GEOGEBRA TOOLS FOR THE POINCARÉ DISK

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Abstract: The Poincaré disk model played an important role in the acceptance and development of hyperbolic geometry. Although exceptionally useful, the pedagogical value of the model can be further enhanced via experimentation in a dynamic geometry environment. The focus of this article is on the creation of custom tools in GeoGebra for constructing hyperbolic lines and circles in the Poincaré disk. In an effort to make this material accessible to a wider audience, the necessary mathematics is also included.

Keywords: Poincaré disk, GeoGebra: custom tools

Introduction

Euclid's Elements was a comprehensive summary of the mathematics known to the ancient Greeks and remained the definitive textbook on geometry for more than 20 centuries. His postulates laid bare many of the assumptions made by the Greek geometers and the fifth postulate, called the parallel postulate, has been of particular interest. All attempts to prove that the fifth postulate was dependent on the first four ended in failure and most of these efforts contributed little to the evolution of geometry. Carl Friedrich Gauss may have been the first mathematician to recognize the independence of the fifth postulate, but he never published his results [7]. Models of hyperbolic geometry came 30 years later and were used to establish the relative consistency of Euclidean and hyperbolic geometry and to prove the independence of the fifth postulate. The Poincaré disk model was originally developed by the Italian mathematician Eugenio Beltrami in 1868 and was reintroduced and popularized by French mathematician Henri Poincaré in 1881. The model is conformal and provides a link to several other branches of mathematics including complex function theory and differential equations. The Poincaré disk model entered the popular culture via the work of the Dutch graphics artist M.C. Escher.

The Poincaré disk and other models made hyperbolic geometry easier to visualize and played a role in its gaining acceptance. Although the models are helpful, students can benefit from the use of dynamic geometry software which provides an environment suitable for experimentation. At present there are several software packages available for this purpose including Cinderella [3] and NonEuclid [2] which both provide tools for working with the Poincaré disk model. These tools are useful, but they are not as well known as GeoGebra [9] and it is natural to wonder whether or not the same tasks could be performed in GeoGebra. Currently, GeoGebra does not provide any built in tools but several custom tools are available on GeoGebraTube. Some of the best include those developed by Maline Christersson
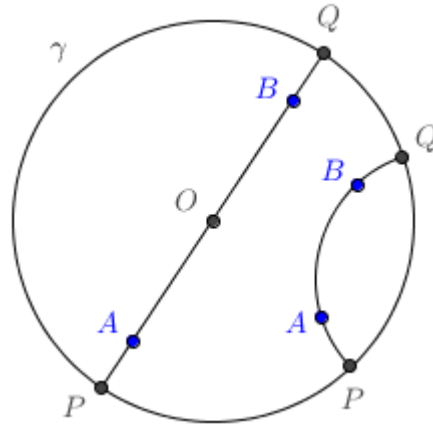
Figure 1.1: Lines in the Poincaré disk model are either open diameters or open circular arcs orthogonal to γ. Each line is shown passing through points A and B and with their ideal endpoints P and Q.

[4] and Stehpan Szydlik [10]. However, both of these tools have some defects. For example, their hyperbolic line tools do not work properly in all cases. The emphasis in this paper is on learning how to create robust custom tools in GeoGebra for constructing hyperbolic lines and circles in the Poincaré disk model. In an effort to make this material accessible to a wider audience, the mathematics needed to construct the tools will also be provided.
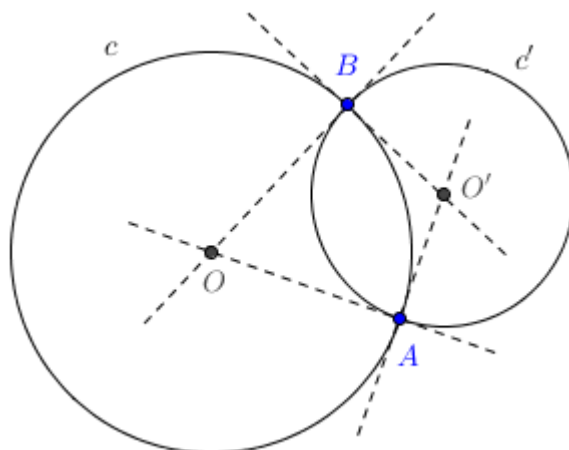
Poincaré disk model

The Poincaré disk model begins with a circle γ with center O in the Euclidean plane. Points in this model are points in the interior of γ. Points on γ are called ideal points. They are not included in the model and are considered to be infinitely far away from points in the interior. There are two types of lines in the Poincaré disk model. The first type are the open diameters of γ and the second type are the open circular arcs orthogonal to γ. Taken together, the two types of lines are called p-lines and they are geodesics. It is worth noting that a diameter may be regarded as the limiting case of a circle whose radius is approaching infinity.

