[MAA 1.6] BINOMIAL THEOREM

SOLUTIONS

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O. Practice questions

1. (a) $(1 \pm x)^3 = 1 \pm 3x + 3x^2 \pm x^3$

(b) $(1 \pm x)^4 = 1 \pm 4x + 6x^2 \pm 4x^3 + x^4$

(c) $(1 \pm x)^5 = 1 \pm 5x + 10x^2 \pm 10x^3 + 5x^4 \pm x^5$

2. (a) $(1+x)^{10} = 1 + 10x + 45x^2 + 120x^3 + \cdots$

(b) $(1+2x)^{10} = 1 + 20x + 180x^2 + 960x^3 + \cdots$

(c) $(2-x)^{10} = 1024 - 5120x + 11520x^2 - 15360x^3 + \cdots$

3. $-120x^7$

4. $45x^{16}$

5. (a) (i) $\left(x - \frac{1}{x}\right)^3 = x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$ (ii) $\left(x - \frac{1}{x}\right)^4 = x^4 - 4x^2 + 6 - \frac{4}{x^2} - \frac{1}{x^4}$

(b) (i) $-\binom{10}{5} = -252$ (ii) $\binom{10}{4} = 210$ (iii) $\binom{10}{6} = 210$

6. (a) $(1+2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4$ (b) $8x^2$ (c) $32x^2$ (d) $71x^2$

7. (a) (show) (b) (i) 10 (ii) 45 (iii) 120

8. (a) (show) (b) $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$

9. (a) 43110144 (b) 51963120
(c) $531441x^{12} - 4251528x^{10} + 15588936x^8 - 34642080x^6 + \cdots$

10. (a) 10264320 (b) 43110144 (c) $531441x^{24} - 425152x^{21} + 15588936x^{18} + \cdots$

11. (a) -3041280 (b) 10264320 (c) $531441x^{36} - 425152x^{32} + 15588936x^{28} + \cdots$

A. Exam style questions (SHORT)

12. $\left(\sqrt{3} + \sqrt{2}\right)^3 = \cdots = 9\sqrt{3} + 11\sqrt{2}$

13. $\left(\sqrt{3} + \sqrt{2}\right)^4 = \cdots = 49 + 20\sqrt{6}$

14.

$$\left(\sqrt{3} - 2\right)^3 = \left(\sqrt{3}\right)^3 + 3\left(\sqrt{3}\right)^2(-2)^1 + 3\left(\sqrt{3}\right)(-2)^2 + (-2)^3$$

$$= 3\sqrt{3} + (-18) + 12\sqrt{3} - 8$$

$$= 15\sqrt{3} - 26 \quad \text{(accept } a = 15,\ b = -26\text{)}$$

15. **METHOD 1**

Using binomial expansion

$$\left(3+\sqrt{7}\right)^3 = 3^3 + \binom{3}{1}3^2\left(\sqrt{7}\right)+\binom{3}{2}3\left(\sqrt{7}\right)^2+\left(\sqrt{7}\right)^3 = 27 + 27\sqrt{7} + 63 + 7\sqrt{7}$$

$$\left(3+\sqrt{7}\right)^3 = 90+34\sqrt{7} \qquad (\text{so } p = 90 \text{ , } q = 34 \text{)}$$

METHOD 2

For multiplying

$$\left(3+\sqrt{7}\right)^2\left(3+\sqrt{7}\right) = \left(9+6\sqrt{7}+7\right)\left(3+\sqrt{7}\right) = 27 + 9\sqrt{7} + 18\sqrt{7} + 42 + 21 + 7\sqrt{7}$$

$$\left(3+\sqrt{7}\right)^3 = 90+34\sqrt{7} \qquad (\text{so } p = 90 \text{ , } q = 34 \text{)}$$

