RECTIFYING THE CIRCUMFERENCE WITH GEOGEBRA

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1 Introducction

The relation between the circumference and the radius of a circle is one of the most important concepts explained during the first years in the academic path of a math student. This relation is not a trivial concept and during centuries has fascinated and get the attention of mathematicians and scientists from the ancient Egypt to the modern supercomputers whose power has been tested by calculating digits of π.
Archimedes created an algorithm for calculating it in his book Measurement of the circle but it was Leonard Euler who in 1737 introduced the use of the Greek letter π to refer to the number we know as *pi*. [1]. Introducing the relation to Elementary and Secondary students for the first time can be a complex task. As a matter of fact the comprehension of pi took centuries to mankind but nowadays the programs of study spend short time explaining it. Nowadays new technologies can be extremely useful in explaining this relation and Geogebra is one of the mathematic digital tools that stands out among the others for this purpose. The construction of the animation explained in this article provides a valuable didactic resource to facilitate the students comprehension of the relation between the circumference and the diameter of a circle.

In 1882, the German mathematician Ferdinand Lindermann proved that π is a transcendental number, meaning it is not a root of any polynomial with rational coefficients. The most important consequence of this is the impossibility of solving one of the three unsolved problems of ancient Greece: *squaring the circle*, also known as *the quadrature of a circle*. The transcendence of pi implies the impossibility of *circling* the square, as well as squaring the circle by compass and straightedge. A circle can be squared but not exclusively by using compass and straightedge. [2] The impossibility of squaring the circle implies the impossibility of rectifying the circumference, as we are going to explain a segment of length equivalent to the length of a circumference can't be constructed by compass and straightedge. The impossibility of solving the other two problems of ancient Greece, trisecting an angle and doubling a cube, is deduced by other methods.

### 1.1 The square of the circle and the rectifying of the circumference

The impossibility of squaring the circle implies the impossibility of rectifying the circumference

Consider the following geometric process:
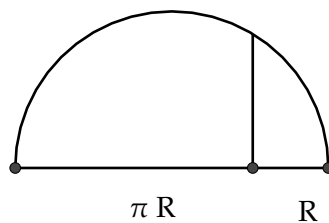
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[1]https://en.wikipedia.org/wiki/Pi
[2]https://en.wikipedia.org/wiki/Squaring_the_circle

1. Construct a line segment of length $\pi \cdot R + R$, and a semicircle with center in its middle point (It is assumed that $R > 0$)

π R   R
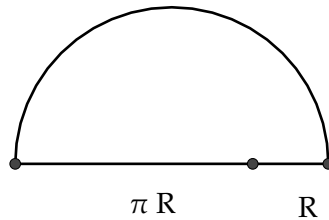
2. Construct a line segment perpendicular to the previous segment with origin in $\pi \cdot R$ and end in the intersection with the semicircumference.
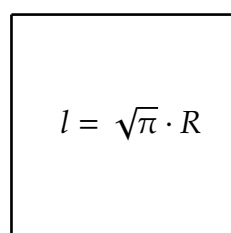

π R   R

3. The drawing shows a right triangle. The angle corresponding to the A vertex is a right angle.

π R   R
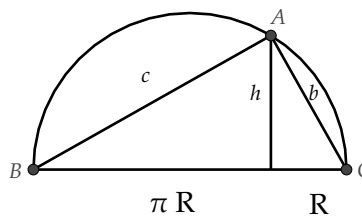
As the line segment h is the height of the triangle using the geometric mean theorem we have that:

$$h^2 = \pi \cdot R \cdot R = \pi \cdot R^2 \implies h = \sqrt{\pi} \cdot R$$

4. Constructing a square of side $l = \sqrt{\pi} \cdot R$

$l = \sqrt{\pi} \cdot R$

5. The surface of the square would be $S = (\sqrt{\pi \cdot R})^2 = \pi \cdot R^2$. Therefore we have constructed a square whose surface is the same than a circle of radius R.

