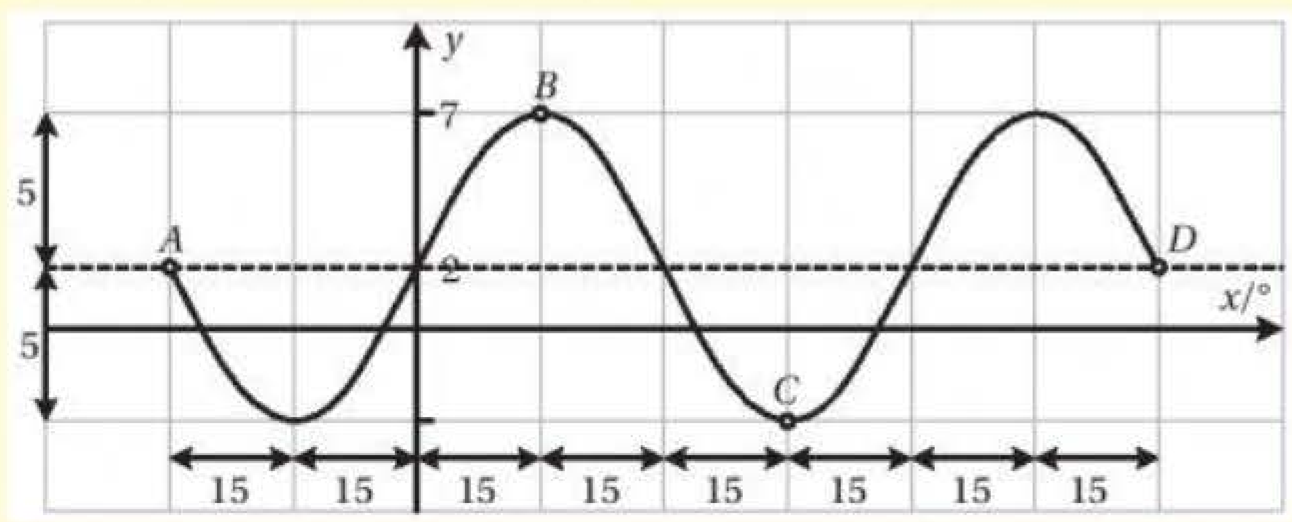
Solution

(i) As the graph crosses the midway line on the y-axis, we take $f(x)$ to be a sine function.

(ii) Let $f(x) = a + b \sin cx$.

To find the values of a, b and c, we need to analyse the graph carefully.

Draw the lines and mark in the values shown.



From the graph, we can determine the following.

[1] The midway line is $y = 2$. Thus, $a = 2$.

[2] The orientation of the curve about the midway line is normal. Thus, b is positive.

[3] The amplitude is $7 - 2 = 5$. Thus, $b = 5$.

[4] The period (width of one repeating block) is $4 \times 15° = 60°$. Thus,

$$\frac{360°}{c} = 60°$$

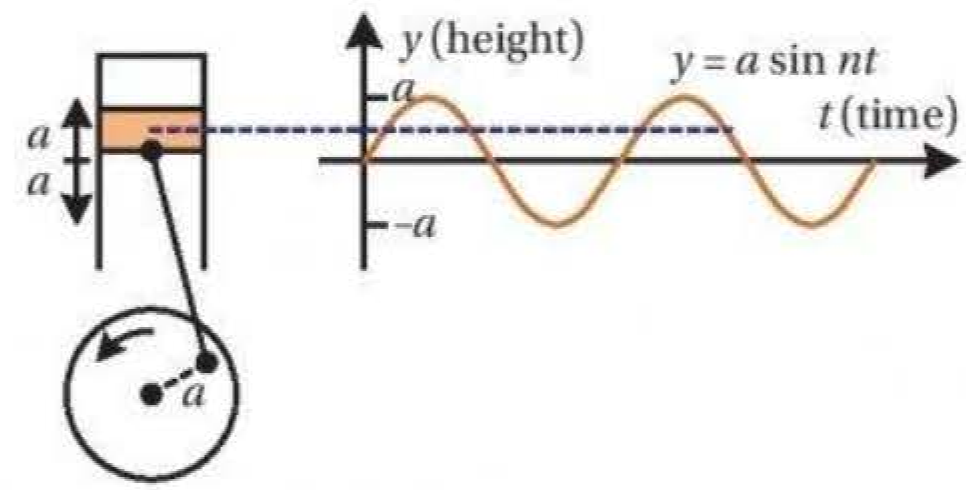
$$c = \frac{360°}{60°} = 6.$$

Thus, the equation of the graph is

$$y = 2 + 5 \sin 6x.$$

(iii) From the graph, the y co-ordinate of C is $2 - 5 = -3$. Thus, $C = (45,-3)$ and $D = (90,2)$.

There are many examples of sine and cosine graphs in real life. One involves the piston engine, which is the most commonly used engine in the world.



The motion of the piston can be described using a sine curve. If y is the height of the piston in the cylinder, measured from its centre point, at a time t, then y can be given by

$$y = a \sin nt$$

where a is the amplitude and n is determined by the speed of the motion.

Exercises 15.2

1. Identify the horizontal midway line of each of the following graphs.
(i) $y = 4 + 3 \sin 2x$ (ii) $y = 2 - 5 \cos x$ (iii) $y = -2 \sin 4x$
(iv) $y = -4 + \sin 3x$ (v) $y = -\frac{1}{2} + \frac{3}{2} \cos 3x$ (vi) $y = \frac{4}{3} - \frac{5}{6} \cos 2x$

2. For each of the following graphs, state whether the orientation of the graph about the midway line is normal or inverted.
(i) $y = 3 \sin 2x$ (ii) $y = -4 \cos 5x$ (iii) $y = 2 - 5 \sin x$
(iv) $y = -2 + 5 \cos 2x$ (v) $y = -0{\cdot}3 + 1{\cdot}2 \sin 0{\cdot}4x$ (vi) $y = 1 - \frac{5}{2} \cos \frac{2}{3}x$

For each of the following graphs,
(a) state the minimum y value,
(b) state the maximum y value,
(c) write down the range of the graph.