Circles orthogonal to γ can be constructed in several different ways. One method is to use the following result from Euclidean geometry which makes use of tangents [11].
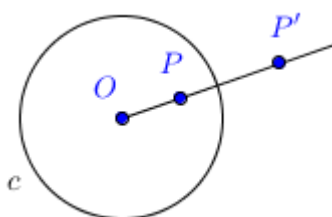
Definition 1.1 *Two circles are orthogonal if their tangent lines are perpendicular at the point of intersection.*

Theorem 1.1 *Let c be a circle with center O and c' be a circle with center O' and suppose the circles intersect at points A and B. Then the circles are orthogonal if and only if the tangent lines to c' at A and B intersect at O and the tangent lines to c at A and B intersect at O'.*
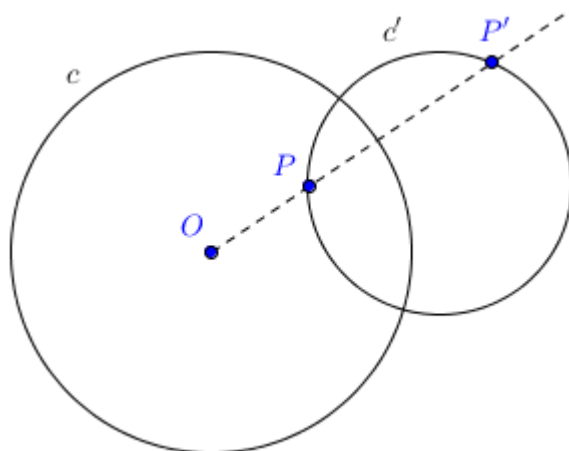
Another method involves using inversive geometry. In this approach, circles orthogonal to γ are constructed using inverse points [5, 6, 7].

Definition 1.2 *The inverse of a point P with respect to a circle c with center at point O and radius r is the unique point P' on the ray OP such that $OP \cdot OP' = r^2$.*

Theorem 1.2 *Let c be a circle with center O and radius r and P be a point that is not on c and that is not the center of c. Define P' as the inverse of P with respect to c. Suppose a circle c' passes through point P. Then the circles c and c' are orthogonal if and only if c' passes through P'.*

Angles in the Poincaré disk model are defined as the Euclidean angle between tangents. The fact that angles are preserved is considered to be an advantage of the disk model over other models of hyperbolic geometry.
Distance is more complicated in the Poincaré disk model and is significantly distorted. The ratio of distances is also distorted. However, the cross ratio defined as a ratio of ratios of distances is preserved [7].
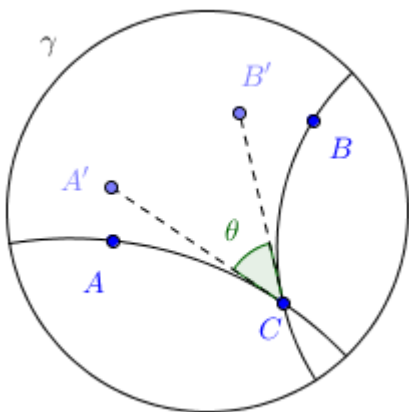
Figure 1.2: The angle between two hyperbolic lines is defined as the Euclidean angle between the tangents at their point of intersection. The angle ACB is defined as the Euclidean angle $A'CB'$.

Definition 1.3 *Suppose that A and B are any two points in the Poincaré disk and let P and Q be the ideal endpoints of the line that connects them. Then the cross ratio is defined as*

$$(AB, PQ) = \frac{AP}{AQ} \div \frac{BP}{BQ}$$

where the AP, AQ, BP and BQ are Euclidean lengths.

Theorem 1.3 *Let c be a circle with center O. If A, B, P, Q are four distinct points different from O and A', B', P', Q' are there inverses with respect to c, then their cross ratio is preserved, $(A'B', P'Q') = (AB, PQ)$.*

Definition 1.4 *The distance between points A and B in the Poincaré disk model is defined in terms of the cross ratio as $d(A, B) = |\ln(AB, PQ)|$.*

A hyperbolic circle with center A and radius r is defined as the set of all points X in the Poincaré disk such that $d(A, X) = r$. The relationship between hyperbolic and Euclidean circles in the Poincaré disk model is captured in the following theorem [1, 7].

Theorem 1.4 *Hyperbolic circles in the Poincaré disk model are also Euclidean circles but their centers are not the same unless the hyperbolic circle is centered at O.*

An isometry is a distance preserving transformation. There are four isometries in the hyperbolic plane including hyperbolic reflections, hyperbolic translations, hyperbolic rotations and the parabolic isometry which has no Euclidean counterpart. In this paper, we focus on the hyperbolic reflections [8] since they are required for the construction of the hyperbolic circle.

