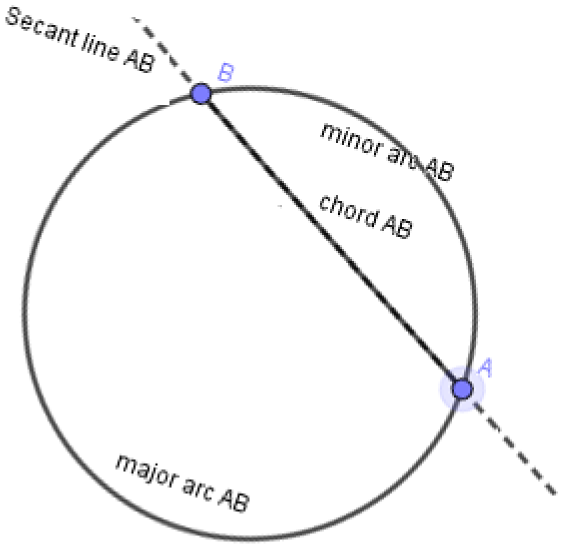
A Random Chord?

By Chris Patterson

Introduction

On 31 July [Steven Strogatz asked Twitter]() "Imagine picking a random chord on a circle. On average, how long is it compared to the circle's diameter?" What is a chord? It is that part of a secant line which is inside the circle. The chord separates the circumference into two arcs.

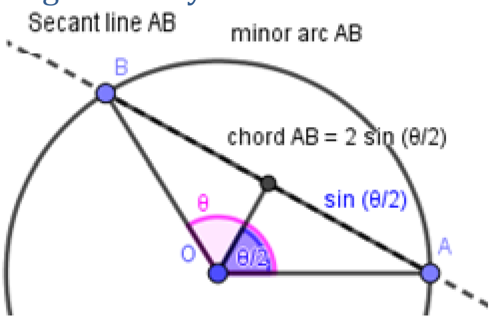


The minimum chord length will be zero, when points A and B coincide. The maximum possible chord is a diameter. Therefore, $0 \leq \overline{AB} \leq D$. It seems I will get *random chords* by drawing *random lines* through *random circles*. I decided this was too much randomness. Instead, I decided to take a different approach, making use of the symmetries of a circle.

A Table of Circle Symmetries

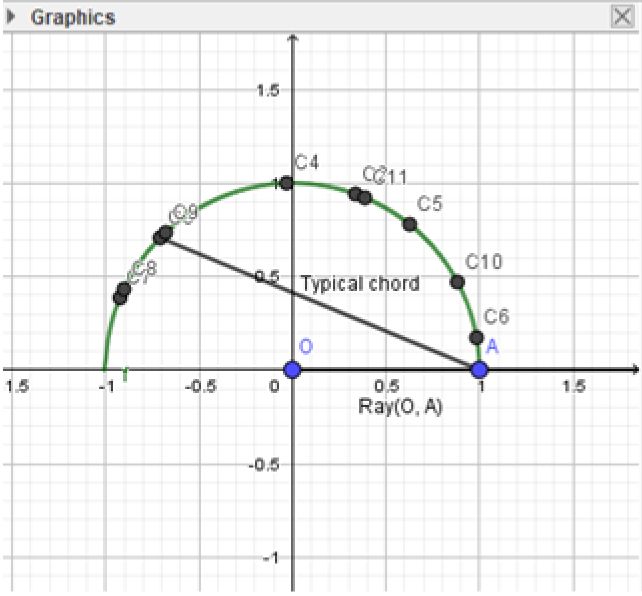
Circle Symmetry	Decision
Circles are congruent after translation	I will centre the circle on the origin.
Circles are similar under enlargement.	My circle will be a unit circle, with radius $r = 1$ radius. ("One radius" could be 2 inches, or 3 km, or 4 Martian miles, almost anything.)
Circles are congruent after rotation.	I will use polar coordinates $(r;\ \theta)$ with one endpoint of the chord on the polar ray. Hence, $A = (1;\ 0)$. Remember, $r = 1$.
Circles are congruent after reflection.	Since the maximum chord length is a diameter, I can work with the semicircle $y = \sqrt{1 - x^2}$

Trigonometry



My next thought was that sines originated in the study of chords. Rather than working with random lines, I will work with random angles. Lines are defined by two points, i.e., four random values (x_1, y_1) and (x_2, y_2). Rotating in an anticlockwise sense from the polar ray, I need a single random number, θ. Moreover, the chord formula, chord $(\theta) = 2\sin\left(\frac{\theta}{2}\right)$ gives me the length of the chord in a very nice form.