3. $y = 1 + 2 \sin 3x$ **4.** $y = 7 - 3 \cos x$ **5.** $y = -4 + 3 \cos 2x$ **6.** $y = 2 - 8 \sin 4x$

7. $y = \frac{1}{2} + \frac{7}{2} \sin 4x$ **8.** $y = \frac{2}{3} - \frac{8}{3} \cos 2x$ **9.** $y = -2{\cdot}4 + 5{\cdot}2 \cos 3x$ **10.** $y = 3{\cdot}7 - 2{\cdot}3 \sin 0{\cdot}2x$

For each of the following graphs, write down the period
(a) in degrees,
(b) in radians.

11. $y = 2 \sin 4x$ **12.** $y = -3 \cos 5x$ **13.** $y = 1 - 4 \cos 3x$ **14.** $y = -4 + 3 \sin 8x$

15. $y = 3 - 2 \sin \frac{2}{3}x$ **16.** $y = \frac{3}{4} - \frac{2}{3} \cos \frac{5}{2}x$ **17.** $y = 2{\cdot}3 + 1{\cdot}8 \sin 0{\cdot}4x$ **18.** $y = -4{\cdot}7 - 3{\cdot}5 \cos 1{\cdot}5x$

Starting with the standard sine graph, $y = \sin x$ or the standard cosine graph, $y = \cos x$, whichever is appropriate, outline a sequence of transformations which will map the standard curve to the given curve.

19. $y = -2 + 3 \sin 4x$ **20.** $y = 4 - 3 \cos 2x$ **21.** $y = \frac{3}{2} - \frac{1}{2} \sin 6x$ **22.** $y = \frac{7}{2} + \frac{3}{2} \cos 4x$

23. A function is defined by $f(x) = 4 + 3 \sin 2x$.

(i) Complete the following table for the graph $y = 4 + 3 \sin 2x$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$2x$									
$\sin 2x$									
$3 \sin 2x$									
$y = 4 + \sin 2x$									

(ii) Construct the graph $y = f(x)$ for $0 \le x \le 2\pi$.

(iii) State the period, the range and the amplitude of the graph.

(iv) If the period is T, state the value of $f\left(\frac{\pi}{4} + 10T\right)$.

(v) Use your graph to find the solutions of $f(x) = 5{\cdot}5$ in the given domain.

24. A function is defined by $f(x) = -1 + 2 \cos 3x$.

(i) Complete the following table for the graph $y = -1 + 2 \cos 3x$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$3x$													
$\cos 3x$													
$2 \cos 3x$													
$y = -1 + 2 \cos 3x$													

(ii) Construct the graph $y = f(x)$ for $0 \le x \le 2\pi$.

(iii) State the period, the range and the amplitude of the graph.

(iv) If the period is T, state the value of $f\left(\frac{5\pi}{6} + 8T\right)$.

(v) Use your graph to find the solutions of $f(x) = -2$ in the given domain.

25. A function is defined by $f(x) = 3 \sin x$.

(i) State the period and the range of the function.

(ii) Without forming a table of values, construct the graph $y = f(x)$ for $-360° \le x \le 360°$.

26. A function is defined by $f(x) = 1 + 4 \sin 2x$.

(i) State the period and the range of the function.

(ii) Without forming a table of values, construct the graph $y = f(x)$ for $-360° \le x \le 360°$.

27. A function is defined by $f(x) = 2 - 3 \cos 6x$.

(i) State the period and the range of the function.

(ii) Without forming a table of values, construct the graph $y = f(x)$ for $0° \le x \le 180°$.

28. A function is defined by $f(x) = -2 + 7 \sin 4x$.

(i) State the period and the range of the function.

(ii) Without forming a table of values, construct the graph $y = f(x)$ for $0° \le x \le 180°$.

29. A function is defined by $f(x) = 2 - 3 \cos 2x$.

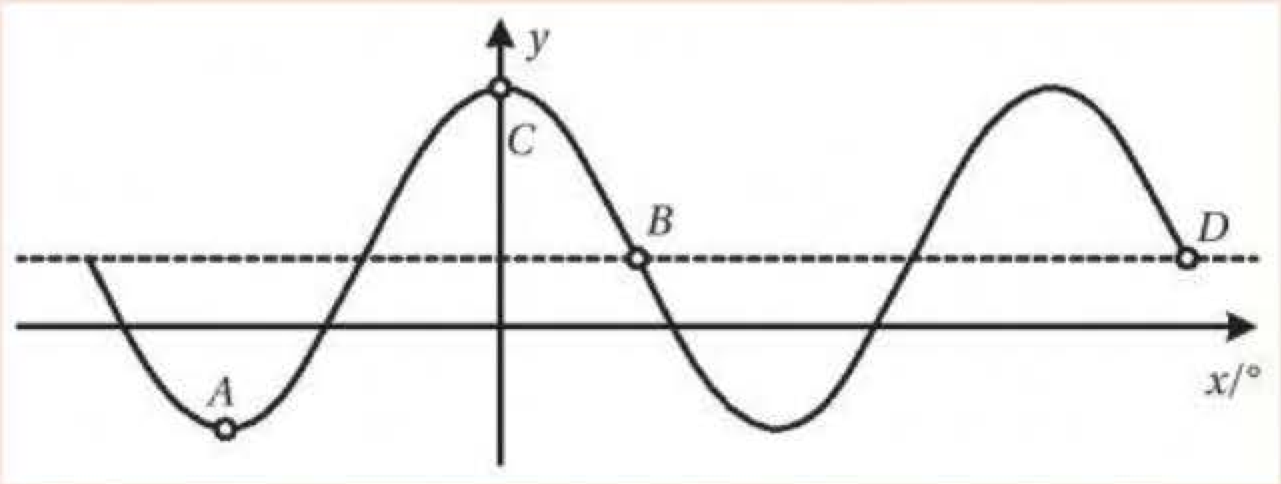
(i) State the period and the range of the function.

(ii) Without forming a table of values, construct the graph $y = f(x)$ for $-2\pi \le x \le 2\pi$.

(iii) Use your graph to estimate the solutions of $f(x) = \frac{1}{2}$ in the given domain.

