PULLEY SOLUTIONS USING RREF

Make a sketch of the physical situation. From this, show free-body diagrams for the masses and the pulleys, indicating the forces and torques on them. Be consistent with the direction of motion. Recognize that in many of these problems the masses will move in opposite directions, but this is not always the case. The pulley will of course rotate in the direction of motion, and if there are two pulleys, they *both* rotate in the same direction.

Having the diagrams, write out the force-balance equations. There will need to be as many equations as variables to be found. Usually the solution will be for the tension(s) in cables, and the acceleration of the system. The masses and the pulley will all have the same linear acceleration, under the usual assumptions for these problems.

The "trick" is to re-arrange the force/torque equations into a linear-systems or matrix format, so that the simultaneous equations can be solved easily, using RREF. Even if algebraic solutions are needed, and RREF cannot be used, arranging things in this manner makes the algebra clearer.

Arrange the equations (which are the rows of the matrix) so that all the terms using a given variable (e.g., the acceleration) are in the same column. If a system variable does not appear in a given equation, put a zero in that column. Then take the coefficients of the variables and build the RREF matrix. The RHS of the equations go in the last (rightmost) column of the RREF matrix. This matrix will be $(n, n+1)$ in size, where n is the number of variables.

For example, consider two masses on either side of a pulley. There are three unknowns: the tension on either side of the pulley, and the system acceleration. The force/torque equations lead to

$$\begin{aligned} T_1 - m_1 g &= m_1 a \\ T_2 - m_2 g &= -m_2 a \\ \left(T_2 - T_1\right) R &= I \frac{a}{R} \end{aligned}$$

Note that the masses go in opposite directions here. Re-writing this in a matrix arrangement, we have

$$\begin{aligned} T_1 + 0 - m_1 a &= m_1 g \\ 0 + T_2 + m_2 a &= m_2 g \\ -T_1 + T_2 - \frac{I}{R^2} a &= 0 \end{aligned}$$

Using the moment of inertia for a thin solid disk, ½ MR^2, we see that the last row can be simplified, and the RREF matrix becomes, with m_p the mass of the pulley (its radius cancels):

$$RREF\begin{pmatrix} 1 & 0 & -m_1 & m_1 g \\ 0 & 1 & m_2 & m_2 g \\ -2 & 2 & -m_p & 0 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ a \end{pmatrix}$$

Note that the rightmost RREF column contains *given* information, so that these matrix elements are numbers. The solution above will work correctly if the pulley is considered to be "massless" as was the case in earlier problems; just use a zero value in the matrix for m_p.

In the setup of these equations we must assume a direction of motion. If the masses are inconsistent with this, all that will happen is that the indicated acceleration will be negative. That just means things move in the opposite direction; the magnitude should still be correct. Check that the tension is larger in the direction of motion, also.

See more RREF results below.

A single mass hanging from a pulley (Example 10.12).

$$RREF\begin{pmatrix} 1 & m_1 & m_1 g \\ 2 & -m_p & 0 \end{pmatrix} = \begin{pmatrix} T \\ a \end{pmatrix} \tag{1}$$

Two masses hanging on either side of a pulley; can use zero for pulley mass, i.e., Atwood (Example 10.16).

$$RREF\begin{pmatrix} 1 & 0 & -m_1 & m_1 g \\ 0 & 1 & m_2 & m_2 g \\ -2 & 2 & -m_p & 0 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ a \end{pmatrix} \tag{2}$$

Two masses, one hanging from a pulley, cable over to another pulley, another mass hangs from it. In this situation the pulleys *A* and *B* might have different masses, although they usually are the same. Note that the radius of the pulley cancels and does not appear here (Example 10.13).

$$RREF\begin{pmatrix} 1 & 0 & 0 & -m_1 & m_1 g \\ 0 & 0 & 1 & m_2 & m_2 g \\ -2 & 2 & 0 & -m_A & 0 \\ 0 & -2 & 2 & -m_B & 0 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ a \end{pmatrix} \tag{3}$$

Two masses, both on a surface, one inclined, one horizontal, with friction (Problem 29).

$$RREF\begin{pmatrix} 1 & 0 & -m_1 & \mu m_1 g \\ 0 & -1 & -m_2 & m_2 g\left[\mu\cos(\theta) - \sin(\theta)\right] \\ -2 & 2 & -m_p & 0 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ a \end{pmatrix} \tag{4}$$

Note that the "AP" problem cannot use these solutions directly since the hoop-plus-rod is not a solid disk, so it would have a different moment of inertia. However, Eq(1) is easily adapted to this problem.

These equations will produce correct results over a wide range of parameter values. However, the results must be checked for reasonableness, since in some extreme cases the solution may be nonsensical. For example, Eq(4) for a nonzero coefficient of friction and a large m_1 (the horizontal mass) will produce a negative acceleration, which is impossible. The model formulation doesn't know that the acceleration cannot be negative; what would happen is that the system would not move at all. ***Check that the results make physical sense*.**

These solutions are just examples. The idea here is not these specific results, but rather the *concept* of obtaining solutions for these kinds of problems, that involve simultaneous equations, in a much simpler, faster way than plowing through a bunch of algebra.