Definition 1.5 *A hyperbolic reflection is either a Euclidean reflection across a diameter or an inversion with respect to a circular arc orthogonal to γ.*

Theorem 1.5 *Hyperbolic reflections are isometries. If points A', B' are reflections of points A, B about a hyperbolic line, then $d(A', B') = d(A, B)$.*

Hyperbolic line tool

Although in theory the defining circle γ can be any circle, for the purpose of creating custom tools in GeoGebra for constructing hyperbolic lines and circles, we will take γ to be the unit circle. When constructing a hyperbolic line passing through two points, we must consider several possibilities depending on whether or not the points lie on a diameter of γ and whether the points are located in the interior or on the boundary of γ. These possibilities can be divided into three cases; diameters, arcs and ideal arcs.
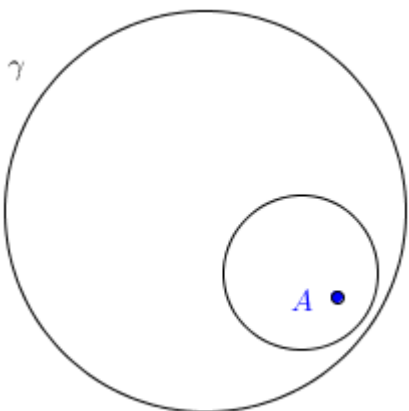
Figure 1.3: A hyperbolic circle in the Poincaré disk is also a Euclidean circle. The hyperbolic center is offset towards the boundary of the disk since distances near the boundary are larger.
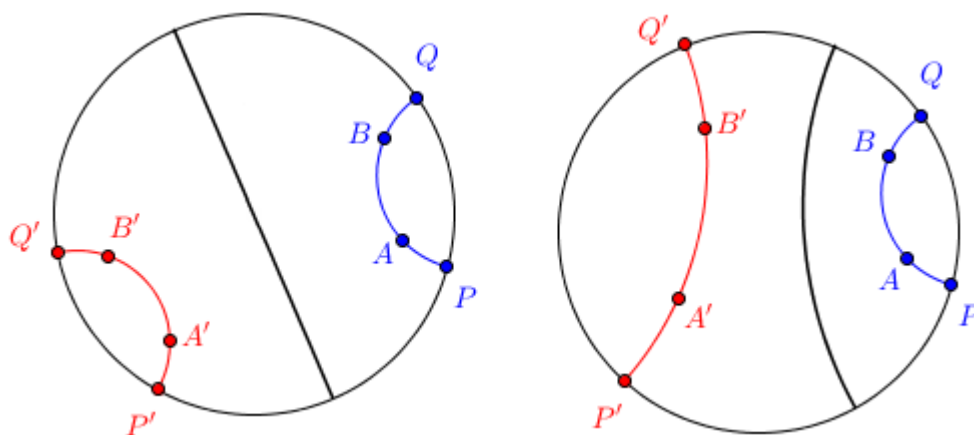

Figure 1.4: Hyperbolic reflections. (a) If the hyperbolic line is a diameter of γ, the hyperbolic reflection is simply the Euclidean reflection about the diameter. (b) If the hyperbolic line is a circular arc orthogonal to γ, then the hyperbolic reflection is an inversion about the hyperbolic line.
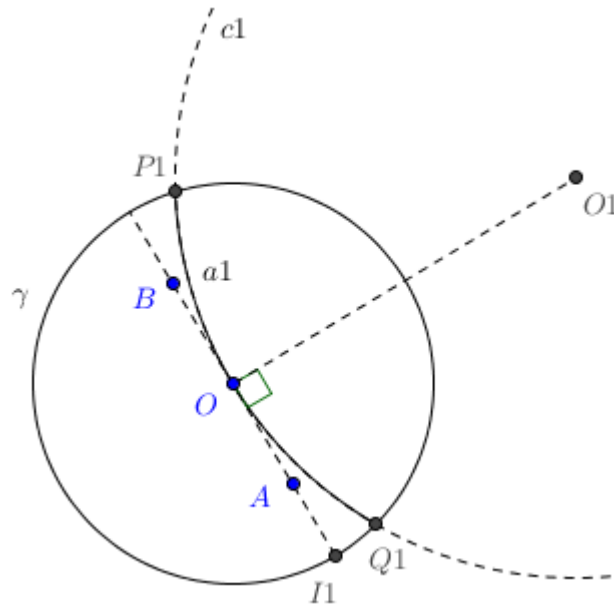
Figure 1.5: Use an arc of circle $c1$ to approximate the diameter passing through points A and B. The circle $c1$ is tangent to the diameter at O and its center $O1$ lies on the perpendicular bisector of the diameter. Define $I1$ as one of the intersections of the diameter and the defining circle γ. Locate $O1$ by rotating $I1$ about O counterclockwise by 90 degrees and then dilating by a factor equal to the radius of $c1$. Find the intersections $P1$ and $Q1$ of $c1$ and γ and use them as the endpoints of the circular arc $a1$. Note that the circle $c1$ is shown with a radius of 2 in order to make the details of the construction easier to see. When the radius is set equal to 1000, the arc $a1$ appears as a straight line.