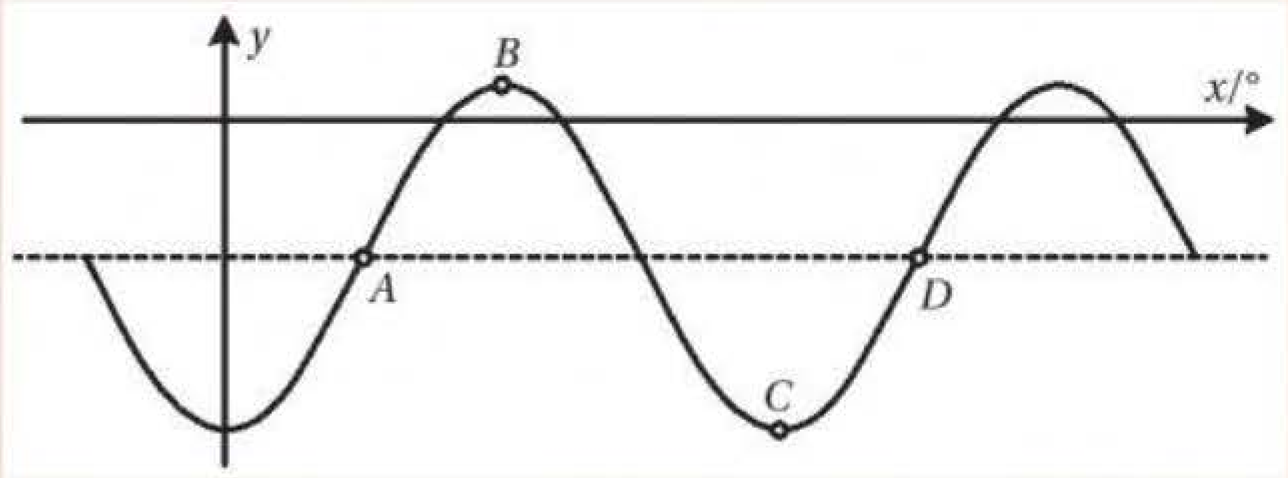
30. A function is defined by $f(x) = -1 - 4\cos 6x$.
(i) State the period and the range of the function.
(ii) Without forming a table of values, construct the graph $y = f(x)$ for $0 \le x \le \pi$.
(iii) Use your graph to estimate the solutions of $f(x) = -3$ in the given domain.

31. The graph of a trigonometric function $y = f(x)$, where x is in degrees, is shown below. The co-ordinates of the points A and B are $(-60,-3)$ and $(30,2)$ respectively.



(i) State whether $f(x)$ is a sine function or a cosine function. Give a reason.
(ii) Write the equation of the graph.
(iii) Find the co-ordinates of the points C and D.

32. The graph of a trigonometric function $y = f(x)$, where x is in degrees, is shown below. The co-ordinates of the points A and B are $(10,-8)$ and $(20,2)$ respectively.



(i) State whether $f(x)$ is a sine function or a cosine function. Give a reason.
(ii) Write the equation of the graph.
(iii) Find the co-ordinates of the points C and D.

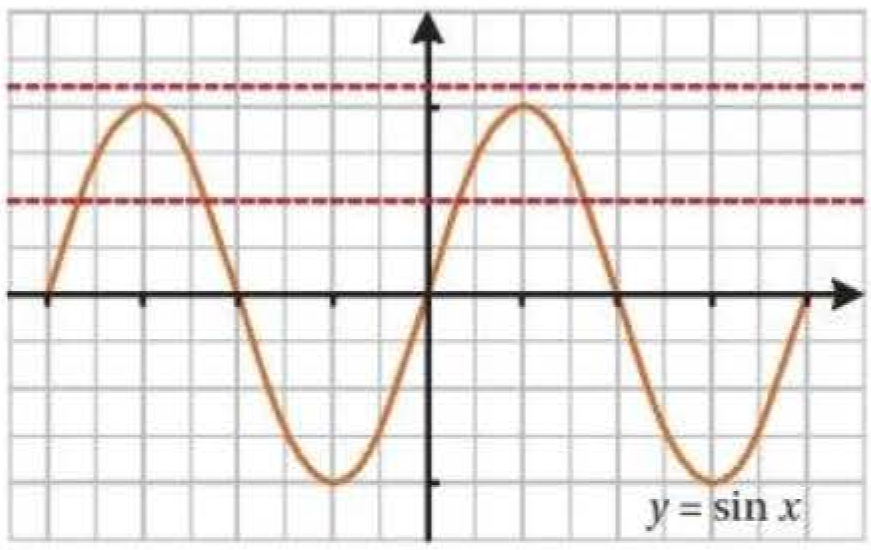
33. A stone was dropped in an artificial pond. The height, y centimetres, of the water above its usual level at a point on the pond wall t seconds after the first wave reaches the wall is given by the equation
$y = 30 + 10\cos 20t$.
(i) What is the range of the height of the water? What is the amplitude of the motion?
(ii) If t is in degrees, calculate the period of the motion.
(iii) For what value of t is the water first at its lowest level?

34. A trigonometric function is given by
$f(x) = a + b\cos cx$,
where $a, b, c \in \mathbb{R}$ and x is in degrees.
The range of the graph $y = f(x)$ is $[-10,50]$ and its period is 72°.
If $b < 0$, find the values of the constants a, b and c.

15.3 Inverse Trig Functions

1. Sine and Sine Inverse

If $f(x) = \sin x$ is defined as a function from $\mathbb{R}$ to $\mathbb{R}$, then it is neither injective (as some horizontal lines intersect the curve $y = \sin x$ more than once) nor surjective (as there are horizontal lines that do not intersect the curve at all). The dashed lines on the diagram below illustrate this.



Thus, this function is not bijective and its inverse is not a function. We need the inverse to be a function. What we can do is restrict the domain and the codomain, so that the resulting function is bijective.