With these initial conditions I produced this Geogebra spreadsheet and its associated graph.



Spreadsheet

	A	B	C	D
1	Trial	Theta	Point B	Length
2	1	1.23	(1; 70.26°)	1.15
3	2	2.36	(1; 135.0...	1.85
4	3	1.6	(1; 91.76°)	1.44
5	4	0.89	(1; 51.15°)	0.86
6	5	0.17	(1; 9.94°)	0.17
7	6	2.75	(1; 157.3°)	1.96
8	7	2.69	(1; 154.3...	1.95
9	8	2.32	(1; 132.6...	1.83
10	9	0.49	(1; 28.06°)	0.48
11	10	1.17	(1; 67.17°)	1.11
12	Sum	15.67		
13	Average	1.57		

Note that the angles in column B are given in radians by the formula $\theta = random()\pi$. Geogebra shows these angles in degrees in column C, but I work with radians throughout. Radians are a very natural measure of angles that arise from a reinterpretation of the circumference formula, $C = 2\pi r$ and greatly simplify the calculus of trigonometric functions. I was reasonably happy with this as an initial exploration. However, I also had doubts:

- Should the length be reported as a decimal number?
- Would I recognise the significance of a particular decimal expansion? Is 1.57 simply 1.57. Or is it $\frac{\pi}{2}$?
- How many trials would be enough?
- Should I consider a statistical solution satisfactory for a problem posed in pure maths?

Algebra

A1 = "Trial"
b = 1
B1 = "Dice Toss"

Spreadsheet

A1 | Trial

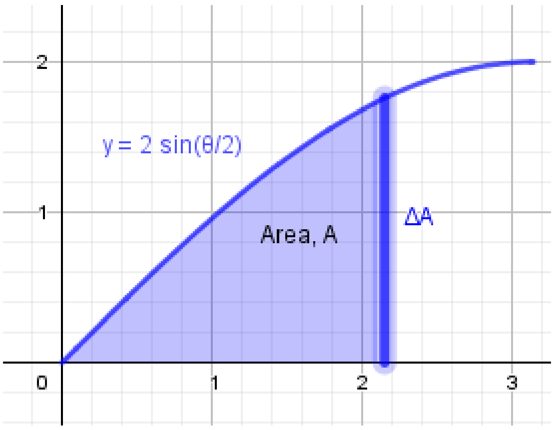
	A	B
1	Trial	Dice Toss
2	1	4
3	2	3
4	3	1
5	4	2
6	5	6
7	6	2
8	7	5
9	8	6
10	9	6
11	10	2
12		3.7
13		

Feeling stuck, I pivoted to a simpler problem. What is the average dice throw if I repeatedly toss a single, fair, six-sided dice? Is a spreadsheet the best way to answer this problem? The entries in column B are randombetween (1, 6). If I recalculated, the entries in column B would change, and therefore the average as well.

A more mathematical method considers the distribution of dice tosses. Assuming the Fates are indifferent, i) the order of tosses is of no consequence, and ii) every face should occur equally often provided I toss long enough. With these ideas in mind, I can create a theoretical table, yielding an exact result. The average toss is three and a half. Of course, no single toss can be 3½, but that is merely a reminder to be careful when thinking about averages.

Toss, x	Probability, $P(x)$	Product: $x\,P(x)$
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	6/6
Sum	6/6	21/6 = 3½

Can I create a similar table for the chord length distribution? The immediate problem is that chord lengths are distributed continuously, unlike the discrete dice tosses. There were only six possible values for the dice toss; there is an infinite number of different chord lengths. A mathematical maxim, when dealing with infinities, try calculus! Accordingly, I turned again to Geogebra to create the graph of $y = 2\sin\left(\frac{\theta}{2}\right)$ on the domain $0 \le \theta \le \pi$ radians.