16. (a) $$\left(\mathrm{e}+\frac{1}{\mathrm{e}}\right)^4 = \binom{4}{0}\mathrm{e}^4+\binom{4}{1}\mathrm{e}^3\left(\frac{1}{\mathrm{e}}\right)+\binom{4}{2}\mathrm{e}^2\left(\frac{1}{\mathrm{e}}\right)^2+\binom{4}{3}\mathrm{e}\left(\frac{1}{\mathrm{e}}\right)^3+\binom{4}{4}\left(\frac{1}{\mathrm{e}}\right)^4$$

$$\left(\mathrm{e}+\frac{1}{\mathrm{e}}\right)^4 = \mathrm{e}^4+4\mathrm{e}^3\left(\frac{1}{\mathrm{e}}\right)+6\mathrm{e}^2\left(\frac{1}{\mathrm{e}}\right)^2+4\mathrm{e}\left(\frac{1}{\mathrm{e}}\right)^3+\left(\frac{1}{\mathrm{e}}\right)^4 = \mathrm{e}^4+4\mathrm{e}^2+6+\frac{4}{\mathrm{e}^2}+\frac{1}{\mathrm{e}^4}$$

(b) $$\left(\mathrm{e}-\frac{1}{\mathrm{e}}\right)^4 = \mathrm{e}^4-4\mathrm{e}^3\left(\frac{1}{\mathrm{e}}\right)+6\mathrm{e}^2\left(\frac{1}{\mathrm{e}}\right)^2-4\mathrm{e}\left(\frac{1}{\mathrm{e}}\right)^3+\left(\frac{1}{\mathrm{e}}\right)^4 = \mathrm{e}^4-4\mathrm{e}^2+6-\frac{4}{\mathrm{e}^2}+\frac{1}{\mathrm{e}^4}$$

Adding gives $2\mathrm{e}^4 + 12 + \frac{2}{\mathrm{e}^4}$

17. $... + 6 \times 2^2(ax)^2 + 4 \times 2(ax)^3 + (ax)^4 = ...+ 24a^2x^2 + 8a^3x^3 + a^4x^4$

18. $$(3x + 2y)^4 = (3x)^4 + \binom{4}{1}(3x)^2(2y) + \binom{4}{2}(3x)^2(2y)^2 + \binom{4}{3}(3x)(2y)^3 + (2y)^4$$

$= 81x^4 + 216x^3y + \mathbf{216x^2y^2 + 96xy^3 + 16y^4}$

19. (i) $$\binom{7}{3} = \frac{7!}{3!\times 4!} = \frac{5\times 6\times 7}{1\times 2\times 3} = 35$$

(ii) $$\binom{200}{2} = \frac{200!}{2!\times 198!} = \frac{199\times 200}{2} = 199\times 100 = 19900$$

20. (a) LHS = $\binom{5}{2}+\binom{5}{3} = \frac{5!}{2!\times 3!}+\frac{5!}{3!\times 2!} = 10+10 = 20$, RHS = $\binom{6}{3} = \frac{6!}{3!\times 3!} = 20$

(b) LHS= $\binom{19}{9}+\binom{19}{10} = \frac{19!}{9!\times 10!}+\frac{19!}{9!\times 10!} = \frac{2\times 19!}{9!\times 10!} = \frac{2\times 19!}{9!\times 10!}$

RHS= $\binom{20}{10} = \frac{20!}{10!\times 10!} = \frac{20\times 19!}{10\times 9!\times 10!} = \frac{2\times 19!}{9!\times 10!}$

21. $(a + b)^{12}$

Coefficient of a^5b^7 is $\binom{12}{5} = \binom{12}{7} = 792$

22. $(5a+b)^7 = \ldots + \binom{7}{4}(5a)^3(b)^4 + \ldots = \dfrac{7\times 6\times 5\times 4}{1\times 2\times 3\times 4} \times 5^3 \times (a^3b^4) = 35 \times 5^3 \times a^3b^4$

So the coefficient is 4375

23. (a) $n = 10$

(b) $a = p$, $b = 2q$ (or $a = 2q$, $b = p$)