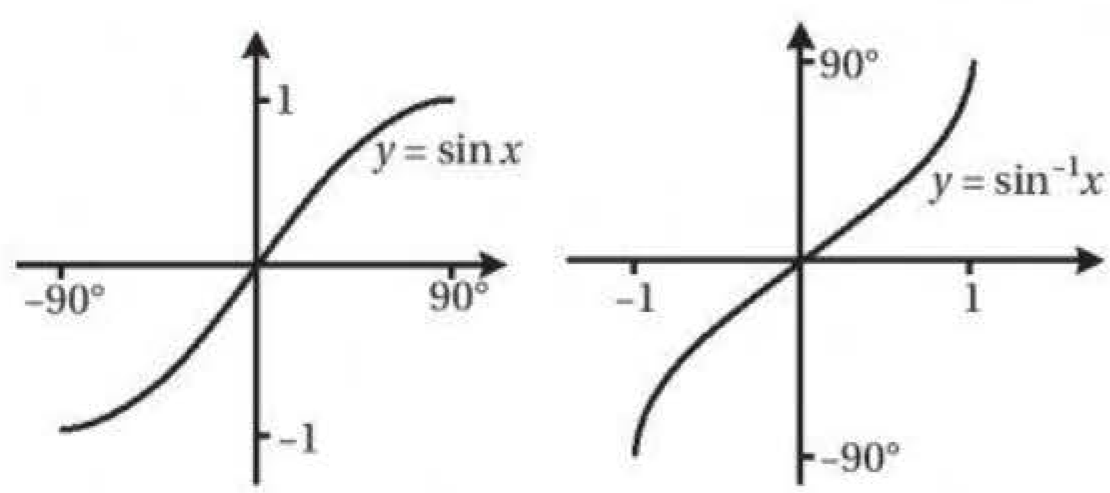
A long time ago, it was agreed that we would take the domain to be $[-90°, 90°]$, or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ if working in radians, to form a bijective function. Then the codomain, which is the same as the range, is $[-1,1]$. This gives us the following bijective function:

$f: [-90°, 90°] \rightarrow [-1,1]: x \rightarrow \sin x.$

The inverse of this is a function, called the inverse sine function, defined as follows:

$f^{-1}: [-1,1] \rightarrow [-90°, 90°]: x \rightarrow \sin^{-1} x.$

The diagram below shows the curves $y = \sin x$ and $y = \sin^{-1} x$ as defined above. Recall that each is the image of the other under reflection in the line $y = x$.



This is why $\sin^{-1} x$ is only defined for $-1 \leq x \leq 1$, and has a value that always lies in the range $-90° \leq \sin^{-1} x \leq 90°$. This is sometimes referred to as the 'principal part' of the inverse sine function, and is why, for example,

$$\sin^{-1} \frac{1}{2} = 30°,$$

and not any other angle whose sine is $\frac{1}{2}$. Your calculator is aware of this definition.

2. Cosine and Cosine Inverse

Similarly, $f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow \cos x$ is not a bijective function, but the function

$f: [0°,180°] \rightarrow [-1,1]: x \rightarrow \cos x$

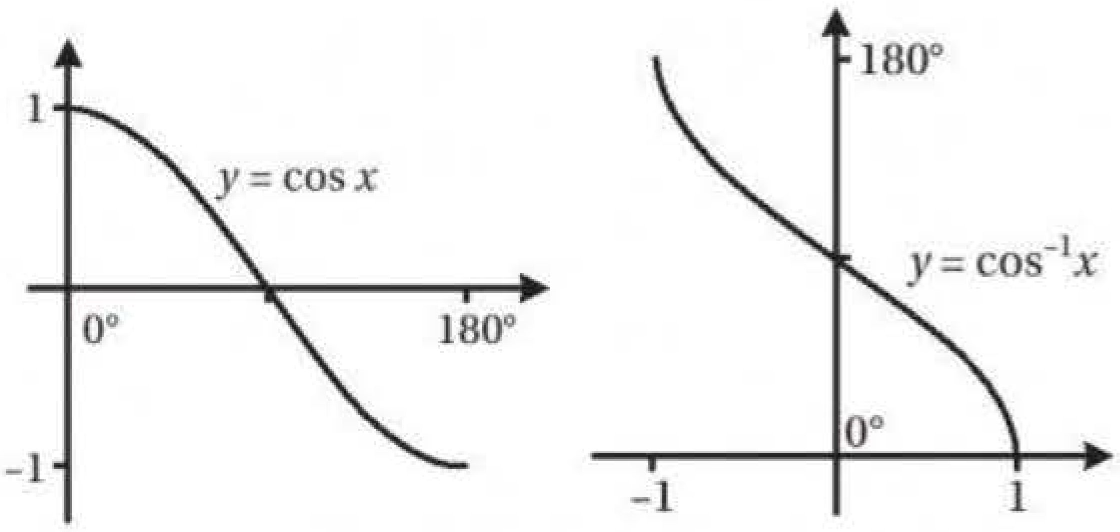
is.

Its inverse is

$f^{-1}: [-1,1] \rightarrow [0°,180°]: x \rightarrow \cos^{-1}x$

which is a function, called the principal part of the inverse cosine function.

The diagram below shows the curves $y = \cos x$ and $y = \cos^{-1}x$ as defined above.



This is why $\cos^{-1}\left(-\frac{1}{2}\right) = 120°$, and not any other angle that has its cosine equal to $-\frac{1}{2}$. You should check that your calculator agrees with this.

3. Tangent and Inverse Tangent

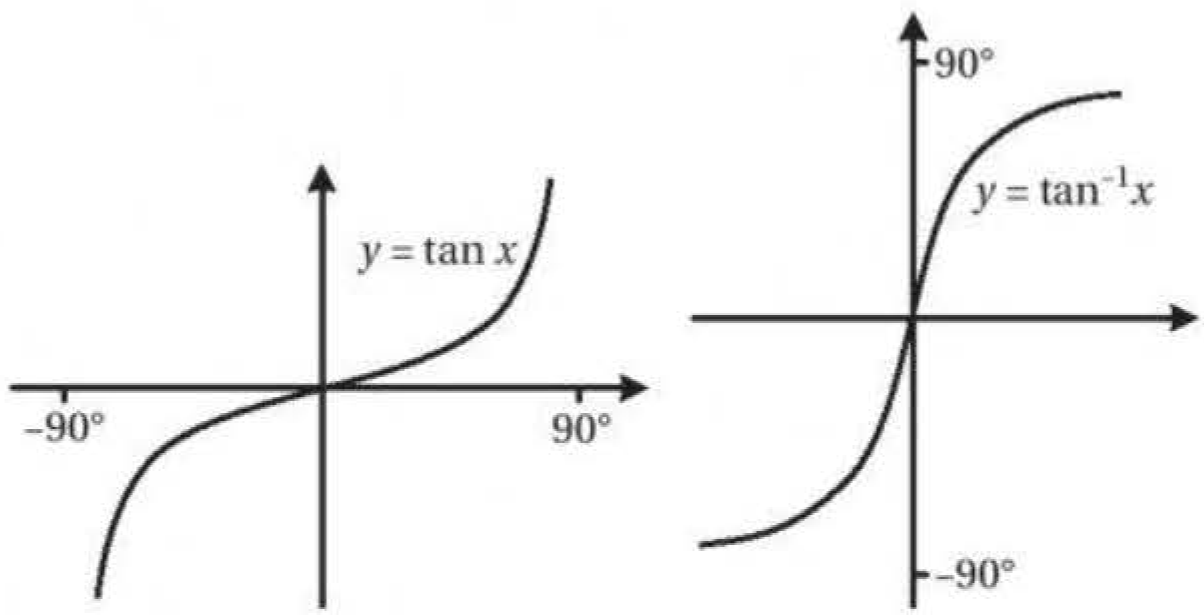
In the last chapter, we saw that $f(x) = \tan x$ was defined only for $\mathbb{R}\backslash\{\pm 90°, \pm 279°, ...\}$, and that its range is $\mathbb{R}$. To form a bijective function, we take the domain to be $-90° < x < 90°$, which can be written as the open interval $(-90°,90°)$. Then the codomain is $\mathbb{R}$, i.e. the set of all real numbers. The bijective function is then $f: (-90°,90°) \rightarrow \mathbb{R}: x \rightarrow \tan x$.