Diameters

If A and B are collinear with O, then the hyperbolic line passing through A and B is a diameter of γ. Although diameters are simple in theory, there are some complications when implementing this case in GeoGebra since in the end all three cases will be combined into a single custom tool. In an ideal world, we could use the Segment [<Point >, <Point >] command to construct the diameter. Unfortunately, the If [<Condition >, <Then >, <Else >] command needed to combine the cases requires that the <Then >and <Else >objects be of the same type. Since segments and arcs do not meet this criterion, the diameter will be approximated using the small arc of a large circle as shown in Figure 1.5.

Begin the construction by creating γ using the Circle[<Point >, <Radius Number >] and two points inside γ using the PointIn[<Region >] command

```
// Poincarè disk --------------
// Construct the unit circle γ
// Create point A inside γ
// Create point B inside γ
// --------------------------

γ: Circle[(0,0), 1]
A: PointIn[Circle[(0,0), 1]]
B: PointIn[Circle[(0, 0), 1]]
```

Then construct the approximate diameter. Note that the Intersect [ ] command has several forms and in the form used below, the third argument is the index of the intersection.
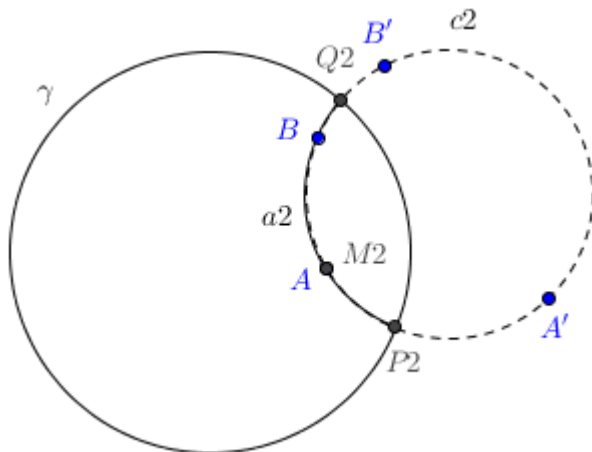
Figure 1.6: Construct the hyperbolic line for the case in which points A and B are not collinear with O and at least one of the points is in the interior of γ. Begin by determining the inverses of A and B with respect to γ. If A is in the interior of γ, then by Theorem 2 the circle $c2$ passing through points A, A' and B is orthogonal to γ. If A is on γ then A and A' coincide (points on the circle of inversion are invariant) and $c2$ can be constructed using points A, B and B'. Determine the intersections $P2$ and $Q2$ of $c2$ and γ and use them as the endpoints of circular arc $a2$. Locate an additional point $M2$ on arc $a2$. If A is on γ, choose $M2 = B$, otherwise choose $M2 = A$. Finally construct arc $a2$ using points $P2$, $M2$ and $Q2$.

```
// Diameters----------------------------------
// Approximate the diameter using the small arc
// of a large circle
// Locate an endpoint I1 of the diameter
// Use I1 to locate the center O1 of circle c1
// Construct circle c1
// Locate endpoint P1 of arc a1
// Locate endpoint Q1 of arc a1
// Construct circular arc a1
//--------------------------------------------

I1: Intersect[Circle[(0,0), 1], Line[A, B],1]
O1: 1000 Rotate[I1, 90°]
c1: Circle[O1, 1000]
P1: Intersect[Circle[(0,0), 1], c1, 1]
Q1: Intersect[Circle[(0, 0), 1], c1, 2]
a1: CircumcircularArc[P1, (0, 0), Q1]
```

If you study the GeoGebra commands in this section, you will notice that γ is purely decorative and is included only to make the unit circle visible during the construction process. If the unit circle is required on the right hand side of any command, it is created using Circle [(0,0), 1] command. If the decorative γ were used instead, then the final tool would require the user to input two points and the circle. In other words, a little extra effort at this stage will result in a tool that requires two inputs instead of three.

Arcs

If A and B are not collinear with O, then the hyperbolic line passing through A and B is the open arc of a circle orthogonal to γ. If in addition, at least one of the points lies in the interior of γ, then the hyperbolic line can be constructed using inversive geometry as described in Figure 1.6.

Prepare for this construction by hiding all of the objects from the previous construction except for points A and B and the defining circle γ. Then construct the circular arc using the GeoGebra code provided below. Note the use of the Reflect[<Object >, <Circle >] command to determine the inverses of A and B with respect to γ and the Length[<Object >] command to determine whether or not A is on γ. The If[<Condition >, <Then >, <Else >] command is used to construct the circle passing through points A and B and orthogonal to γ. If A is on γ, then the circle is constructed using A, B and B', otherwise it is constructed using points A , A' and B.

```
// Arcs--------------------------------------------
// Construct a circular arc using Theorem 2
// Determine the inverse of point A with respect to γ
// Determine the inverse of point B with respect to γ
// Construct circle c2 passing through A and B and
// orthogonal to γ (if A is on γ, use points A, B, B',
// otherwise use points A, B, A')
// Locate endpoint P2 of arc a2
// Locate endpoint Q2 of arc a2
// Locate an additional point M2 on arc a2
// (if A is on γ, set M2 = B, otherwise set M2 = A)
// Construct circular arc a2 using points P2, M2 and Q2
//--------------------------------------------------

A': Reflect[A, Circle[(0, 0), 1]]
B': Reflect[B, Circle[(0, 0), 1]]
c2: If[Length[A] == 1, Circle[A, B, B'], Circle[A, B, A']]
P2: Intersect[c2, Circle[(0,0), 1], 1]
Q2: Intersect[c2, Circle[(0, 0), 1], 2]
M2: If[Length[A] == 1, B, A]
a2: CircumcircularArc[P2, M2, Q2]
```

