Rational Functions and Their Asymptotes

About this Lesson
This lesson provides a comprehensive review of the characteristics of rational functions, including *x*- and *y*-intercepts, horizontal and vertical asymptotes, while emphasizing the concept that a horizontal asymptote describes only the end behavior of the function. Several of the functions in the lesson actually cross their horizontal asymptotes at values in the "interior" of the domain.

Prior to working this lesson, students should have some introductory experience with graphing rational functions, using intercepts and horizontal and vertical asymptotes to determine the basic shape of the graph.

This lesson is included in Module 8 – Limits.

Objectives
Students will

- determine key features of rational functions algebraically.
- graph rational functions with and without vertical asymptotes.
- recognize that horizontal asymptotes describe only the end behavior of rational functions.
- determine the locations, if any, where a rational function intersects its horizontal asymptote(s).

Level
Pre-Calculus

Common Core State Standards for Mathematical Content
This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (⋆) at the end of a specific standard indicates that the high school standard is connected to modeling.

Explicitly addressed in this lesson

Code	Standard	Level of Thinking	Depth of Knowledge
F-IF.7d (LTF extends to include intersecting the horizontal asymptote)	(+) Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.⋆	Analyze	III
A-SSE.3a	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. Factor a quadratic expression to reveal the zeros of the function it defines.⋆	Apply	II

Code	Standard	Level of Thinking	Depth of Knowledge
A-APR.3	Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.	Apply	II

Common Core State Standards for Mathematical Practice

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. LTF incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

Implicitly addressed in this lesson

Code	Standard
1	Make sense of problems and persevere in solving them.
2	Reason abstractly and quantitatively.
3	Construct viable arguments and critique the reasoning of others.
6	Attend to precision.

LTF Content Progression Chart

In the spirit of LTF's goal to connect mathematics across grade levels, the Content Progression Chart demonstrates how specific skills build and develop from sixth grade through pre-calculus. Each column, under a grade level or course heading, lists the concepts and skills that students in that grade or course should master. Each row illustrates how a specific skill is developed as students advance through their mathematics courses.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_06.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_07.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_A1.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_GE.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_A2.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_PC.LI_H.01)
				Write equations for vertical and horizontal asymptotes. (200_A2.LI_H.06)	Write equations for vertical, horizontal, and slant asymptotes. (200_PC.LI_H.06)

Connection to AP*

AP Calculus Topic: Limits

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Materials and Resources

- Student Activity pages
- Graphing calculators
- Applet that can be used to change the values of a rational function and thereby visualize and investigate its horizontal and vertical asymptotes and its graph: https://www.mrperezonlinemathtutor.com/CARFILES/algebra/Applet_Rational_Functions_Horizontal_Vertical_Asymptotes.html

Assessments

The following types of formative assessments are embedded in this lesson:

- Students engage in independent practice.
- Students summarize a process or procedure.

The following additional assessments are located on the LTF website:

- Limits – Pre-Calculus Free Response Questions
- Limits – Pre-Calculus Multiple Choice Questions

Teaching Suggestions

The lesson is divided into two parts. Part 1 introduces new levels of sophistication to the characteristics of rational functions and is designed to be teacher-led with the entire class. Part 2 incorporates functions with a variety of characteristics and may be assigned to groups or as independent practice.

A key point of emphasis in the lesson is that horizontal asymptotes describe the behavior of the function for large absolute values of x (end behavior) and that there may be points on the function where the y-values equal the value of the horizontal asymptote. In other words, a function may actually cross its horizontal asymptote at some locations in the interior of its domain.

If students have not been previously introduced to the skills of determining the vertical and horizontal asymptotes of a rational function, see the Algebra 2 lessons, "Rational Functions – Short Run Behavior," "Rational Functions – Long Run Behavior," and "Rational Functions Exploration" on the LTF website.

Modality

LTF emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using these representations to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

Answers

Part 1

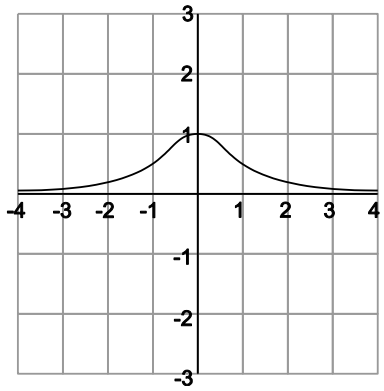
1. a. $y = 0$

   b. none

   c. no x-intercept, y-intercept = 1

   d. see graph

   e. $y = 1$, The smallest value for the denominator is when $x = 0$ or at the point (0, 1)



2. No, for example, $f(x) = \dfrac{x^2 - 1}{x - 1}$ has no vertical asymptotes.

3. a. $y = 1$

   b. $x = 0$

   c. no x or y intercept

   d. $f(-1.5) = 1$ The horizontal asymptote is $y = 1$.

   e.