The inverse function is then

$f^{-1}: \mathbb{R} \rightarrow (-90°,90°): x \rightarrow \tan^{-1}x$,

called the principal part of the inverse tangent function.

The diagram below shows the curves $y = \tan x$ and $y = \tan^{-1}x$, as defined above.



Hence, for example, $\tan^{-1}1 = 45°$, and not any other angle whose tan is 1.

Exercises 15.3

1. (i) Calculate sin 60° and sin 120°. What do you notice?
   (ii) Explainw hy $\sin^{-1}\frac{\sqrt{3}}{2} = 60°$ and not 120°.

2. (i) List five angles whose sine is $-\frac{1}{2}$.
   (ii) Find $\sin^{-1}\left(-\frac{1}{2}\right)$ and explain why its value is not the other angles you have listed in part (i).

3. (i) List five angles whose cosine is $\frac{\sqrt{3}}{2}$.
   (ii) Find $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and explain why its value is not the other angles you have listed in part (i).

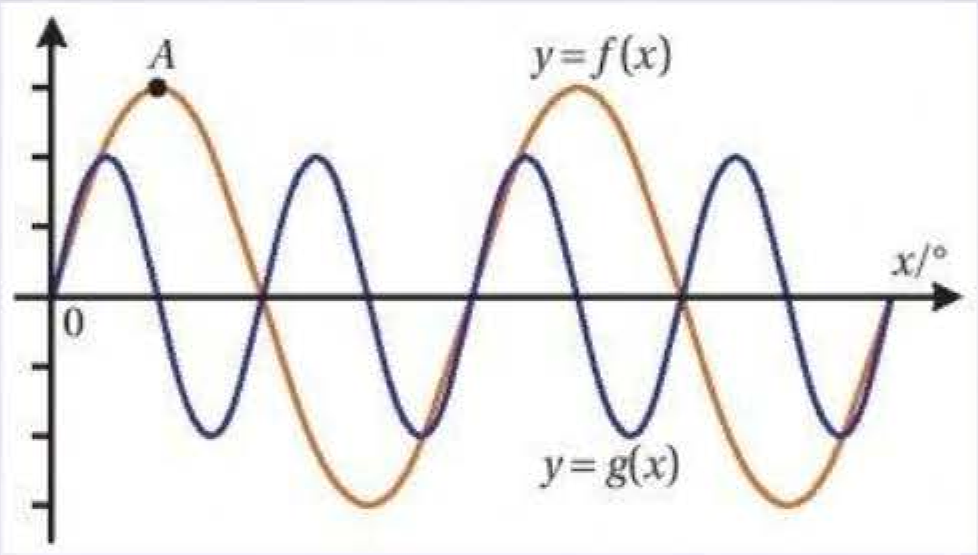
4. (i) List five angles whose tangent is $\frac{1}{\sqrt{3}}$.
   (ii) Find $\tan^{-1}\frac{1}{\sqrt{3}}$ and explain why its value is not the other angles you have listed in part (i).

Revision Exercises 15

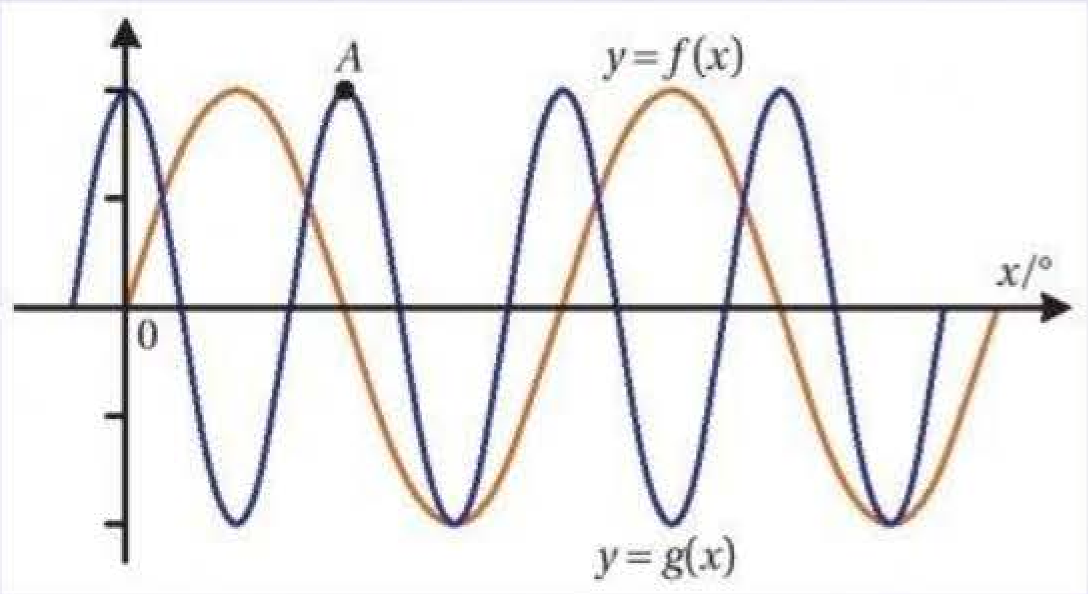
1. A trigonometric graph has period 90° and range [−12,12]. The graph contains the point (180,0), where the x co-ordinate is in degrees. If y is the variable on the vertical axis, write down the possible equations of the graph.

