The Mathematics of Diagnostics Tomography

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What is Tomography?

Tomography is a "radiologic technique for obtaining clear X-ray images of deep internal structures by focusing on a specific plane within the body."

(Encyclopedia Britannica)

What is Computed Tomography?

- Simply refers to tomography done using data collected from X-rays and a computer.

- All X-ray imaging -such as Computed Tomography- is based on **the absorption of X-rays** as they pass through the different parts of a patient's body.

- Based on the variable absorption of X-rays by different tissues, computed tomography (CT) imaging, also known as "CAT scanning" (Computerized Axial Tomography), provides **a form of imaging known as cross-sectional imaging.**

(US Food and Drug Administration)

What is Computed Tomography?

- We can think of cross-sectional images or "slices" of anatomy like the slices in a loaf of bread.

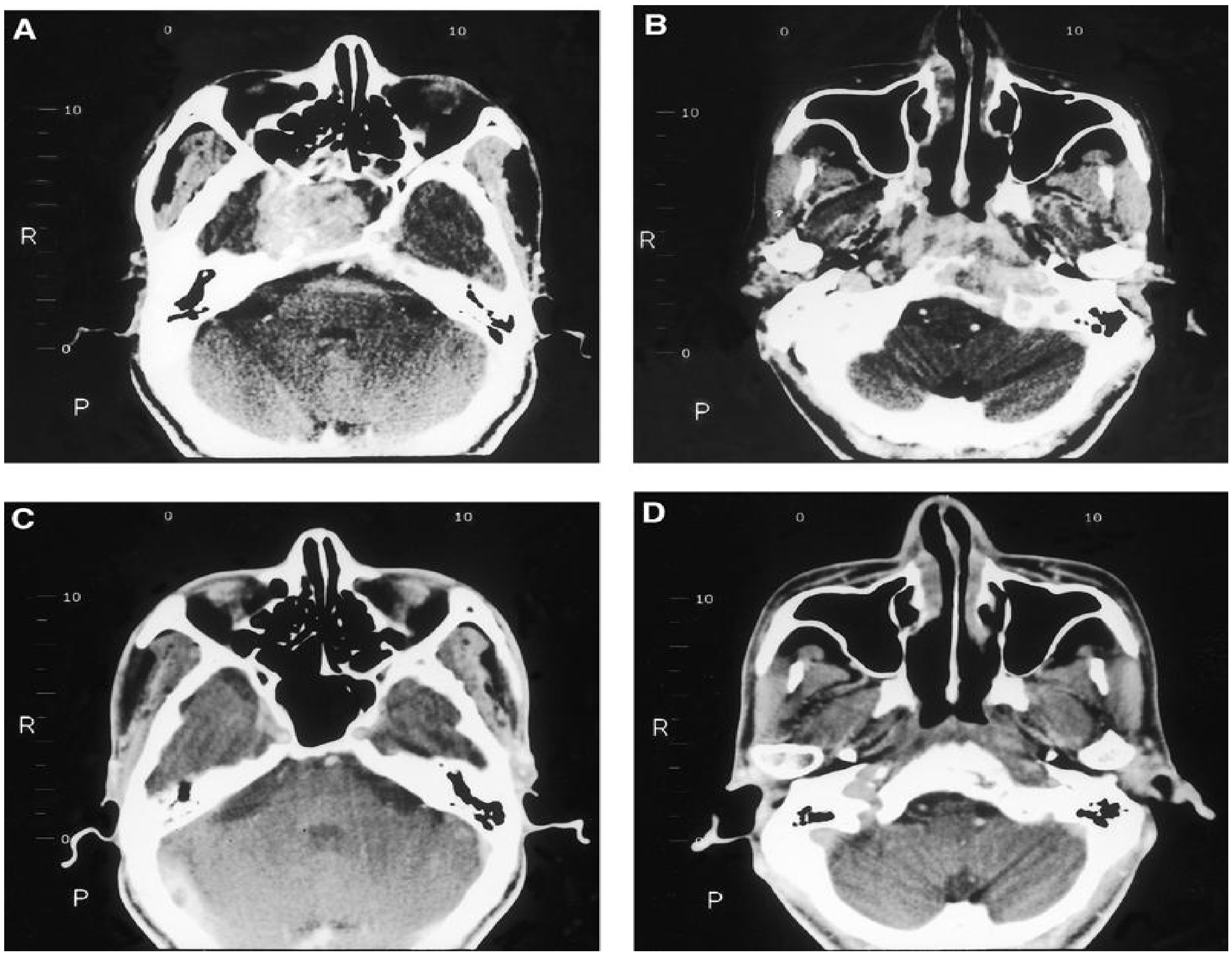
(US Food and Drug Administration)

Conventional X-Rays

(www.safetynewsalert.com)

(Percision-Upper Cervical Center of New Jersey)

For a tomograph, or a cross section, the beams are projected in the same plane as the picture.