Ideal arcs

Consider the situation for which A and B are not collinear with O and both points lie on γ. In this case, the previous construction fails since both points are invariant under inversion about γ and inversion cannot be used to obtain the third point needed to construct the circle. The solution is to use a result from Euclidean geometry as illustrated in Figure 1.7.

Prepare for this construction by hiding all of the objects from the previous constructions except for points A and B and the defining circle γ. Then construct arc $a3$ using the GeoGebra code provided below. Note that the most obvious method for constructing the tangents would be to use the Tangent[<Point >, <Conic >] command. However if this command is used, the final hyperbolic line tool will fail when imported into another GeoGebra file (most likely due to the fact that the Tangent command was designed to construct multiple tangents from a point to a conic). To work around this problem, unique tangents to γ at A and B are constructed using the fact that the tangent to a circle is perpendicular to the radius of the circle.

```
// Ideal arcs----------------------------------------
// Construct ideal arcs using Theorem 1
// Construct tangent t3a to γ at point A
// Construct tangent t3b to γ at point B
// Locate the center O3 of circle c3
// (intersection of the tangents)
// Construct circle c3 passing through A and B
// and orthogonal to γ
// Locate an additional point M3 on arc a3 by finding
// the intersection of circle c3 with the ray from O3
// to the midpoint of segment AB
// Construct circular arc a3 using points A, M3, B
// --------------------------------------------------
```
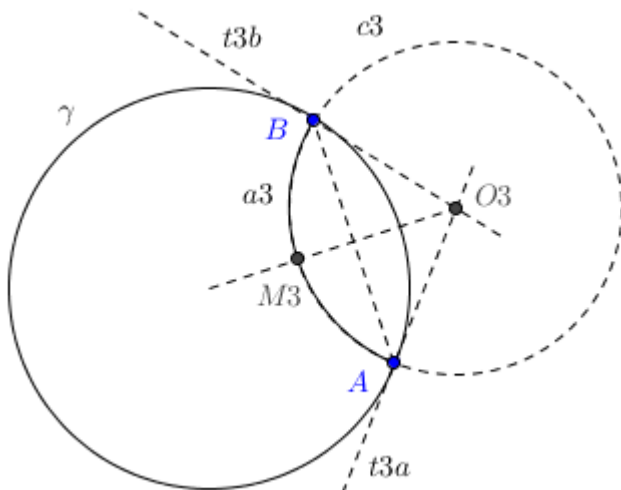
Figure 1.7: Construct the hyperbolic line passing through points A and B for the case in which A and B are not collinear with O and both points lie on γ. Begin by constructing tangents $t3a$ and $t3b$ to γ at points A and B respectively. By Theorem 1, the tangents intersect at the center $O3$ of a circle $c3$ which passes through A and B and is orthogonal to γ. Construct $c3$ using $O3$ as the center and either point A or B. Use A and B as the endpoints of arc $a3$. Determine an additional point $M3$ on arc $a3$ by constructing a ray from $O3$ to the midpoint of segment AB and locating the intersection of this ray with $c3$. Then use points A, $M3$, B to construct arc $a3$.

```
t3a: PerpendicularLine[A, Line[(0, 0), A]]
t3b: PerpendicularLine[B, Line[(0, 0), B]]
O3: Intersect[t3a, t3b]
c3: Circle[O3, A]
M3: Intersect[c3, Ray[O3, Midpoint[A, B]]]
a3: CircumcircularArc[A, M3, B]
```
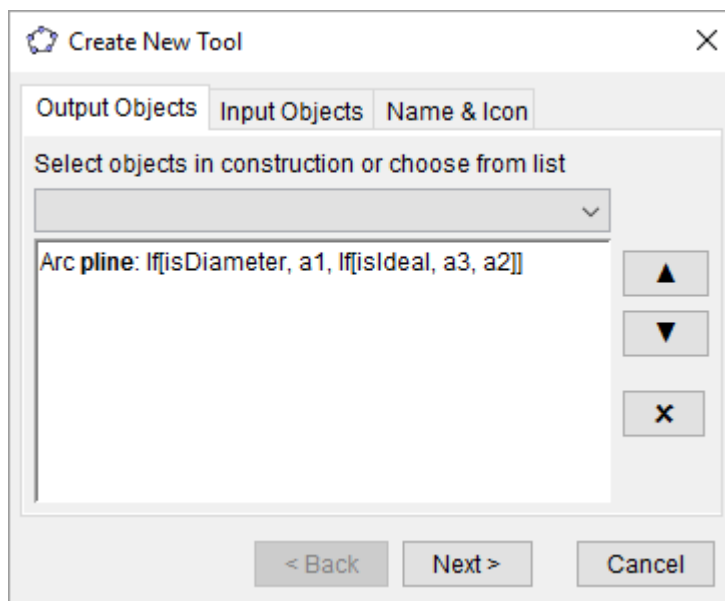
Create the hyperbolic line tool