2. A sine graph is defined for $x \geq 0$. Its first maximum occurs at the point (45,12) and its second minimum occurs at the point (75,4).
   (i) State the range of the graph.
   (ii) By drawing a rough graph, calculate the period.
   (iii) Find the co-ordinates of the third maximum point.

3. Two trigonometric graphs, $y = f(x)$ and $y = g(x)$, are shown in the diagram opposite. The co-ordinates of the point A are (30,6), where the x co-ordinate is in degrees.
   (i) Find the period and the range of $y = f(x)$ and write down its equation.
   (ii) Find the period and the range of $y = g(x)$ and write down its equation.
   (iii) For the section of the graph shown, find all the values of x for which $f(x) = g(x)$.

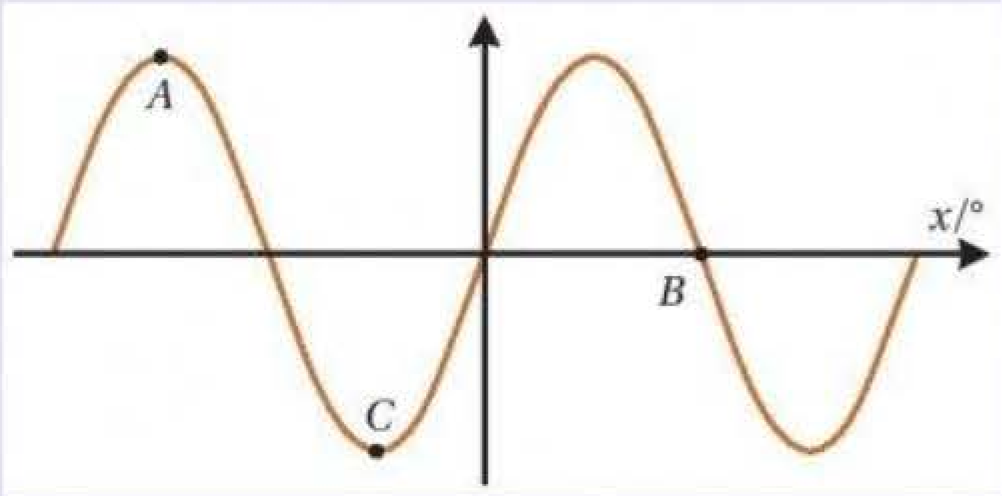


4. Two trigonometric graphs, $y = f(x)$ and $y = g(x)$, are shown in the diagram opposite.
   The co-ordinates of the point A are (90,8), where the x co-ordinate is in degrees.
   (i) Find the period and the range of $y = f(x)$ and write down its equation.
   (ii) Find the period and the range of $y = g(x)$ and write down its equation.
   (iii) Use the graph to state two values of x for which $f(x) - g(x) = 0$.

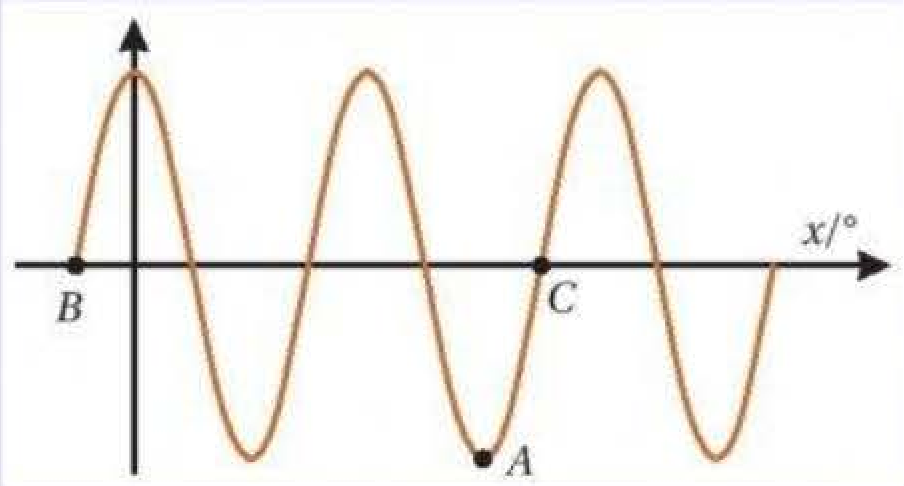


5. A trigonometric graph has two consecutive peaks at (0,6) and $\left(\frac{\pi}{4},6\right)$ where the x co-ordinates are in radians. The lowest points on the graph have a y co-ordinate of -2. Write down the equation of the graph.

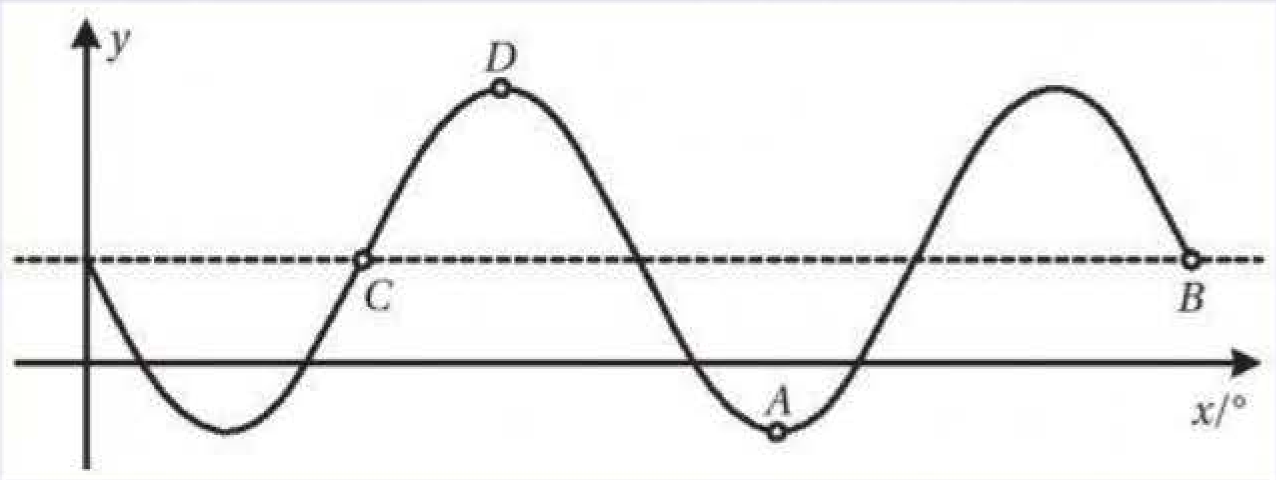
6. The diagram opposite shows a trigonometric graph $y = f(x)$ where x is in degrees. If the co-ordinates of the point A are $(-135,8)$,
   (i) state the period and the range of $f(x)$,
   (ii) write down an expression for $f(x)$,
   (iii) find the co-ordinates of B and C.