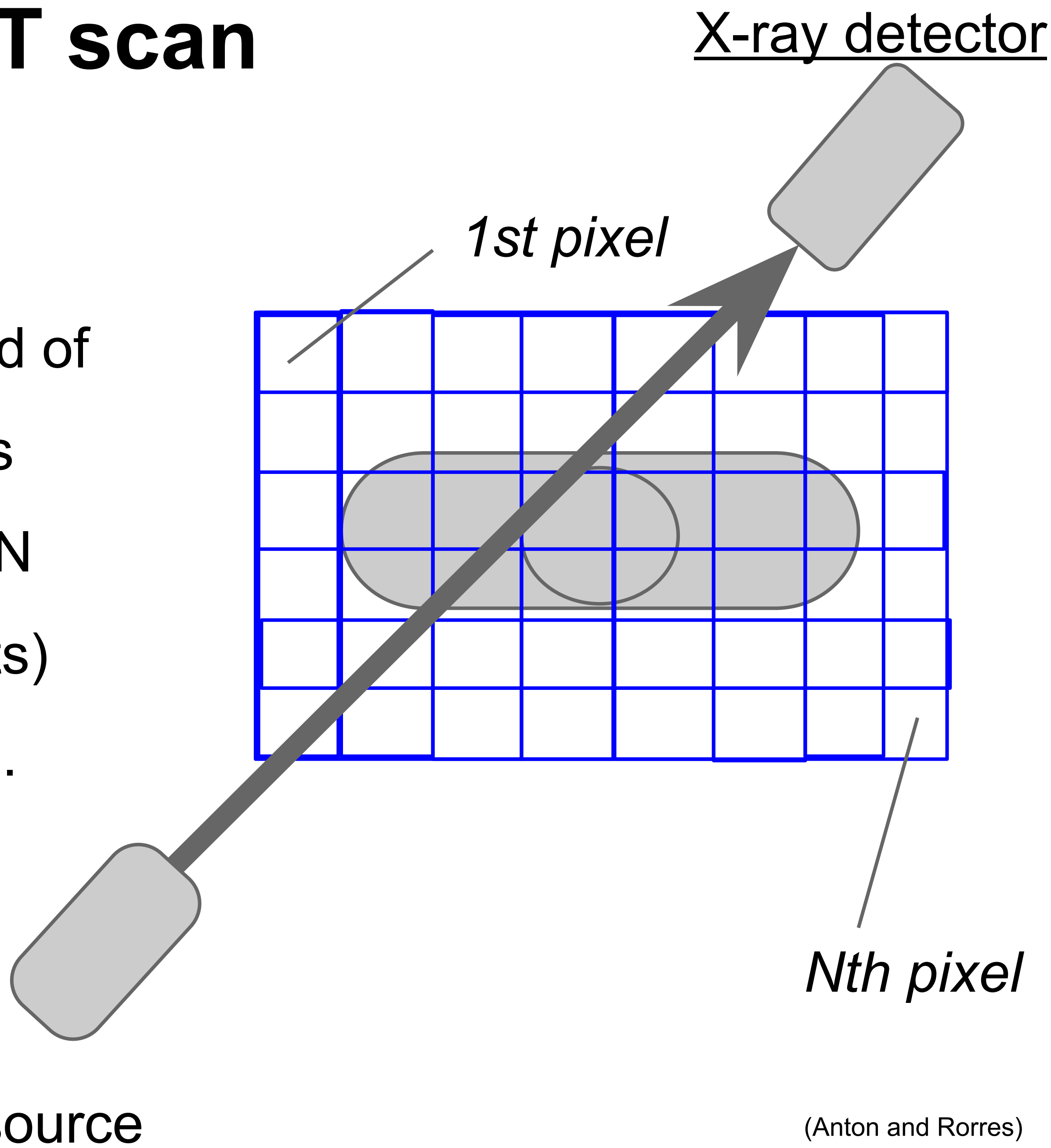
(British Journal of Cancer)

AIM:

To understand the mathematics behind how a cross section of the human body is reconstructed on a computer screen from many individual photon beam measurements.