It seems that we achieved our goal: the square of the circle. Where is the problem? Have we used strictly compass and straightedge to draw the previous construction? Lindemann did not proved that the square of the circle was impossible, he proved that it is not possible by using only compass and straightedge. If in the previous construction the drawing of a line segment of length $\pi \cdot R$ using straightedge and compass was possible, the squaring of the circle would be possible. Knowing that the squaring of the circle is not possible we can conclude that the rectifying of the circumference is not possible.

1.2 Rectifying the circumference with Geogebra

Geogebra is based on the straightedge and the compass. Therefore it is not possible with this software to solve the problem of rectifying the circumference. However, as we are going to show it is possible to obtain an acceptable approximation to the rectify of the circumference.

2 Explanation of the problem and its elements

2.1 The final construction

The purpose of this document is the explanation of the process of unfolding a circumference to become a line segment on the OX axis using Geogebra. The final construction can be found at `https://www.geogebra.org/m/YZ5wBUDD` and it is inspired by a construction of the Geogebra institute of HongKong by Anthony C.M. OR that can be found here: `https://www.geogebra.org/m/fyqAUV22`.
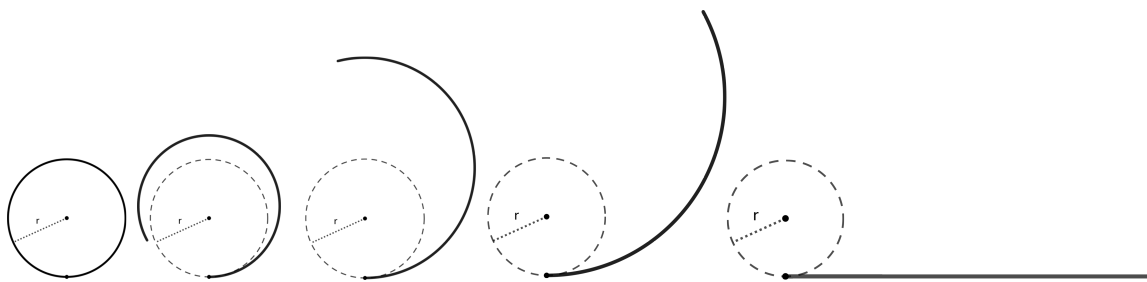

Figure 1: Animation

In order to understand the animation a sequence is showed in Figure 1

2.2 Key element definition

Let's define some key elements of the construction on figure 2

1. r es the radius of the circumference whose lenght is going to be unfold becoming a line segment. The length of the line segment is the circumference which is going to be rectify.
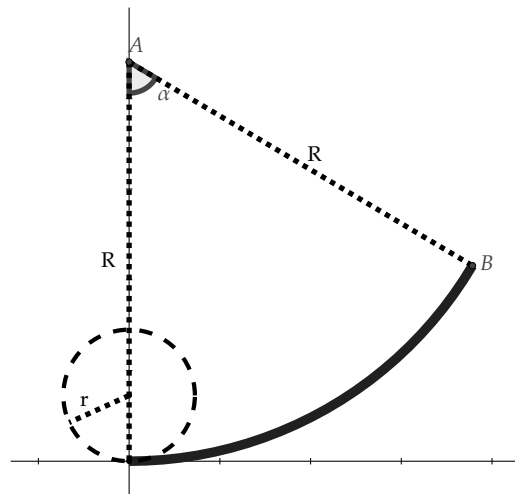
Figure 2: Key Elements

2. R is the radius of the arc of circle that creates the animation. It is an arc of a fixed lenght of a circumference with a variable radius.

3. The point A is the center of the circumference of radius R, which moves along the positive *X-axis*.

4. The point B is the endpoint of the arc of circle that is going to be unfolded

3   Analyisis of the animation elements

3.1   The point A

A slider in Geogebra is a control to modify a specific value by the user. In this particular case we will build a slider for the m variable that would change from 0 to 1 allowing to draw the point A giving the coordinates $(0, \frac{r}{1-m})$ to it.