(c) $\binom{10}{5}p^5(2q)^5$

24. The required term is $\binom{10}{7}2^{10-7}\,3^7$

$= 2\,099\,520$

25. (a) 12 terms

(b) (10th term, $r = 9$, $\binom{11}{9}(x)^2\,(2)^9$)

$\binom{11}{9}\,(x)^2\,(2)^9 = 55\times 2^9\,x^2 = 28160x^2$

26. Term involving x^3 is $\binom{5}{3}\,(2)^2(-x)^3 = -40x^3 \Rightarrow$ The coefficient is -40

27. Required term is $\binom{8}{5}(3x)^5(-2)^3$

Therefore the coefficient of x^5 is $56 \times 243 \times -8 = -108864$

28. $\binom{8}{3}(2)^5\,(-3x)^3$

Term is $-48\,384x^3$

28. The coefficient of x^3 is $\binom{8}{3}\left(\dfrac{-1}{2}\right)^3$

The coefficient of x^3 is -7

30.

Term in $x^3 = {}^6C_3 \times 2^3\left(\dfrac{-3x}{2}\right)^3$

$= 20\times 8\times\left(-\dfrac{27x^3}{8}\right)$

$= -540x^3$

(Coefficient of $x^3 = -540$)

31. $\binom{8}{5}\left(\dfrac{2}{3}x\right)^3\,(-3)^5 = -4032x^3$

32. $a = \binom{10}{8}\times 2^2 = 45 \times 4 = 180$

33. $\binom{7}{2}5^2(2x^2)^5 = 16\,800\,x^{10}$

34. (a) 6 terms

(b) the fourth term is $\binom{5}{3}(-2)^3\left(x^2\right)^2 = 10\,(-8)\,x^4 = -80x^4$ hence $A=-80$

35. (a) 7 terms

(b) $\binom{6}{3}\left(x^3\right)^3(-3x)^3$

$\binom{6}{3}=20, (-3)^3=-27$

Term is $-540x^{12}$

36. $\binom{5}{2}(3x^2)^3\left(\frac{-2}{x}\right)^2\ \binom{5}{2},\ 27x^6, \frac{4}{x^2};\ 10(3x^2)^3\left(\frac{-2}{x}\right)^2 = 10\times27\times4\,x^4$

term = $1080x^4$

37. (a) 10

(b) $\binom{9}{6}(3x^2)^3\left(-\frac{1}{x}\right)^6 = 84\times3^3x^6\,\frac{1}{x^6}$

constant =2268

38. $\binom{9}{3}x^6\left(\frac{-2}{x^2}\right)^3 = 84x^6\left(\frac{-8}{x^6}\right) = -672$

39. $\binom{6}{3}\left(x^2\right)^3\left(-\frac{2}{x^2}\right)^3 = -160$

40.

$\binom{5}{2}(3x)^3\left(\frac{-2}{x}\right)^2$

$=10(27x^3)\left(\frac{4}{x^2}\right)$

$=1080x$

Coefficient is 1080

41. $\binom{10}{3}2^7(ax)^3 \quad \left(\text{accept}\binom{10}{7}\right)$

$120\times2^7a^3 = 414\,720$

$a^3 = 27 \Leftrightarrow a=3$

42. $T_{r+1} = \binom{7}{r}x^{7-r}\left(\frac{1}{ax^2}\right)^r = \binom{7}{r}\left(\frac{1}{a}\right)^r x^{7-3r}$

$7-3r=1 \Rightarrow r=2$

Now, $\binom{7}{2}\frac{1}{a^2} = \frac{7}{3} \Rightarrow a^2 = 9 \Rightarrow a = \pm3$

43. (a) $(x-2)^4 = x^4 + 4x^3(-2) + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4$

$(x-2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16$

(b) finding coefficients, 3×24 $(= 72)$, $4 \times (-8)(= -32)$

term is $40x^3$

44. (a) $2^4 + 4(2^3)x + 6(2^2)x^2 + 4(2)x^3 + x^4$, $(4 + 4x + x^2)(4 + 4x + x^2)$