How does a CT scan work?

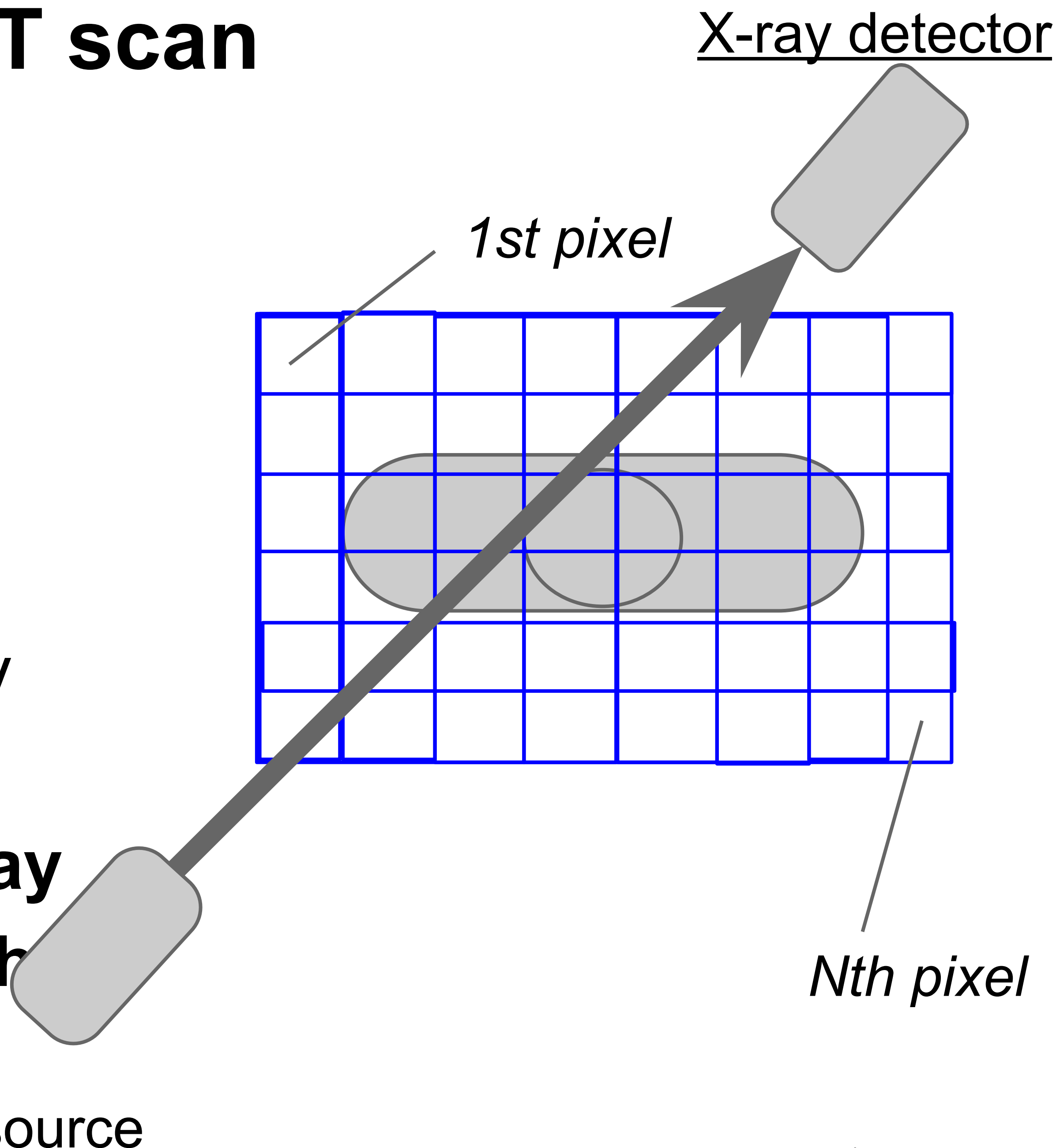
First, we divide the field of view in which the cross section is located into N pixels (picture elements) numbered 1 through N.



(Anton and Rorres)

How does a CT scan work?