As we said before, the point A moves away from the origin of the coordinate system on the positive *Y-axis* as the circumference is being unfold.

The point moves from the center of the circle that is going to be rectify ((0,r) coordinates) moving away from the origin of the coordinate system as the point moves along the *Y-axis*, therefore its first coordinate would be always cero, morevover $r > 0$ and r approaches $+\infty$. It can be seen that when the arc of circle had been completely unfolded, the value of m will be 1 and therefore the value of the second coordinate of point A will be undefined as we defined it as $\frac{r}{1-m}$). At any point before reaching the horizontal position, the point A would be on the positive *Y-axis* As we said before a way to represent this situation is to create a slider in Geogebra called m.

$$A(0, \frac{r}{1-m})$$

Point A coordinates

Notice that $\lim_{m \to 1^-} \frac{r}{1-m} = +\infty$. Funtion $f(x) = \frac{r}{1-m}$ grows approaching $+\infty$ as m approaches 1 from the left. The slider will do the point A to move along the *Y-axis* from coordinate (0,r) to (0,+∞)

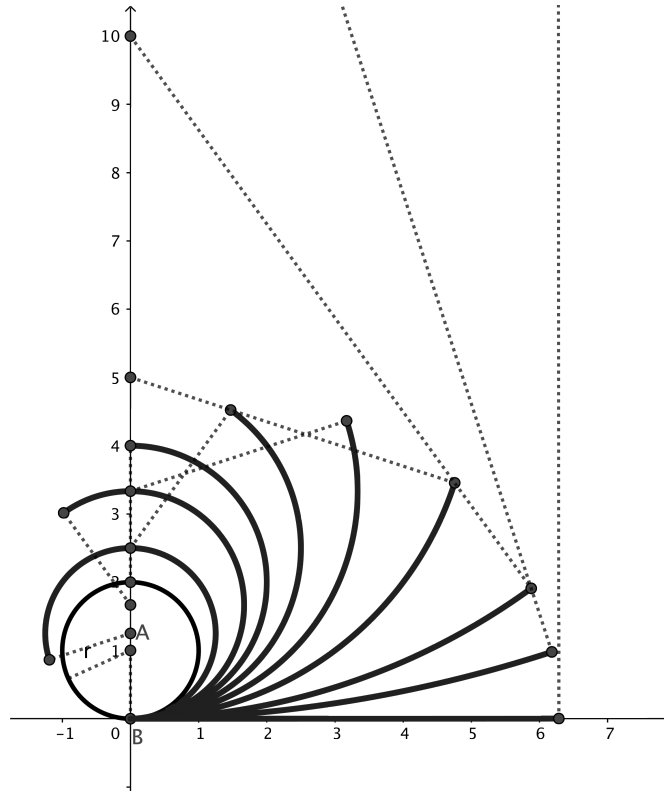The behaviour of the point A can be understood better looking figure 3

Figure 3: The point A moves from (0,r) along the positive Y-axis

3.2 Point B

The point B is the endpoint of the arc of circle that is going to be rectified. In other words, the circumference becomes an arc whose endpoints are the origin of the coordinate system (0,0) and the point $B(x_1, x_2)$

The coordinates of point B are defined in figure 4. In this figure can be seen the right triangle ABC and the α angle whose sin is $sin(\alpha) = \frac{x_1}{R}$. Knowing that $R - x_2$ is the value of the adjacent side to α in that triangle, we have that $cos(\alpha) = \frac{R - x_2}{R}$.

3.3 The angle α

In the figure 5 it can bee seen the circumference to be rectified with radius r, and the arc in which is transformed. This arc has a radius R where one of the endpoints is the point A and the other endpoint is (0,0).

