AREA OF PARABOLIC SEGMENT AT Y-AVERAGE

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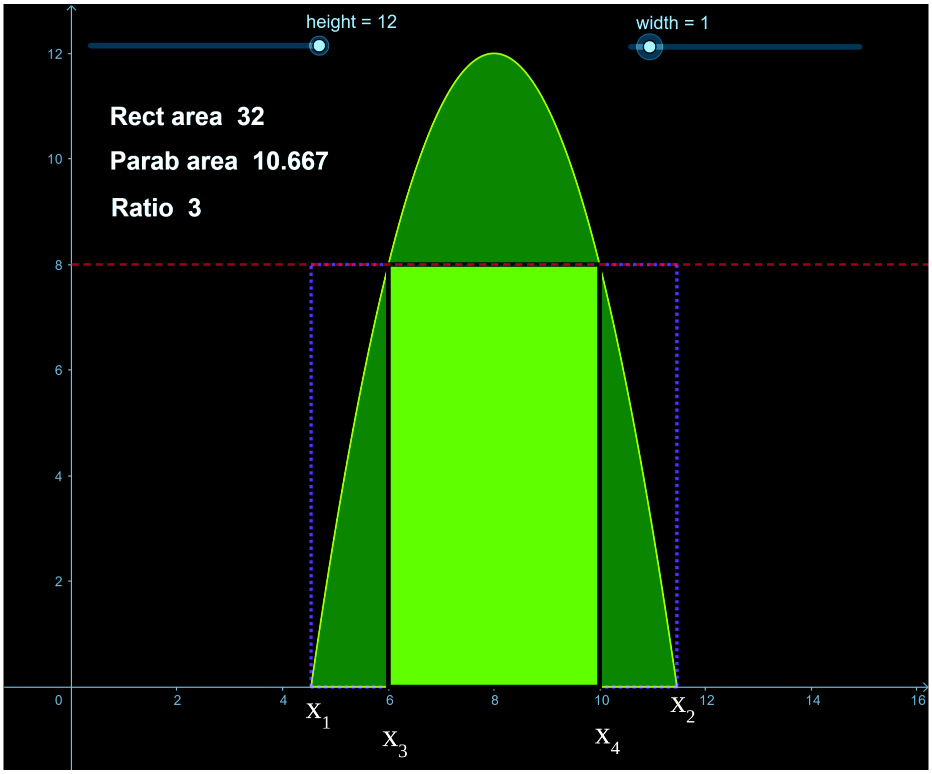

The concave-down parabola in the figure is given by

$$y(x) \;=\; k \,-\, a\left(\,x - h\right)^2$$

with the parameters a and k adjusted by sliders, and h is set at 8. The points of intersection with the x-axis (roots) are

$$x_1 \;=\; h \,-\, \sqrt{\frac{k}{a}} \qquad\qquad x_2 = h \,+\, \sqrt{\frac{k}{a}}$$

Then the area under the parabola and above the x-axis (darker shade) will be

$$A_{parabola} \;=\; \int_{x_1}^{x_2} \left[\; k \,-\, a\left(x - h\right)^2 \;\right] dx \;=\; \frac{4}{3}\sqrt{\frac{k^3}{a}}$$

Using this, the average y-value (red dashed line) is the remarkably simple result

$$\overline{y} \;=\; \frac{A_{parabola}}{x_2 - x_1} \;=\; \frac{2}{3}\,k$$

The dotted blue rectangle has its height at the y-average; the area of this rectangle is the same as that of the parabola:

$$A \;=\; \overline{y}\left(x_2 - x_1\right) \;=\; \frac{2}{3}k\left[\left(h \,+\, \sqrt{\frac{k}{a}}\right) \,-\, \left(h \,-\, \sqrt{\frac{k}{a}}\right)\right] \;=\; \frac{4}{3}\sqrt{\frac{k^3}{a}}$$

Thus, a simple method of estimating the area of a given concave-down parabola above the x-axis is: (1) find the vertex height; (2) take 2/3 of that value; (3) find the distance between the roots; (4) multiply the last two results.

The points of intersection of the horizontal red line, at the average y-value, with the parabola can be shown to be

$$x_3 = h - \sqrt{\frac{k}{3a}} \qquad x_4 = h + \sqrt{\frac{k}{3a}}$$

Then the "top" area (a parabolic segment), under the parabola and above the red horizontal y-average line, is

$$A_{top} = \int_{x_3}^{x_4} \left[k - a\left(x-h\right)^2 - \bar{y} \right] dx = \frac{4}{9}\sqrt{\frac{k^3}{3a}}$$

Next, the area of the rectangle (lighter shade, black outline) defined by dropping perpendicular lines from the intersection points to the x-axis is

$$A_{rect} = \bar{y}\left(x_4 - x_3\right) = \frac{2}{3}k\left[\left(h + \sqrt{\frac{k}{3a}}\right) - \left(h - \sqrt{\frac{k}{3a}}\right)\right] = \frac{4}{3}\sqrt{\frac{k^3}{3a}}$$

Finally, the ratio of this interior rectangle area to that of the parabola "top" is readily seen to be 3. This constant ratio is independent of the "height" and "width" of the parabola, as well as its location along the x-axis.