1. Divide field into N-pixels
2. Send X-rays through cross section to X-ray detector
3. **Determine X-ray density of each pixel**



(Anton and Rorres)

Modes of Scanning:

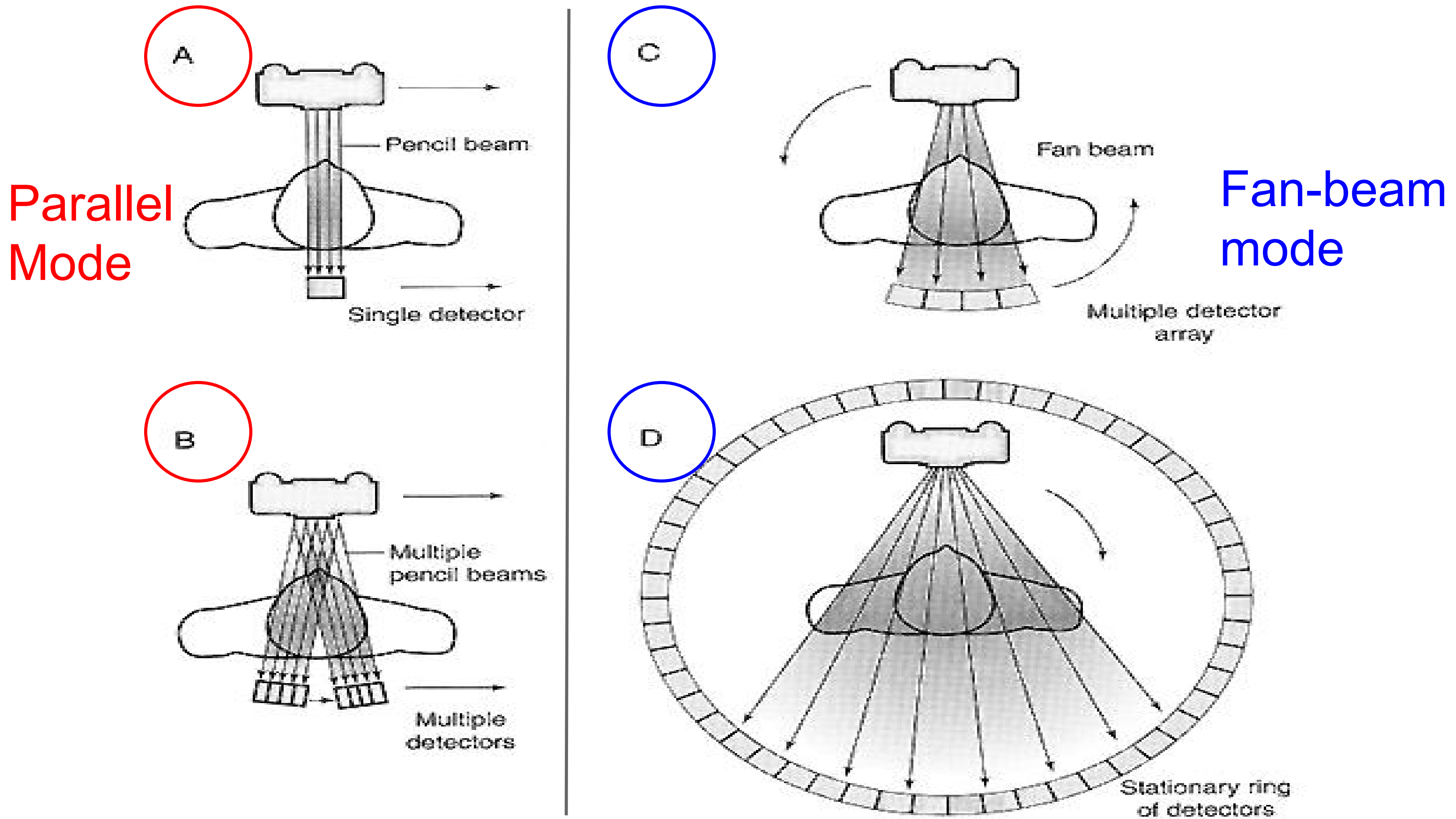


FIGURE 1-11 Four basic scanning methods or systems: **A**, first generation; **B**, second generation; **C**, third generation; **D**, fourth generation.

(capone.mtsu.edu)

The Three Basic Steps of a CT Scan:

1. The field of view containing the cross section is scanned with X-rays.
2. **The data from the X-rays is used to compute the X-ray density of each of the pixels in the field of view.**
3. The pixels are colored shades of gray proportional to their X-ray density, producing the cross sectional image on the video monitor.

