A simple proof of the right hand rule

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Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, the cross product $\mathbf{u}\times\mathbf{v}$ is defined as the vector $\langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle$. Because this is somewhat complicated, a typical calculus textbook introduces the symbolic determinant

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$ and $\mathbf{k} = \langle 0, 0, 1\rangle$, which provides a quick proof that $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, and that $\mathbf{u}\times\mathbf{v} = 0$ if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

The cross product satisfies another property called the *right hand rule*, meaning that as the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, your thumb points in the direction of $\mathbf{u}\times\mathbf{v}$. Although very important in applications, most current calculus textbooks do not prove it. An exception is Colley [1], which defines the cross product through its geometric properties, including right-handedness, then proves its algebraic properties, including the determinant formula. Because of its length, such a proof is not suitable for classroom presentation. Once in a while, curious students may ask for a direct proof, and it may be difficult for an unprepared instructor to come up with one quickly. This note provides a proof simple enough to be presented in a lecture.

For $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$, the determinant of the vector

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triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is defined by

$$\det(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

It is a good exercise for students to check that $\det(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}) = \|\mathbf{u} \times \mathbf{v}\|^2$. We show that the positivity of this determinant implies the right hand rule.

The key to the proof is to relax the definition of the right hand rule. Given two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$, let $\mathbf{n}(u, v)$ be the unit vector in the direction that your thumb points as the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$. Given three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ that are not coplanar, we say that the vector triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ satisfies the *relaxed right hand rule* if $\mathbf{n}(u, v)$ forms an acute angle with $\mathbf{w}$. Of course, if $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then the relaxed right hand rule is the same as the right hand rule. Using a right hand drawing convention, the vector triple $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ satisfies the right hand rule, while the triple $(\mathbf{i}, \mathbf{j}, \mathbf{i} + \mathbf{k})$, for example, satisfies only the relaxed rule. Note that if you draw $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ left-handed, the terms 'right' and 'left' have to be interchanged.

The benefit of introducing the relaxed right-hand rule is the following lemma.

Lemma. *Adding a scalar multiple of one vector to another does not change the relaxed right-handedness of a triple* $(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

Proof. To begin, suppose we add a scalar multiple of $\mathbf{u}$ to $\mathbf{v}$. Observe that the vectors $\mathbf{v}$ and $\mathbf{v} + c\mathbf{u}$ are coplanar with $\mathbf{u}$, and always lie on the same side of $\mathbf{u}$, regardless of whether c is positive or negative. Thus, the direction of the thumb does not change, see Figure 1, so the angle between $\mathbf{w}$ and the direction of the thumb is unchanged. Similarly, if we add a scalar multiple of $\mathbf{v}$ to $\mathbf{u}$, the angle between $\mathbf{w}$ and the direction of the thumb does not change.

Next, suppose we add a scalar multiple of $\mathbf{u}$ or $\mathbf{v}$ to $\mathbf{w}$. Since $\mathbf{n}(u, v)$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{n}(u, v) \cdot (\mathbf{w} + \alpha\mathbf{u} + \beta\mathbf{v}) = \mathbf{n}(u, v) \cdot \mathbf{w}$, and so the acuteness of the angle between the thumb and $\mathbf{w} + \alpha\mathbf{u} + \beta\mathbf{v}$ is unchanged.

Finally, suppose we add a scalar multiple $\mathbf{w}$ to either $\mathbf{u}$ or $\mathbf{v}$, say $\mathbf{u}$. Because the vectors $\mathbf{n}(u + \alpha w, v)$, $\mathbf{n}(u, v)$ and $\mathbf{n}(\alpha w, v)$ are all orthogonal to $\mathbf{v}$, they are coplanar. We can write

$\mathbf{n}(u+\alpha w, v) = c(\alpha)\mathbf{n}(u,v) + d(\alpha)\mathbf{n}(w,v)$. Note that if you move your whole hand slightly, the direction of the thumb only changes slightly. In other words, $c(\alpha)$ and $d(\alpha)$ are continuous functions of α with $c(0) = 1$. If $c(\alpha) < 0$ for some α, then by intermediate value theorem of continuous functions, there exists $\beta \neq 0$ such that $c(\beta) = 0$. This gives $\mathbf{n}(u+\beta w, v) = \mathbf{n}(w,v)$. Thus $\mathbf{n}(u+\beta w, v)$ is orthogonal to $\mathbf{u}+\beta\mathbf{w}$, $\mathbf{w}$ and $\mathbf{v}$, which is a contradiction. Hence $c(\alpha)$ is always positive, and consequently the inner products $\mathbf{n}(u+\alpha w, v)\cdot\mathbf{w}$ and $\mathbf{n}(u,v)\cdot\mathbf{w}$ have the same sign. Hence, the acuteness of the angle between the thumb and $\mathbf{w}$ is unchanged. □

Theorem. *A vector triple* $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ *satisfies the relaxed right-hand rule if and only if it has a positive determinant.*

Proof. By adding a scalar multiple of one row vector to another, we can change a square non-singular matrix into a diagonal matrix. These row operations neither change the determinant value nor, thanks to our lemma, change the handedness of the vector triple. Thus, we can find vectors $\mathbf{a} = a\mathbf{i}$, $\mathbf{b} = b\mathbf{j}$ and $\mathbf{c} = c\mathbf{k}$ such that the vector triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ has the the same handedness and the same determinant value as the original triple. However, it is straightforward to check, given in the right-hand drawing convention, that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ satisfies the right hand rule if and only if $\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) > 0$. □

Corollary. *The cross product satisfies the right hand rule.*

Proof. For any two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$, because $\det(\mathbf{u}, \mathbf{v}, \mathbf{u}\times\mathbf{v}) = \|\mathbf{u}\times\mathbf{v}\|^2 > 0$, by the theorem above, the vector triple $(\mathbf{u}, \mathbf{v}, \mathbf{u}\times\mathbf{v})$ satisfies the relaxed right hand rule. However, $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, the relaxed right hand rule is the same as the right hand rule. □

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References

[1] Colley, S. J. *Vector Calculus*, 4th ed., Pearson, Boston MA, 2012.

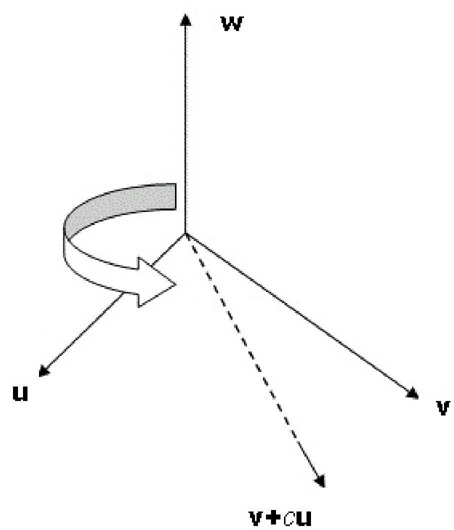

Figure 1 – The triple $(\mathbf{u}, \mathbf{v} + c\mathbf{u}, \mathbf{w})$ has the same *relaxed* right-handedness as $(\mathbf{u}, \mathbf{v}, \mathbf{w})$.