$(2 + x)^4 = 16 + 32x + 24x^2 + 8x^3 + x^4$

(b) finding coefficients 24 and 1

term is $25x^2$

45. (a) $(2 + x)^5 = 25 + 5(2)^4(x) + 10(2)^3x^2 + 10(2)^2x^3 + 5(2)x^4 + x^5$

$= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

(b) Let $x = 0.01$

$\Rightarrow (2.01)^5 = 32 + 0.8 + 0.00\,8 + 0.00\,00\,4 + 0.00\,00\,00\,1 + 0.00\,00\,00\,00\,01$

$= 32.8080401001$

(c) Let $x = -0.01$

$\Rightarrow (1.99)^5 = 32 - 0.8 + 0.00\,8 - 0.00\,00\,4 + 0.00\,00\,00\,1 - 0.00\,00\,00\,00\,01$

$= 31.2079600999$

46.

. **METHOD 1**

constant term: $\binom{5}{0}(-2x)^0\binom{7}{0}x^0 = 1$

term in x: $\binom{7}{1}x + \binom{5}{1}(-2x) = -3x$

term in x^2: $\binom{7}{2}x^2 + \binom{5}{2}(-2x)^2 + \binom{7}{1}x\binom{5}{1}(-2x) = -9x^2$

METHOD 2

$$(1-2x)^5(1+x)^7 = \left(1 + 5(-2x) + \frac{5\times 4(-2x)^2}{2!} + \ldots\right)\left(1 + 7x + \frac{7\times 6}{2}x^2 + \ldots\right)$$
$$= (1 - 10x + 40x^2 + \ldots)(1 + 7x + 21x^2 + \ldots)$$
$$= 1 + 7x + 21x^2 - 10x - 70x^2 + 40x^2 + \ldots$$
$$= 1 - 3x - 9x^2 + \ldots$$

47. Given $(1 + x)^5(1 + ax)^6 = 1 + bx + 10x^2 + \ldots + a^6x^{11}$

$\Leftrightarrow$ $(1 + 5x + 10x^2 + \ldots)(1 + 6ax + 15a^2x^2 + \ldots) = 1 + bx + 10x^2 + \ldots + a^6x^{11}$

$\Leftrightarrow$ $1 + (6a + 5)x + (15a^2 + 30a + 10)x^2 + \ldots = 1 + bx + 10x^2 + \ldots + a^6x^{11}$

$\Leftrightarrow$ $6a + 5 = b$ ①

$15a^2 + 30a + 10 = 10$ ②

② $\Rightarrow$ $15a^2 + 30a = 0$

$\Leftrightarrow$ $15a(a + 2) = 0$

$\Rightarrow$ $a = -2$

Substitute into ① $b = -7$

Note: *$a \neq 0$ since $a \in \mathbb{Z}^*$*

48.

(a) evidence of expanding

e.g. $2^4+4(2^3)x+6(2^2)x^2+4(2)x^3+x^4$, $(4+4x+x^2)(4+4x+x^2)$

$(2+x)^4=16+32x+24x^2+8x^3+x^4$

(b) finding coefficients 24 and 1

term is $25x^2$

49 The terms of $(2+x)^4$ involve $x^4,\ x^3,\ x^2,\ x^1,\ x^0=1$

The terms of $(1+x^2)^5$ involve $x^{10},\ x^8,\ x^6,\ x^4,\ x^2,\ x^0=1$

The combinations that give x^3 are shown above.

The first combination is $\binom{4}{3}2^1x^3\ \times\ \binom{5}{5}1^5(x^2)^0\ =\ 8x^3\times\ \ 1\ =\ 8x^3$

The second combination is $\binom{4}{1}2^3x^1\ \times\ \binom{5}{4}1^4(x^2)^1\ =\ 32x\ \times\ 5x^2\ =\ 160x^3$

Since $8x^3+160x^3=168x^3$, the coefficient of x^3 is 168.