(Anton and Rorres)

AIM (simplified):

To determine the X-ray density of each pixel.

What *type* of mathematics is involved?

The construction of a cross section requires the solution of a large linear system of equations.

Certain algorithms, called algebraic reconstruction techniques (ARTs) can be used to solve these linear systems whose solutions yield the cross sections in digital form.

i.e. the cross section that appears on the computer screen is **<u>the solution to a linear system of equations.</u>**

(Anton and Rorres)

Understanding a Linear System of Equations

A linear equation is an equation in which all variables have exponent equal to one.

For example,

$3x - 2y = 7$

A general linear equation with *n* variables is written as the following:

$$a_1 x_1 + a_2 x_2 + ... + a_n x_n = b$$

(Anton and Rorres)

Understanding a Linear System of Equations

A solution to a linear equation is a set of n real numbers:

$$s_1, s_2, \ldots s_n$$

such that when s_i is substituted for x_i the equation is valid.

(Anton and Rorres)

Understanding a Linear System of Equations

A <u>system of linear equations</u> - or simply <u>a linear system</u> - is a set of linear equations, **all with the same variables**.

$$
\begin{aligned}
a_{1,1}\mathbf{x}_{\mathbf{1}} + a_{1,2}\mathbf{x}_{\mathbf{2}} + \ldots + a_{1,n}\mathbf{x}_{\boldsymbol{n}} &= b_1 \\
a_{2,1}\mathbf{x}_{\mathbf{1}} + a_{2,2}\mathbf{x}_{\mathbf{2}} + \ldots + a_{2,n}\mathbf{x}_{\boldsymbol{n}} &= b_2 \\
&\vdots \\
a_{m,1}\mathbf{x}_{\mathbf{1}} + a_{m,2}\mathbf{x}_{\mathbf{2}} + \ldots + a_{m,n}\mathbf{x}_{\boldsymbol{n}} &= b_m
\end{aligned}
$$

(Anton and Rorres)

Understanding a Linear System of Equations

A general linear system of m equations and n variables is written as follows:

$$a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n = b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + ... + a_{m,n}x_n = b_m$$

m equations

n variables

(Anton and Rorres)

Understanding a Linear System of Equations

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a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n &= b_2 \\
&\vdots \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n &= b_m
\end{aligned}
$$

Each coefficient can be written in the form

$a_{i,j}$

in which
i = row number
j = column number

(Anton and Rorres)

Understanding a Linear System of Equations

Once more: **a linear system of equations is a set of equations with *all the same variables.***

$$a_{1,1}\boldsymbol{x}_{\mathbf{1}} + a_{1,2}\boldsymbol{x}_{\mathbf{2}} + \ldots + a_{1,n}\boldsymbol{x}_{\boldsymbol{n}} = b_1$$

$$a_{2,1}\boldsymbol{x}_{\mathbf{1}} + a_{2,2}\boldsymbol{x}_{\mathbf{2}} + \ldots + a_{2,n}\boldsymbol{x}_{\boldsymbol{n}} = b_2$$