Why? Although I cannot add up every y-value, I can find the area between $y = 2\sin\left(\frac{\theta}{2}\right)$ and $y = 0$ from $x = 0$ to $x = \theta$. One typical area, A, is shown shaded in light blue together with a small adjoining darker blue rectangle, ΔA. The width of this rectangle is $\Delta\theta$, a small increment in θ, perhaps 0.1 radians. The height of the rectangle is $y + \Delta y$. The area of a rectangle is length times width. Hence,

$$\begin{aligned}\Delta A &= \Delta\theta \times (y + \Delta y)\\ &= y\Delta\theta + \Delta y\,\Delta\theta\end{aligned}$$

As I make $\Delta\theta$ smaller, Δy also becomes smaller. Importantly, both the terms of the sum, $y\Delta\theta$ and$\Delta y\,\Delta\theta$, become smaller, and the latter term is always much smaller than the former. See the table below.

Spreadsheet

D11

	A	B	C	D	E
1	Δθ	y	Δy	yΔθ	ΔyΔθ
2	0.1	1.75	0.0545351882	0.175	0.0054535188
3	0.04	1.75	0.0278564977	0.07	0.0011142599
4	0.007	1.75	0.0124996448	0.01225	0.0000874975
5	0.0002	1.75	0.0092754385	0.00035	0.0000018551
6	0.00003	1.75	0.0091945726	0.0000525	0.0000002758
7	...				
8					

Eventually, $\Delta y\ \Delta\theta$ becomes of no consequence. This yields the result I want:

$$\Delta A \to dA = y\,d\theta + 0 = y\,d\theta.$$

We call 'dee-A' and 'dee-theta' infinitesimals. (Disclaimer: the argument above is not a proof; please consult any standard calculus text.) I can rearrange $dA = y\,d\theta$ to make y the subject of the equation:

$$\frac{dA}{d\theta} = y$$

Hence, $\frac{dA}{d\theta} = 2\sin\left(\frac{\theta}{2}\right)$.

Amazing! Now I can simply look for an already solved problem in differential calculus and I will have a formula for the area, A, as a function of the angle, θ. *This insight, in the late 1600s, by Newton and Leibnitz changed everything.* If you know the chain rule and remember $\frac{d\cos x}{dx} = -\sin x$ you will have already jumped ahead of me; $A = -4\cos\left(\frac{\theta}{2}\right) + k$. Or, if you have yet to learn calculus, perhaps this essay will provide sufficient motivation. After doing a little bit of arithmetic, I obtain the result that

$$A = \left(-4\cos\left(\frac{\pi}{2}\right) + k\right) - \left(-4\cos(0) + k\right) = 0 - (-4) + k - k = 4 \text{ units}$$

Since A is an area, the units will be the product of the y-axis units and the x-axis units. The y-axis is measured in radiuses. Remember $r = 1$ was an initial choice and the x-axis is measured in radians, or, in a sense, anticlockwise distances along the unit circle circumference in fractions of π. Thus, the average chord will be Area, A, divided by the angle, θ. Some more calculus: $\int_0^{\pi} 1\,d\theta = \pi - 0 = \pi$. As this calculation is peripheral to my essay, I will again suggest you consult a textbook for further details.

Drum roll! My average random chord has a length of $\frac{4}{\pi}$. (I was right to mistrust decimal expansions.) Using the approximation $\pi = \frac{22}{7}$, the average chord is $\frac{14}{11}$ units. As $r = 1$, D must be 2 units. The average chord is about $\frac{7}{11}$ of a diameter.

Reflection:

Upon arriving at a solution, it is good to go back and look at the starting point, which was a vague notion of *random chords* obtained by drawing *random lines* through *random circles*. As this seemed intractable, I created a similar problem I believed I could solve. The tools I had 'at hand' included symmetries, polar coordinates, trigonometry, probability distributions, radians, and *especially* calculus.

I have no doubt that my solution is correct, given the constraints I imposed. The larger question, however, is *did those constraints diminish or change the original problem?* My suspicions were that *other* procedures for picking a random chord would generate *other* distributions with averages *othe*r than $\frac{4}{\pi}$. Twitter confirmed these suspicions. Dr Strogatz was not simply setting a problem in calculus; he was asking us to think about the foundations of probability theory.

Feedback

Your comments are welcome, particularly

• constructive criticism,

• suggestions for future blog posts.