7. The diagram opposite shows a trigonometric graph $y = f(x)$ where x is in degrees. If the co-ordinates of the point A are $(135,-4)$,
   (i) state the period and the range of $f(x)$,
   (ii) write down an expression for $f(x)$,
   (iii) find the co-ordinates of B and C.



8. The diagram below shows a trigonometric graph $y = f(x)$ where x is in degrees. The point A has co-ordinates $(100,-20)$ and the y co-ordinate of B is 30.



   (i) State the period and the range of $f(x)$.
   (ii) Write the equation of the graph.
   (iii) Find the co-ordinates of the points C and D.

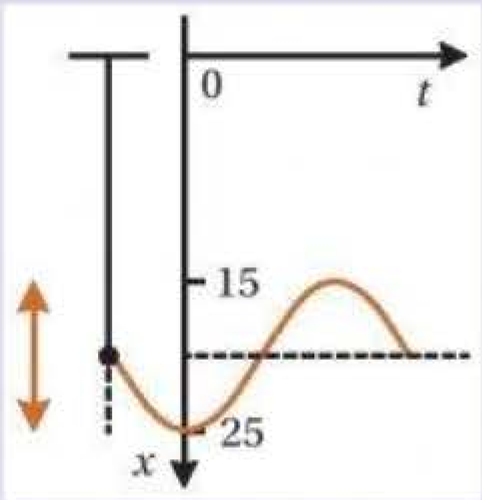
9. The average height of water in a harbour varies with the tide and is given by a sinusoidal curve, i.e. a sine curve or a cosine curve. If x m represents the height t hours after a maximum height of 12 m, then the next lowest height of 7 m occurs 6 hours later. Express x as a function of t, where the angle is in degrees.

10. A particle hangs in equilibrium at the end of an elastic string. It is then pulled down below its equilibrium position and released. Then it bobs up and down about the equilibrium position. If x cm represents its distance below the point where the string is attached after t seconds, then

$$x = a + b \cos ct,$$

where the angle is in radians.

(i) If the particle is pulled down until the length of the string is 25 cm and let go, it then moves up and down so that the length of the string varies between 25 cm and 15 cm. It takes $\frac{\pi}{2}$s econds to go from $x = 25$ to $x = 25$ the next time. Find the values of the constants a, b and c.



(ii) Find the time taken for the particle to pass the equilibrium position for the third time.

Chapter 11

Exercises 11.1, Pages 213–215

1. 120° **2.** 25° **3.** 51° **4.** (i) 124° (ii) 62° **5.** (ii) 35° **6.** (i) 62° (ii) 118° **7.** (i) 56° (ii) 26° **9.** (ii) 20° **12.** (i) 110° **15.** (i) 44°

Exercises 11.2, Pages 217–219

6. (ii) $10\sqrt{2}$ **7.** 4 **10.** (iii) 30° **12.** (ii) 0·75

Exercises 11.3, Pages 221–222

5. 17 cm **8.** (ii) $20\sqrt{2}$ cm

Revision Exercises 11, Pages 222–224

10. (iii) proportional sides in similar triangles theorem

Chapter 12

Exercises 12.1, Pages 234–236

7. (i) $2 < x < 18$ **8.** (i) $3 < x < 21$

Exercises 12.2, Pages 239–240

3. (iii) outside the triangle **12.** (iii) outside the triangle **13.** (iii) yes

Chapter 13

Exercises 13.1, Pages 247–248

6. (iii) yes

Exercises 13.2, Pages 241–254

1. (i) 2·8 (ii) 7·2 (iii) 3·5 (iv) similar **2.** (i) $\frac{5}{3}$ (ii) 3·6 (iii) 4·5 **3.** (i) $\frac{7}{4}$ (ii) 5 (iii) 3 **4.** (i) $\frac{7}{2}$ (ii) $\frac{21}{2}$ (iii) 8·75 **5.** (i) $\frac{5}{2}$ (ii) 6·25 (iii) 4·5 **6.** 2 **7.** (i) $\frac{3}{2}$ (ii) 4 (iii) 31·2975 **8.** (i) $\frac{3}{2}$ (ii) 6 (iii) 3 (iv) $4\sqrt{10}$ **9.** (i) 90° (ii) $\frac{3}{2}$ (iii) 4·5 **10.** (i) 2 (ii) 2 (iv) $2\sqrt{10}$ **11.** (i) $\frac{5}{3}$ (ii) 7·5 (iii) 5·4 (iv) 46·25 **12.** (i) $\frac{5}{3}$ (ii) 31·5 (iii) 56

Revision Exercises 13, Pages 254–256

1. (i) 2 (iii) 16 (iv) 21 **2.** (iii) 21 cm^2 **3.** (iii) 4 cm^2 (iv) $\frac{5}{4}$ **4.** (iii) 24·3 cm^2 **5.** (i) $\frac{7}{4}$ (ii) 6 (iii) 32 **6.** (i) $\frac{9}{4}$ (ii) 6·25 (iii) 12 **7.** (i) $\frac{3}{2}$ (ii) 6 (iii) 12 **8.** (ii) 2 (iv) 84 **9.** (i) A (ii) $\frac{8}{5}$ (iii) 8 **10.** $\frac{9}{4}$

Chapter 14

Exercises 14.1, Page 259

1. 45·4° **2.** 31·25° **3.** 146·2667° **4.** 103·9167° **5.** 42·4758° **6.** 127·3633° **7.** 241·6617° **8.** 64·8092° **9.** 34°33′36″ **10.** 145°16′12″ **11.** 318°50′24″ **12.** 163°43′12″ **13.** 53°27′21·6″ **14.** 248°46′55·2″ **15.** 84°17′34·8″ **16.** 106°54′18″ **17.** $\frac{5\pi}{4}$ **18.** $\frac{2\pi}{3}$ **19.** $\frac{5\pi}{3}$ **20.** $\frac{4\pi}{3}$ **21.** $\frac{2\pi}{5}$ **22.** $\frac{7\pi}{4}$ **23.** $\frac{4\pi}{5}$ **24.** $\frac{4\pi}{9}$ **25.** 210° **26.** 330° **27.** 108° **28.** 260° **29.** 40° **30.** 315° **31.** 22·5° **32.** 252°