$$\vdots$$

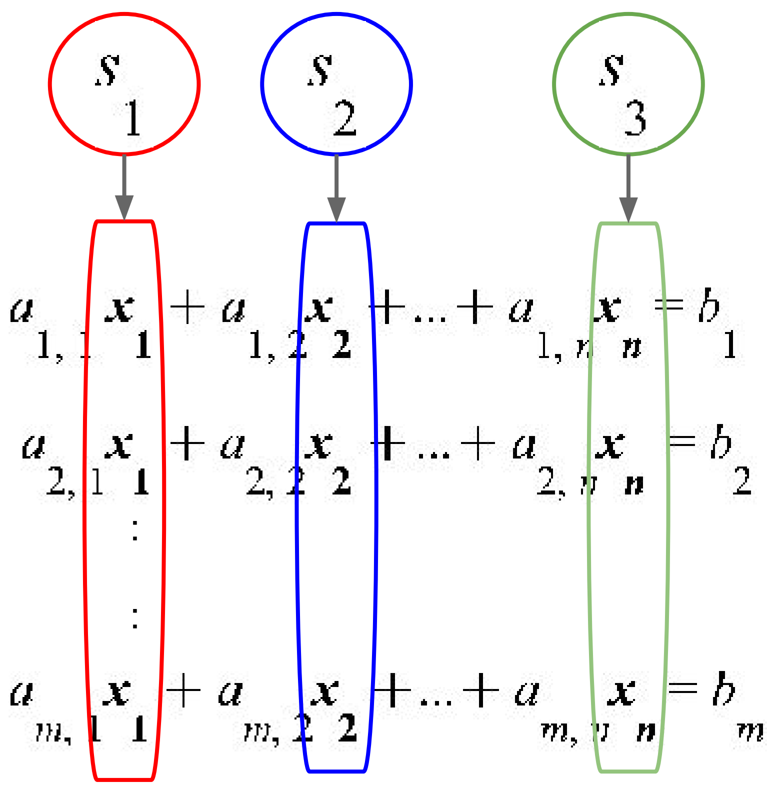
$$\vdots$$

$$a_{m,1}\boldsymbol{x}_{\mathbf{1}} + a_{m,2}\boldsymbol{x}_{\mathbf{2}} + \ldots + a_{m,n}\boldsymbol{x}_{\boldsymbol{n}} = b_m$$

(Anton and Rorres)

The Solution to a Linear System of Equations

An ordered sequence of numbers $s_{1}, s_{2}, \ldots s_{n}$ is a solution to a linear system *if it is a solution to each and every equation in the system, simultaneously.*



(Anton and Rorres)

A Note About Solutions

The solution to a linear system of equations can either be

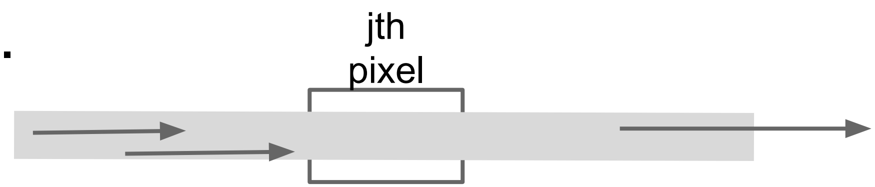
consistent
-meaning it
has exactly one or infinite solutions

inconsistent
-meaning it does not have solution, but an approximate solution can be found

(Anton and Rorres)

Determining the X-ray Density of Each Pixel

- This diagram shows a single pixel with an X-ray beam of roughly the same width as the pixel passing squarely through it.



- The photons constituting the X-ray beam are absorbed by the tissue within the pixel at a rate proportional to the X-ray density of the tissue.
- Quantitatively, the X-ray density of the jth pixel as denoted by x_j and is defined by

$$x_j = \ln\left(\frac{\text{photons entering the jth pixel}}{\text{photons leaving the jth pixel}}\right)$$

(Anton and Rorres)

Determining the X-ray Density of Each Pixel

- If the X-ray beam passes through an entire row of pixels, then the number of photons leaving one pixel is equal to the number of photons entering the next.
- If the pixels are numbered 1, 2, ...n, then the additive property of the logarithmic function gives us:

$$x_1 + x_2 + \ldots + x_n = \ln\left(\frac{\textit{photons entering the jth pixel}}{\textit{photons leaving the jth pixel}}\right)$$

- Except that instead of photons entering and leaving the jth pixel, it is the first pixel of the row over the number of photons leaving the nth pixel of the row.

(Anton and Rorres)

Determining the X-ray Density of Each Pixel

- Thus, due to the additive property of the logarithmic function, to determine the total X-ray density of a row of pixels, we simply sum the individual pixel densities.

Determining the X-ray Density of Each Pixel

- Next we consider the *beam density* of the *ith beam scan*
- Beam density is denoted by bi and represented by

$$b_i = \ln\left(\frac{\textit{number of photons of the ith beam entering the detector without the cross section in the field of view}}{\textit{number of photons of the ith beam entering the detector with the cross section in the field of view}}\right)$$

obtained from clinical calibration

(Anton and Rorres)

Determining the X-ray Density of Each Pixel

- For each beam that passes squarely through a row of n pixels

(*fraction of photons of the beam that pass through the row of pixels without being absorbed*) = (*fraction of photons of the beam that pass through the cross section without being absorbed*)

- Thus, for each jth beam we get the following equation:

Determining the X-ray Density of Each Pixel

$$x_{j_1} + x_{j_2} + \ldots + x_{j_i} = b_i$$

where bi is known from the clinical and calibration measurements, and x1, x2, ...xn are unknown pixel densities that must be determined.