Use the Boolean variables $isDiameter$ and $isIdeal$ to distinguish between the three cases. The $isDiameter$ variable tests whether or not points A and B are collinear with O and $isIdeal$ tests whether or not both A and B are on γ. Then use nested If[$<$Condition$>$, $<$Then$>$, $<$Else$>$] commands to combine the three cases. Note that in traditional programming languages, only the objects which are needed would be constructed. In this construction, however, the objects used in all three cases are always created and the nested If statement is used to display the valid object.
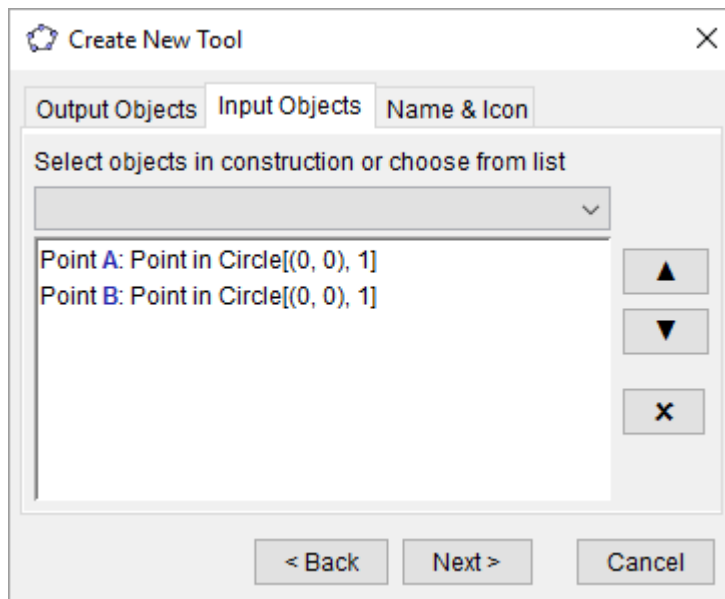
```
// Combine cases----------------------------------------
// Boolean isDiameter is true if A, B and O are collinear,
// otherwise false
// Boolean isIdeal is true if A and B are on γ,
// otherwise false
// Combine diameters, arcs and ideal arcs into a single
// hyperbolic line command
//-----------------------------------------------------

isDiameter: AreCollinear[(0,0), A, B]
isIdeal: Length[A] == 1 & Length[B] == 1
pline: If[isDiameter, a1, If[isIdeal, a3, a2]]
```
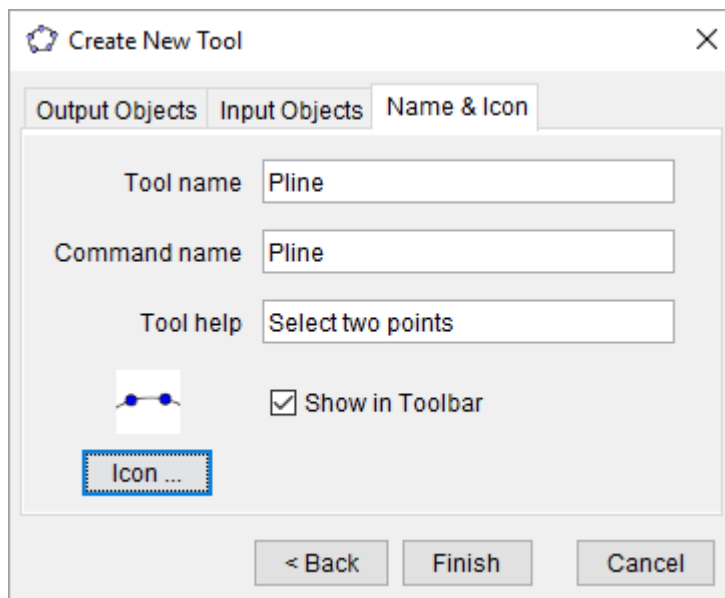
We are finally ready to create a custom tool for creating hyperbolic lines in the Poincaré model. Be sure to hide all of the objects except A, B and $pline$. For $pline$, it is best to turn off the label. Then go to the main menu in GeoGebra and select Tools/Create New Tool. In the Output Objects tab, use the drop-down menu to select $pline$.