Exercises 14.2, Pages 263–264

1. 0·9511 **2.** 0·8290 **3.** 0·8693 **4.** 0·9816 **5.** 0·4147 **6.** 0·8799 **7.** 1·9444 **8.** 0·2089 **9.** $A = 48·59°$ **10.** $A = 52·85°$ **11.** $A = 47·16°$ **12.** $A = 53·13°$ **13.** $A = 20·16°$ **14.** $A = 38·33°$ **15.** $A = 57·48°$ **16.** $A = 78·60°$ **17.** (i) 12·19 (ii) 9·21 **18.** (i) 6·18 (ii) 3·29 **19.** (i) 8·59 (ii) 13·96 **20.** (i) 38·7° (ii) $\sqrt{41}$ **21.** (i) 39° (ii) 14·5 **22.** (i) 36° (ii) $\sqrt{20}$ **23.** $\frac{\sqrt{21}}{5}$ **24.** $\frac{3}{4}$ **25.** $\frac{m}{\sqrt{m^2+n^2}}$ **26.** $\frac{t}{\sqrt{2t+1}}$

Exercises 14.3, Page 270

1. 0·7314 **2.** −0·5095 **3.** −0·6157 **4.** −2·4751 **5.** 0·7880 **6.** −0·5878 **7.** 0·4877 **8.** −0·5446 **9.** −0·8948 **10.** −0·8159 **11.** 0·7590 **12.** −0·3209 **13.** $\frac{\sqrt{3}}{2}$ **14.** $-\frac{1}{\sqrt{2}}$ **15.** $-\frac{1}{\sqrt{3}}$ **16.** $-\frac{\sqrt{3}}{2}$ **17.** $\sqrt{3}$ **18.** $-\frac{\sqrt{3}}{2}$ **19.** $\frac{\sqrt{3}}{2}$ **20.** $-\frac{1}{2}$ **21.** $-\sqrt{3}$ **22.** $\frac{1}{\sqrt{2}}$ **23.** $\frac{1}{\sqrt{2}}$ **24.** −1 **25.** $\frac{1}{2}$ **26.** $\frac{1}{\sqrt{2}}$ **27.** $\sqrt{3}$ **28.** $\frac{\sqrt{3}}{2}$ **29.** $\sqrt{3}$ **30.** $-\frac{\sqrt{3}}{2}$ **31.** $\frac{\sqrt{3}}{2}$ **32.** $-\frac{\sqrt{3}}{2}$ **33.** $-\frac{\sqrt{3}}{2}$ **34.** $-\frac{\sqrt{3}}{2}$ **35.** 1 **36.** $\frac{1}{2}$ **37.** (i) 1 (ii) 1 (iii) 1 **38.** (i) 1 (ii) 1 (iii) 1

Revision Exercises 14, Pages 270–271

1. 5·5°, 1 : 05 : 27 **2.** 66° **7.** (ii) 39° (iii) 64·7 cm **8.** (i) 42·26 m (ii) 106·08 m

Chapter 15

Exercises 15.1, Page 276

1. (iii) 360°, [−1,1] (iv) −330°, −210°, 30°, 150° **2.** (iii) −1, 1, [−1,1] (iv) $\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$ **3.** (iii) 360°, [−1,1] (iv) 120°, 240°, 480° **4.** (iii) 2π, [−1,1] (iv) $-\pi < x < 0$ and $\pi < x < 2\pi$

Exercises 15.2, Pages 282–284

1. (i) $y = 4$ (ii) $y = 2$ (iii) $y = 0$ (iv) $y = -4$ (v) $y = -\frac{1}{2}$ (vi) $y = \frac{4}{3}$ **2.** (i) normal (ii) inverted (iii) inverted (iv) normal (v) normal (vi) inverted **3.** (a) −1 (b) 3 (c) [−1,3] **4.** (a) 4 (b) 10 (c) [4,10] **5.** (a) −7 (b) −1 (c) [−7,−1] **6.** (a) −6 (b) 10 (c) [−6,10] **7.** (a) −3 (b) 4 (c) [−3,4] **8.** (a) −2 (b) $\frac{10}{3}$ (c) $\left[-2, \frac{10}{3}\right]$ **9.** (a) −7·6 (b) 2·8 (c) [−7·6, 2·8] **10.** (a) 1·4 (b) 6 (c) [1·4, 6] **11.** (a) 90° (b) $\frac{\pi}{2}$ **12.** (a) 72° (b) $\frac{2\pi}{5}$ **13.** (a) 120° (b) $\frac{2\pi}{3}$ **14.** (a) 45° (b) $\frac{\pi}{4}$ **15.** (a) 540° (b) 3π **16.** (a) 144° (b) $\frac{4\pi}{5}$ **17.** (a) 900° (b) 5π **18.** (a) 240° (b) $\frac{4\pi}{3}$ **19.** [1] Translate vertically −2. [2] Stretch vertically by a factor of 3 about the midway line. [3] Shrink horizontally by a factor of 4 about the y-axis. **20.** [1] Translate vertically 4. [2] Invert the graph about the midway line, $y = 4$. [3] Stretch vertically by a factor of 3 about the midway line. [4] Shrink horizontally by a factor of 2 about the y-axis. **21.** [1] Translate vertically $\frac{3}{2}$. [2] Invert the graph about the midway line, $y = \frac{3}{2}$. [3] Stretch vertically by a factor of $\frac{1}{2}$ (or shrink by a factor of 2) about the midway line. [4] Shrink horizontally by a factor of 6 about the y-axis. **22.** [1] Translate vertically $\frac{7}{2}$. [2] Stretch vertically by a factor of $\frac{3}{2}$ about the midway line, $y = \frac{7}{2}$. [3] Shrink horizontally by a factor of 4 about the y-axis. **23.** (iii) π, [1,7], 3 (iv) 7 (v) $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ **24.** (iii) $\frac{2\pi}{3}$, [−3,1], 2