(Anton and Rorres)

Determining the X-ray Density of Each Pixel

If we set

$$a_{i,j} = \begin{cases} 1, & \text{if } j = j_1, j_2, \ldots j_i \\ 0, & otherwise \end{cases}$$

In other words, if the beam passes squarely through the pixel

then we can rewrite the equation as the following:

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,N}x_N = b_1$$

Each 1 is on the left, can be replaced with i, giving us the ith beam equation.

(Anton and Rorres)

Determining the X-ray Density of Each Pixel

- Either a 1 or a 0 is not the best way to represent the coefficient a, however.
- This is because the beams of a scan do not either squarely pass through a pixel or entirely miss it.
- Instead, a typical beam passes diagonally through each pixel in its path.
- We have three ways to determine a; each one with increasing accuracy.

(Anton and Rorres)

Determining $a_{i,j}$

$$a_{i,j} = \left(\begin{array}{l} 1, \quad \text{if the ith beam passes through center} \\ 0, \text{ otherwise} \end{array} \right)$$

$$a_{i,j} = \left(\frac{\text{length of the center line of the ith beam that lies in the jth pixel}}{\text{width of the jth pixel}} \right)$$

$$a_{i,j} = \left(\frac{\text{area of the ith beam that lies in the jth pixel}}{\text{area of the ith beam that would lie in the jth pixel if it passed through squarely}} \right)$$

(Anton and Rorres)

Determining the X-ray Density of Each Pixel

- Using any of the three methods to define the a's in the ith beam equation, we can write a set of M beam equations in a complete scan as

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,N}x_N &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,N}x_N &= b_2 \\
&\vdots \\
a_{M,1}x_1 + a_{M,2}x_2 + \ldots + a_{M,N}x_N &= b_M
\end{aligned}
$$

A linear system of equations!

(Anton and Rorres)

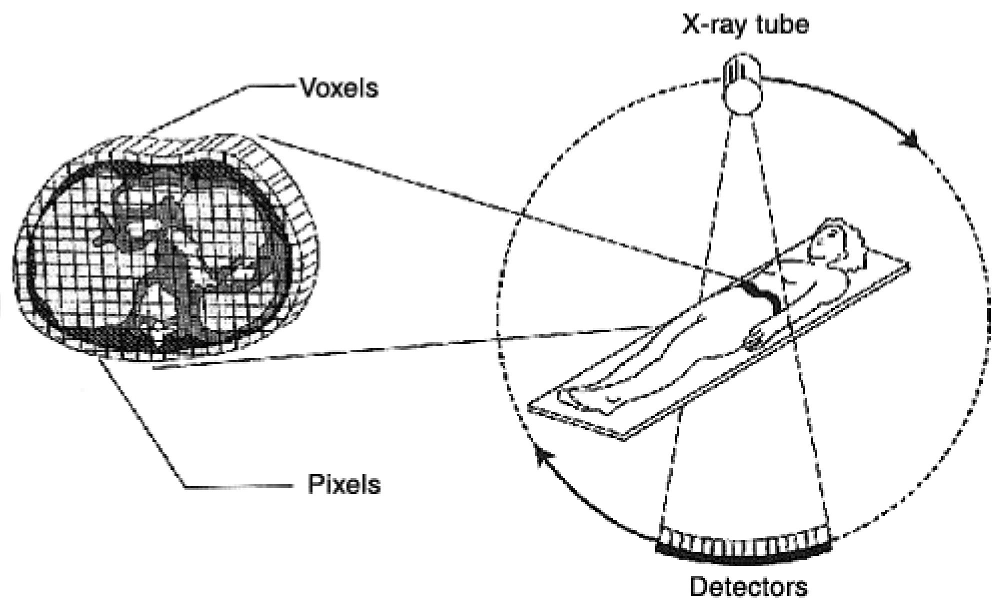
Determining the X-ray Density of Each Pixel

- In this way we have a linear system of M equations (the M beam equations) in N unknowns (the N pixel densities that field of view containing the cross section was divided into)
- The solution to this linear system of equations -determined using Algebraic Reconstruction Techniques (ARTs)- gives us the X-ray densities of each of N pixels.
- The pixels are then shaded gray proportional to their X-ray densities.

(Anton and Rorres)

Determining the X-ray Density of Each Pixel

- And we get a cross section, or a tomograph.



(Anton and Rorres), (US Food and Drug Administration), (www.physicscentral.com)

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Thank you for listening.

Questions?