Use the Next button to move to the Input Objects tab and use the drop-down menu to select points A and B.

Use the Next button to move to the Name & Icon tab and provide a name for the tool and a command name. Provide a tool tip such as "Select two points" to provide users with information on how to use the tool.

Although it is not necessary, it is possible to create an icon. If you decide to do this, consider using a separate GeoGebra Worksheet to create a suitable image and then use screen capturing software such as Jing (free) or SnagIt (commercial) to create an image file. Then press the Icon button and navigate to the image file and press Open. Press the Finish button and you now have a tool for creating lines in the disk model.

This custom tool can be used both as a tool and as the command Pline[<Point >, <Point >] in the Input bar. In addition, the custom tool can be saved to your computer as GGT file and then imported into other GeoGebra files. To do this, select Tools/Manage Tools, then select Save As and press Save. Then open a new GeoGebra file and use File/Open to import the GGT file. Be sure to do this prior to beginning the hyperbolic circle construction, since this approach hides all of the details of construction of the hyperbolic line tool and allows you to focus on constructing the hyperbolic circle. This works since all of the details of the construction are contained in the GGT file and they are not displayed in the Algebra window.

Hyperbolic circle tool

Hyperbolic circles in the Poincaré disk model are also Euclidean circles but in general the hyperbolic center is not the same as the Euclidean center. The key to constructing a hyperbolic circle with center at points A and passing through point B is to locate its Euclidean center C and then construct a Euclidean circle with center C passing through B. The construction of the hyperbolic circle is broken into three cases depending on whether or not $A = O$ and whether or not the A and B are collinear with O.

Hyperbolic center at the origin

If $A = O$, then by Theorem 4 the Euclidean center of the circle is also at O and the hyperbolic circle can be constructed using a Euclidean circle centered at O and passing through B . Begin the construction by creating γ and two points inside γ.
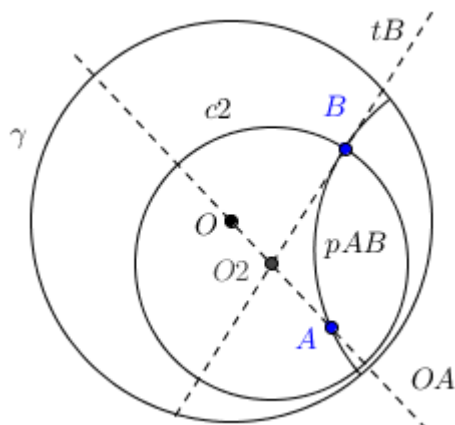
Figure 1.8: Construction of the hyperbolic circle $c2$ with hyperbolic center $A \neq O$ and passing through B for the case in which points A, B and O are not collinear. All hyperbolic lines passing through the center A are orthogonal to the circle. Since line OA is one such line, it is orthogonal to $c2$ and the center of $c2$ must lie on this line (if a line cuts a circle orthogonally, the line must be a diameter of the circle). In addition, $c2$ is orthogonal to the hyperbolic line pAB (circular arc) passing through points A and B . By Theorem 1, the Euclidean center $O2$ of $c2$ must lie on the tangent tB to the hyperbolic line pAB at point B . Thus the Euclidean center $O2$ of $c2$ is located at the intersection of line OA and tangent tB. The hyperbolic circle $c2$ can then be constructed using a Euclidean circle centered at $O2$ and passing through B.

```
// Poincarè disk --------------
// Construct the unit circle γ
// Create point A inside γ
// Create point B inside γ
// ---------------------------

γ: Circle[(0, 0), 1]
A: PointIn[Circle[(0, 0), 1]]
B: PointIn[Circle[(0, 0), 1]]
```

Then construct the circle.

```
// Center at the origin ------------------------
// Construct hyperbolic circle c1 using Theorem 4
// ---------------------------------------------

c1: Circle[A, B]
```

Points not on a diameter

If $A \neq O$ and points A and B are not collinear with O, then the hyperbolic circle centered at A and passing through B can be constructed using the fact that every hyperbolic line passing through A must be orthogonal to the circle as explained in Figure 1.8.