Being the central angles proportional to the length of the correspondent arc, we will have (notice that the angles are measured in radians):
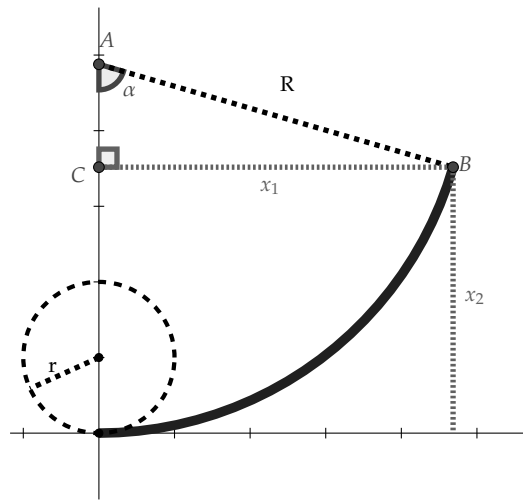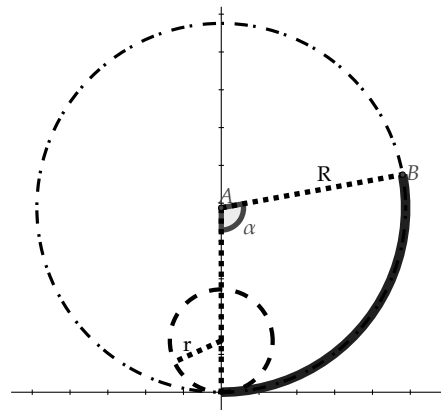
Figure 4: Point B coordinates

Figure 5: Ratio of radiuses

$$\frac{\alpha}{2\pi} = \frac{2\pi r}{2\pi R}$$

Solving for α we obtain that $\alpha = \frac{2\pi r}{R}$ measured in radians. Remember that when we talked about the point A we defined it with the coordinates $(0, \frac{r}{1-m})$. It means that the radius of the big circumference with center in A is $\frac{r}{1-m}$. By substituting in the previous expression: $\alpha = \frac{2\pi r}{R} = \frac{2\pi r}{\frac{r}{1-m}} = 2\pi \cdot (1-m)$.

$\alpha = \frac{2\pi r}{R} = \frac{2\pi r}{\frac{r}{1-m}} = 2\pi \cdot (1-m)$.

Volviendo a las coordenadas del punto B:

The coordinate x_1 can be calculated using the expression $sin(\alpha) = \frac{x_1}{R}$

$$sin(\alpha) = \frac{x_1}{R} \implies x_1 = sin(\alpha) \cdot R = sin(2\pi \cdot (1-m)) \cdot \frac{r}{1-m} = \frac{r \cdot sin(2\pi \cdot (1-m))}{1-m}$$

$$x_1 = \frac{r \cdot sin(2\pi \cdot (1-m))}{1-m}$$

The coordinate x_2 can be calculated solving the expression $cos(\alpha) = \frac{R-x_2}{R}$

$$cos(\alpha) = \frac{R - x_2}{R} \implies x_2 = R - cos(\alpha) \cdot R = R \cdot (1 - cos(\alpha)) = \frac{r}{1-m} \cdot (1 - cos(2\pi \cdot (1 - m)))$$

$$x_2 = \frac{r \cdot (1 - cos(2\pi \cdot (1 - m)))}{1 - m}$$

3.4 Analytic proof of the relation between the coordinate x_1 and the length of the circumference when $m \to 1^-$

It can be proved that when $m \to 1^-$ the coordinate x_1 becomes the length of the circumference. Remembering that $\lim\limits_{x \to 0^-} \frac{sin(x)}{x} = \lim\limits_{x \to 0^+} \frac{sin(x)}{x} = 1$, and knowing that x is measured in radians. We have that:

(Be aware that if $m \to 1$ then $(1 - m) \to 0$)

$$\lim_{m \to 1^-} \frac{r \cdot sin(2\pi \cdot (1 - m))}{1 - m} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2\pi r \cdot \lim_{m \to 1^-} \frac{sin(2\pi \cdot (1 - m))}{2\pi(1 - m)} = 2\pi r$$

In the same way it is true for $m \to 1^+$, therefore:

$$\lim_{m \to 1} \frac{sin(2\pi r \cdot (1 - m))}{1 - m} = 2\pi r$$

For the calculation of this limit L'Hôpital's rule can be used, but that would assume the use of derivatives is known.