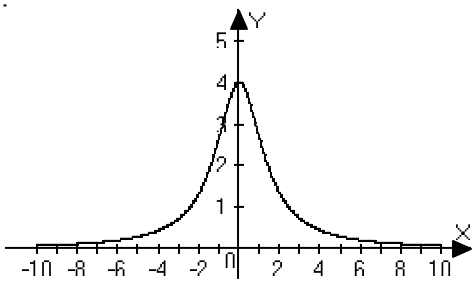
x	0	–0.5	–1	–1.5	–2	–2.5	–3	–4	–5	–6	–100
$f(x)$	none	9	2	1	.75	.68	.6667	.6875	.72	.75	.9803

4. Yes, the horizontal asymptote describes the function's behavior at the ends as x approaches positive or negative infinity. The horizontal asymptote does not describe the function's behavior in the middle of the domain.

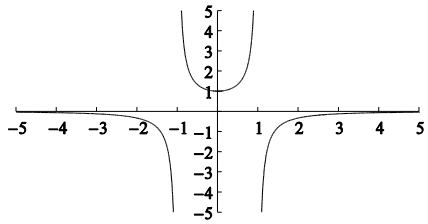
5. Set the function equal to the value of the horizontal asymptote and solve for x. If x is defined, then the function crosses the horizontal asymptote at this value of x.

Part 2

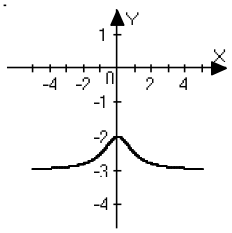
6. a. none
   b. $y = 0$
   c. none
   d. 4
   e. does not cross
   f. see graph



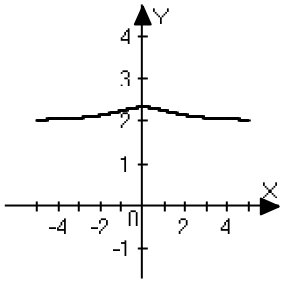
7. a. $x = -1, x = 1$
   b. $y = 0$
   c. none
   d. 1
   e. does not cross
   f. see graph



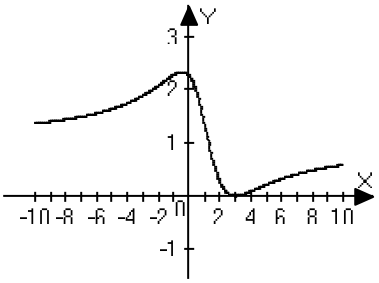
8. a. none
   b. $y = -3$
   c. none
   d. -2
   e. does not cross
   f. see graph



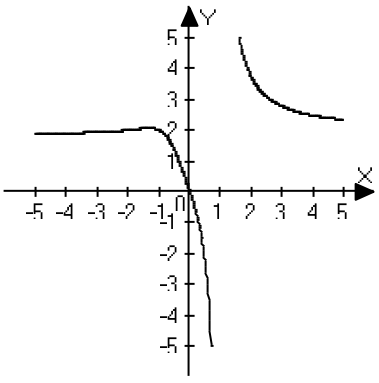
9. a. none
   b. $y = 2$
   c. none
   d. 7/3
   e. does not cross
   f. see graph



10. a. none
    b. $y = 1$
    c. 3
    d. 9/4
    e. 5/4
    f. see graph



11. a. $x = 1$
    b. $y = 2$
    c. 0
    d. 0
    e. $-2, -1$
    f. see graph



Rational Functions and Their Asymptotes

Part 1

1. Given the function $y = \dfrac{1}{x^2 + 1}$
   a. What are the horizontal asymptote(s)?
   b. What are the vertical asymptote(s)?
   c. What are the *x*- and *y*-intercepts?
   d. Graph the function.
   e. What is the maximum value of the function? Explain how you found this value.

2. Do all rational functions have vertical asymptotes? Explain your answer.

3. Given the function $f(x) = \dfrac{x^2 + 2x + 3}{x^2}$
   a. What are the horizontal asymptote(s)?
   b. What are the vertical asymptote(s)?
   c. What are the *x*- and *y*-intercepts?
   d. Determine $f(-1.5)$. What is interesting about this value?
   e. Complete the table of values.

x	0	–0.5	–1	–1.5	–2	–2.5	–3	–4	–5	–6	–100
$f(x)$											

   f. Use your graphing calculator to confirm your findings.

4. Can the graph of a rational function cross its horizontal asymptote? Explain your answer.

5. How can you determine if and where a function crosses its horizontal asymptote?

Part 2

For questions 6 – 11, complete parts a – f.

a. What is the equation of each vertical asymptote?

b. What is the equation of each horizontal asymptote?

c. Determine all *x*-intercepts.

d. Determine the *y*-intercept.

e. Does the function cross its horizontal asymptote? If so, where? Show the work that leads to your answer(s).

f. Graph each function.

6. $y = \dfrac{8}{x^2 + 2}$

7. $y = \dfrac{-1}{x^2 - 1}$

8. $y = \dfrac{1}{x^2 + 1} - 3$

9. $y = \dfrac{2x^2 + 7}{x^2 + 3}$

10. $y = \dfrac{x^2 - 6x + 9}{x^2 - 2x + 4}$

11. $y = \dfrac{2x^3 + x^2 + 3x}{x^3 - 1}$