Begin by hiding $c1$ from the previous case and then implement the construction using the following GeoGebra code. Although the most obvious way to construct the tangent is to use the Tangent[<Point >, <Conic >] command, we forego this approach since the command fails for some locations of B on γ (returns undefined, most likely since the
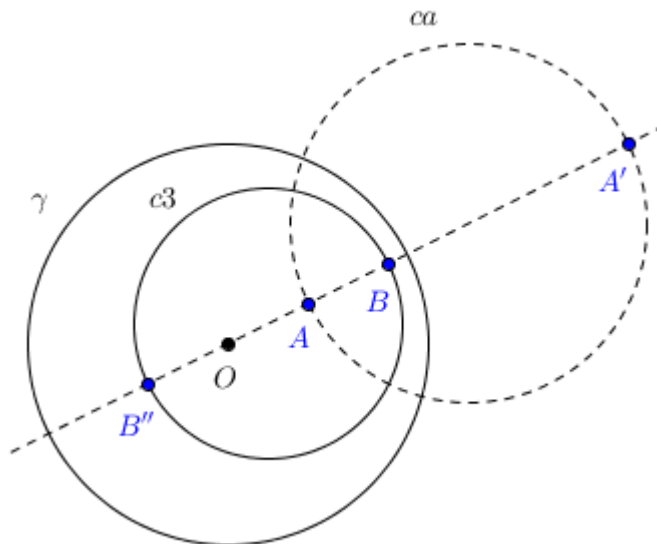
Figure 1.9: Construction of the hyperbolic circle $c3$ with hyperbolic center A and passing through B for the case in which $A \neq O$ and A and B are collinear with O. Begin by determining the inverse of A with respect to γ and then construct circle ca passing through A and A' and centered at the midpoint of segment AA'. By Theorem 2, ca is orthogonal to γ and is therefore a hyperbolic line in the Poincaré disk. Define A'' and B'' as the inverses of A and B with respect to ca (be careful since in all previous constructions, the inverses have been with respect to γ). To finish the construction, we use the distance metric. By Theorem 5, $d(A'', B'') = d(A, B)$. Furthermore since A is invariant after inversion with respect to ca, we obtain $d(A, B'') = d(A, B)$. Hence, the hyperbolic circle $c3$ can be constructed using a Euclidean circle centered at the midpoint of segment BB'' and passing through the point B.

numerical algorithms occasionally fail to find the tangent at the endpoint of an arc). The approach taken here is to use the fact that the tangent to a circle is perpendicular to its radius.

```
// Points not on a diameter ---------------------------
// Construct hyperbolic circle c2 using Theorem 1
// Construct line OA
// Construct hyperbolic line pAB through A and B
// Construct tangent tB to the hyperbolic line pAB at B
// Locate the center O2 of hyperbolic circle c2
// (intersection of line OA and tangent tB)
// Construct the hyperbolic circle c2
// ---------------------------------------------------

OA: Line[(0,0), A]
pAB: Pline[A, B]
tB: PerpendicularLine[B, Line[Center[pAB], B]]
O2: Intersect[OA, tB]
c2: Circle[O2, B]
```

Points on a diameter

If $A \neq O$ and points A, B and O are collinear, then the previous construction fails since the intersection of OA and tB is no longer defined. In this case the circle can be constructed using the distance metric as explained in Figure 1.9.

Begin by hiding all of the objects from the previous construction except points A and B and the defining circle γ and then move the A and B onto a diameter. Use the following GeoGebra code to implement the construction.

```
// Points on a diameter -------------------------------
// Construct hyperbolic circle c using Theorems 2 and 5
// Determine the inverse of point A with respect to γ
// Construct circle ca through points A and A' centered
// at midpoint of segment AA'
// Determine the inverse of point B with respect to ca
// Construct the hyperbolic circle c3
// ---------------------------------------------------

A': Reflect[A, Circle[(0, 0), 1]]
ca: Circle[Midpoint[A, A'], A]
B'': Reflect[B, ca]
c3: Circle[Midpoint[B, B''], B]
```

Create the hyperbolic circle tool

Combine the three cases by defining the Boolean values $Origin$ and $Collinear$ and using nested If[<Condition >, <Then >, <Else >] statements.

```
// Combine cases----------------------------------------
// Boolean isOrigin is true if A = O and false otherwise
// Boolean isCollinear is true if A, B and O are
// collinear and false otherwise
// Combine cases into a single hyperbolic circle command
//-----------------------------------------------------

isOrigin: A == (0,0)
isCollinear = AreCollinear[(0, 0), A, B]
pcircle: If[isOrigin, c1, If[isCollinear, c3, c2]]
```

Hide all of the objects except A, B and $pcircle$ and turn off the label for $pcircle$. Then use Tools/Create New Tool to create a custom tool, $Pcircle$, which accepts points A and B as inputs and outputs $pcircle$. Provide a helpful tooltip such as "Select center, then point" and create an icon for the tool if you wish.

Conclusion

The primary focus of this paper has been on the construction of custom tools in GeoGebra for creating hyperbolic lines and hyperbolic circles in the Poincaré disk model. The hyperbolic line tool is robust and works well in all cases including when points A and B are in the interior of γ, on the boundary of γ and when they are on the same diameter. The hyperbolic circle is similarly robust and works for the full range of positions of the center point A and point B within and on γ. Tools for creating hyperbolic segments and rays can be constructed in a manner similar to that of hyperbolic lines with some modifications. If the reader wishes to create a more complete suite of tools for working with the Poincaré disk, then it is worth noting that the construction of hyperbolic perpendicular bisectors, hyperbolic midpoints, hyperbolic perpendicular lines and hyperbolic angle bisectors mimic their corresponding Euclidean compass and straightedge constructions.

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