3.5 In order to calculate the coordinate x_2 it can be proved that it approaches 0 when $m \to 1$ either from right or left. That implies that the rectified circumference will be on the positive X-axis

$$\lim_{m \to 1^-} \frac{r \cdot (1 - cos(2\pi \cdot (1 - m)))}{1 - m} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (^1) \lim_{m \to 1^-} \frac{r \cdot \left(2sin^2 \left[\frac{2\pi \cdot (1-m)))}{2} \right] \right)}{1 - m} =$$
$$2 \cdot r \cdot \lim_{m \to 1^-} \frac{sin(\pi \cdot (1 - m))}{1 - m} \cdot \frac{sin(\pi \cdot (1 - m))}{1} =$$

$$= 2\pi \cdot \lim_{m \to 1^-} \frac{sin(\pi \cdot (1 - m))}{\pi \cdot (1 - m)} \cdot \frac{sin(\pi \cdot (1 - m))}{1} = 2\pi \cdot 1 \cdot 0 = 0$$

The identity $1 - cos(x) = 2sin^2(\frac{x}{2})$ has been used in (^{1})
The same result will be obtained for $m \to 1^-$, so

$$\lim_{m \to 1} \frac{r \cdot (1 - cos(2\pi r \cdot (1 - m)))}{1 - m} = 0$$

4   Concluding remarks

The point A can not exist when $m = 1$. Therefore we can wonder why when $m = 1$ the line segment is being drawn. The answer is that when $m = 1$ the coordinates of the point A are not defined and we forced the software to draw a line segment of an approximate length $2\pi r$. This can be done by using the visibility condition of a Geogebra object, that can be found in the *advanced* section of the property dialog. In that section we can include conditions to show an object and in this particular case the condition will be $m = 1$. In the animation we will add also three whole circles and the proportional part of a fourth one with the same condition to show the relation in a graphic way.

5   Related constructions

The previous work explained here can be used to build other constructions with some didactic value. The next construction can help in the explanation and comprehension when calculating the area of a circumference. The Geogebra file can found in the link `https://www.geogebra.org/m/BmdpMWez` and it shows a circumference of radius r and the construction of a triangle with the same surface, whose base is πr and whose height is r. The same explanation we did in the first animation should be done here because as we explained the circumference can't be rectified by compass and straightedge.

The animation build in this section uses the previous analysis of this document but draws multiple concentric circumferences that are unfolded until being completely horizontal. Each circumference has been divided into two arcs of the same length as can be seen in the figure 6.

The Geogebra file has been designed using a n slider that controls the number of circumferences allowing to increase the number of them as it is shown in the figure 7

In the figure 8 can be seen the final triangle with an height equal to the radius of the circumference and whose base is $2\pi r$ [3]. Therefore its area can be calculated as:

$$A = \frac{base \cdot height}{2} = \frac{2\pi r \cdot r}{2} = \pi \cdot r^2$$

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[3] As it was said before the base is not really $2\pi r$ long as the circumference can't be rectified by compass and straightedge.
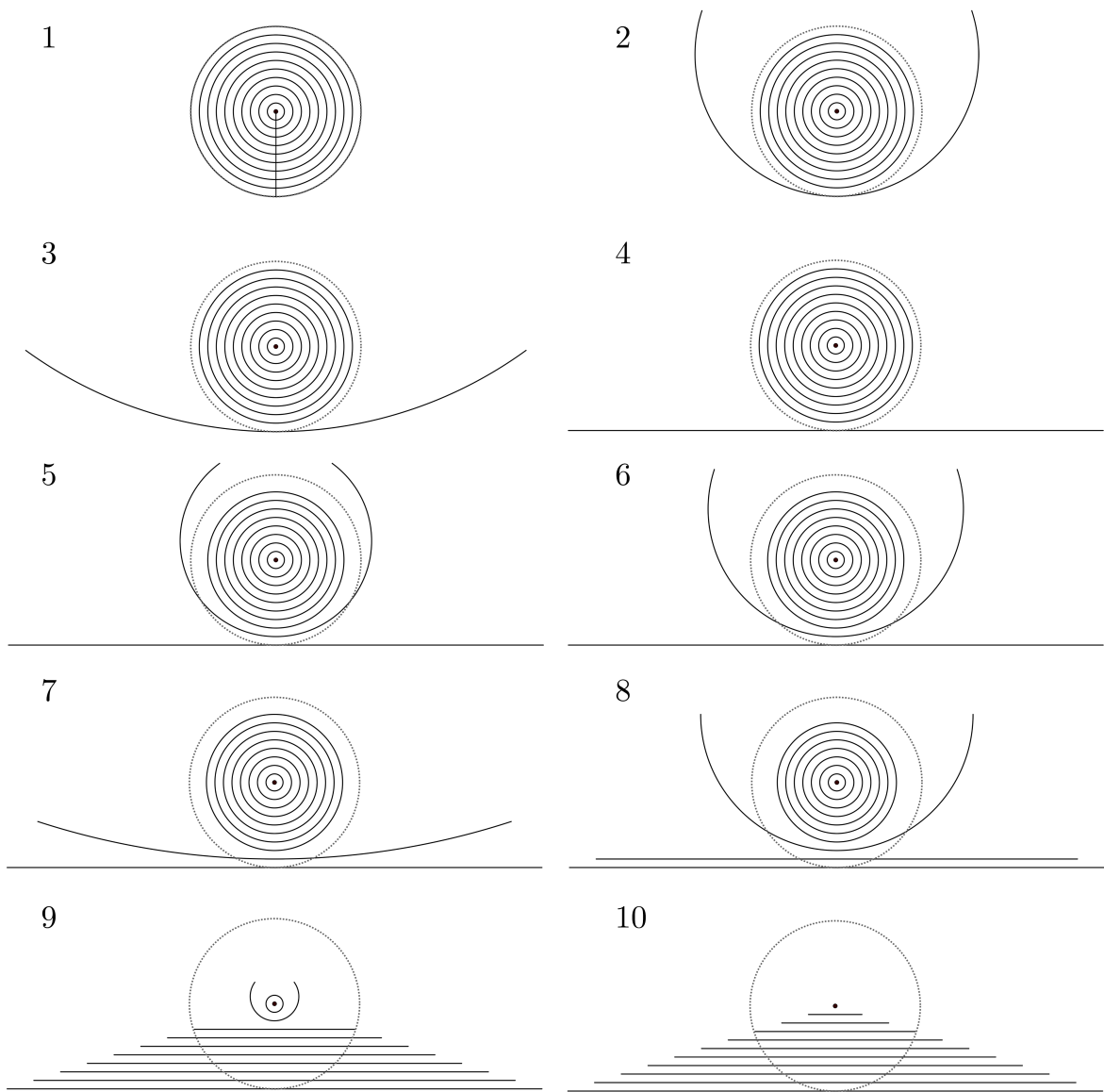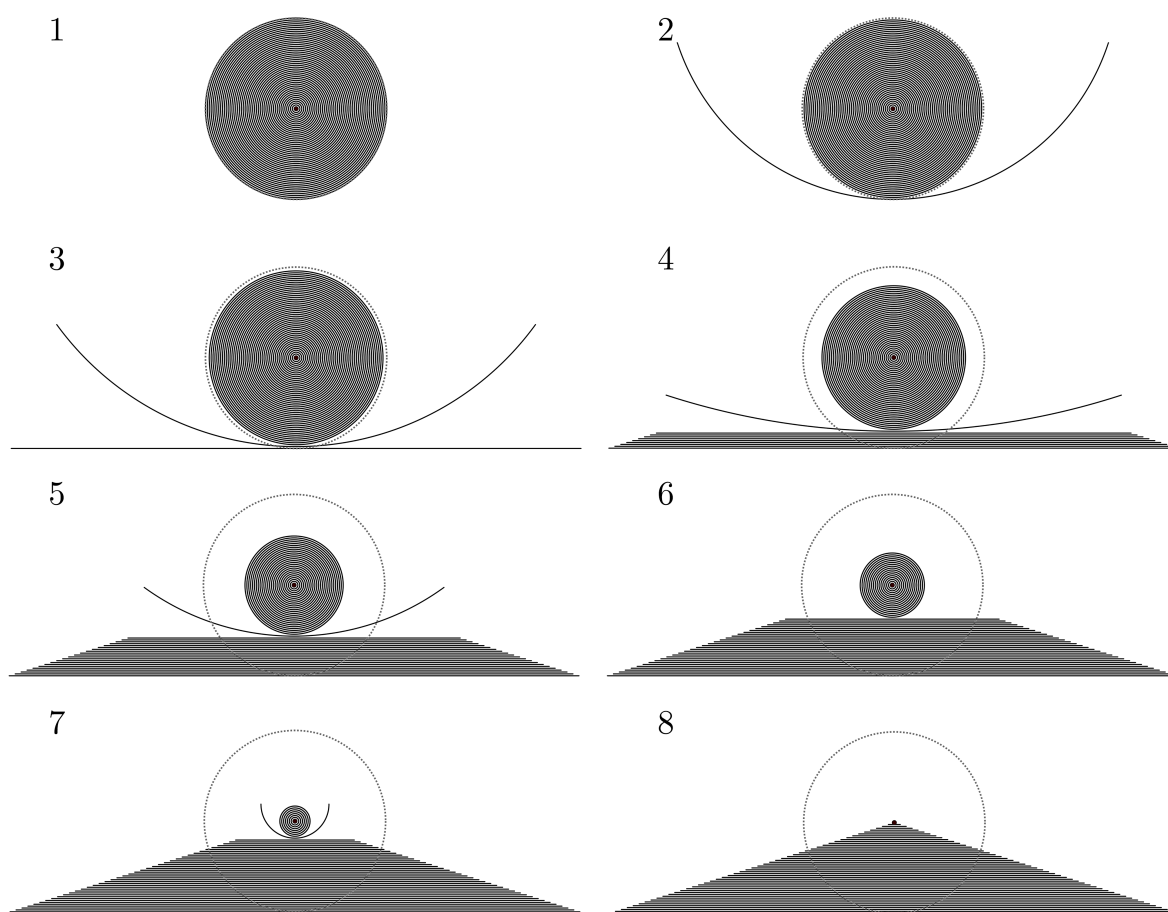
Figure 6: Related constructions

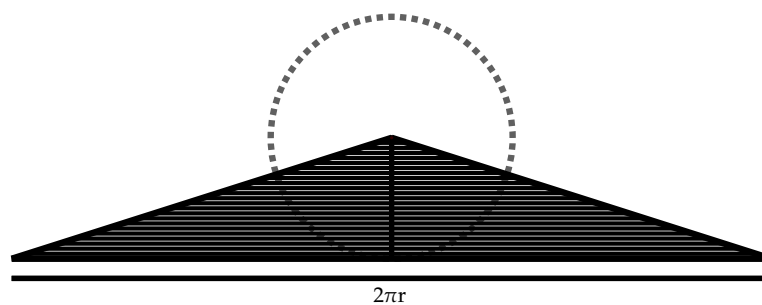Figure 7: Construction increasing the number of circumferences

$$2\pi r$$

Figure 8: Area of the triangle