50. The terms of $(2+x)^4$ involve $x^4,\ x^3,\ x^2,\ x^1,\ x^0=1$

The terms of $\left(2x+\dfrac{1}{x}\right)^5$ involve $x^5,\ x^3,\ x^1,\ x^{-1},\ x^{-3},\ x^{-5}$

The combinations that give constant term are shown above.

The first combination is $\binom{4}{3}2^1x^3\ \times\ \binom{5}{4}(2x)^1(\dfrac{1}{x})^4\ =\ 8x^3\times\ \dfrac{10}{x^3}\ =\ 80$

The second combination is $\binom{4}{1}2^3x^1\ \times\ \binom{5}{3}(2x)^2(\dfrac{1}{x})^3\ =\ 32x\times\ \dfrac{40}{x}\ =1280$

Hence the constant term is 80+1280 = 1360.

51. (a) $2x^2+x-3=(2x+3)(x-1)$

Note: Accept $2\left(x+\dfrac{3}{2}\right)(x-1)$. Either of these may be seen in (b) and A1 should be awarded.

(b) **EITHER**

$(2x^2+x-3)^8=(2x+3)^8(x-1)^8$

$=(3^8+8(3^7)(2x)+\dots)((-1)^8+8(-1)^7(x)+\dots)$

coefficient of $x=3^8\times8\times(-1)^7+3^7\times8\times2\times(-1)^8=-17\,496$

OR

$(2x^2+x-3)^8=(3-(x-2x^2))^8=3^8+8(-(x-2x^2)(3^7)+\dots)$

coefficient of $x=8\times(-1)\times3^7=-17\,496$

52. (b) 61236 and 30618

53. We expand up to the third term

$$(1+ax)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(ax) + \binom{n}{2}1^{n-2}(ax)^2 + \cdots$$

$$= 1 + nax + \frac{n(n-1)}{2}a^2x^2 + \cdots$$

Hence $na = 24$ and $\frac{n(n-1)}{2}a^2 = 252 \Leftrightarrow (n^2-n)a^2 = 504$

The solution if the system is $a = 3$ and $n = 8$.

54. (a) coefficient of x^3 is $\binom{n}{3}\left(\frac{1}{2}\right)^3 = 70$

$$\frac{n!}{3!(n-3)!} \times \frac{1}{8} = 70$$

$$\Rightarrow \frac{n(n-1)(n-2)}{48} = 70$$

$n = 16$

(b) $\binom{16}{2}\left(\frac{1}{2}\right)^2 = 30$

B. Exam style questions (LONG)

55. (a) $(1+1)^4 = 2^4 = 1 + \binom{4}{1}(1) + \binom{4}{2}(1^2) + \binom{4}{3}(1^3) + 1^4$

$$\Rightarrow \binom{4}{1} + \binom{4}{2} + \binom{4}{3} = 16 - 2 = 14$$

(b) For $x = -1$

$$0 = 1 - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + 1 \Rightarrow \binom{4}{1} - \binom{4}{2} + \binom{4}{3} = 2$$

(c) The expansion of $(1+x)^9$ gives

$$(1+1)^9 = 1 + \binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \ldots + \binom{9}{8} + 1$$

$$\Rightarrow \binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \ldots + \binom{9}{8} = 2^9 - 2 = 510$$

(d) For $x = -1$

$$1 - \binom{9}{1} + \binom{9}{2} - \binom{9}{3} + \binom{9}{4} - \binom{9}{5} + \binom{9}{6} - \binom{9}{7} + \binom{9}{8} - 1 = 0$$

$$\Rightarrow \binom{9}{1} + \binom{9}{3} + \binom{9}{5} + \binom{9}{7} = \binom{9}{2} + \binom{9}{4} + \binom{9}{6} + \binom{9}{8}$$

56. (a) $a^2 + b^2 = S^2 - 2P$ (b) $a^3 + b^3 = S^3 - 3SP$ (c) $a^4 + b^4 = S^4 - 4PS^3 - 12SP^2 + 6P^2$