The Thébault configuration keeps on giving

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Given a parallelogram, construct squares outwardly on its sides; hereafter we will call this the 'Thébault configuration'. Our name derives from Thébault's celebrated first problem, which states that the centres of these newly constructed squares also form a square.

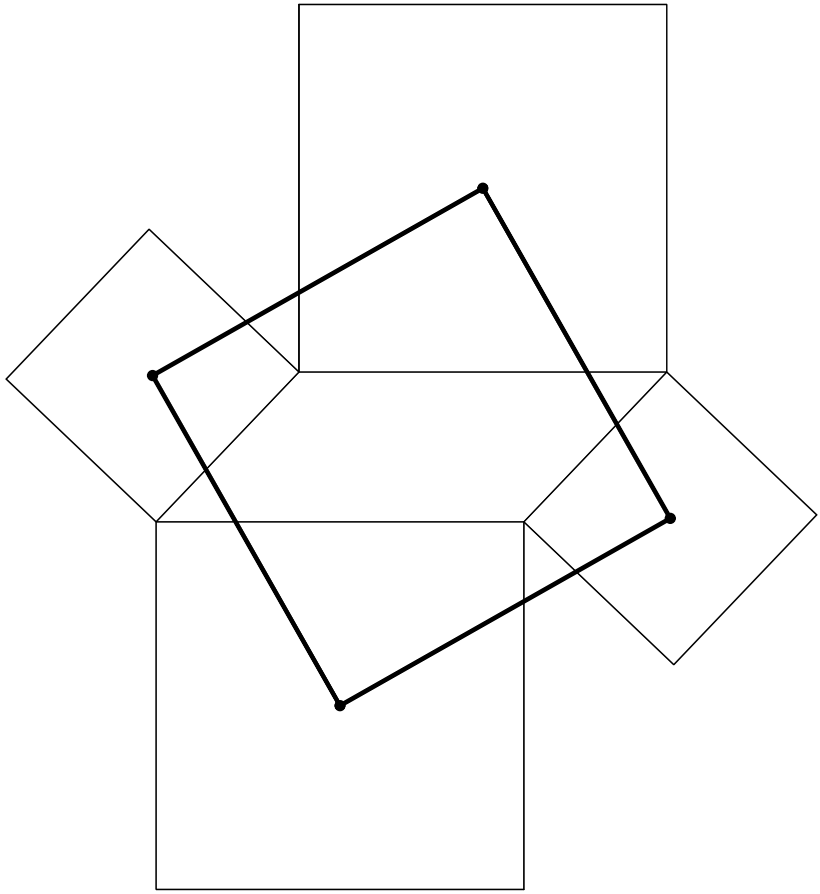

FIGURE 1

In [1], it was shown that yet another square lurks inside the Thébault configuration, at the midpoints of the segments joining the squares' adjacent 'free vertices'.

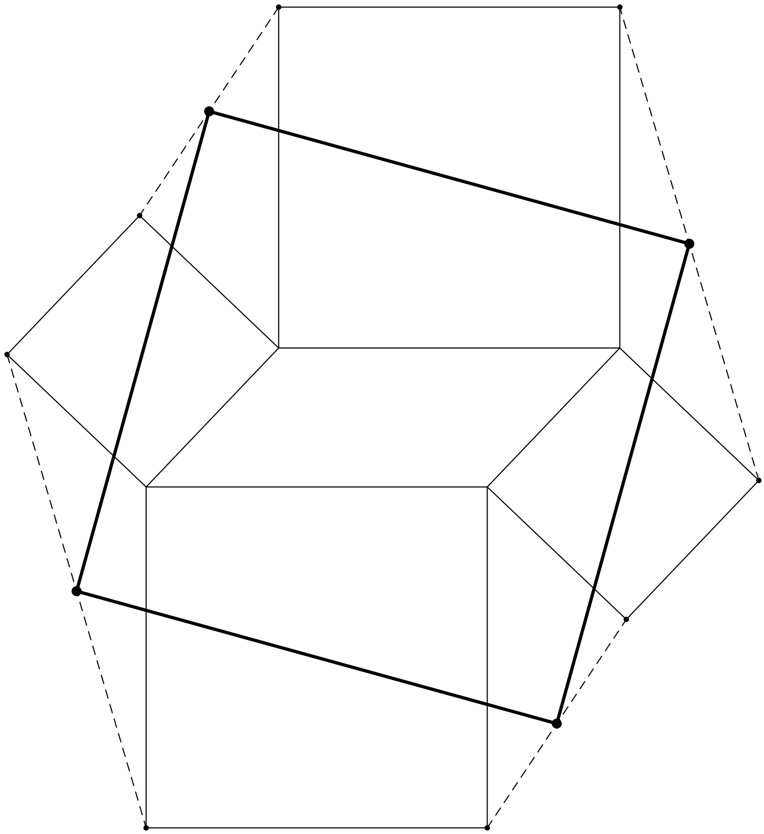

FIGURE 2

Both of the above results are also true for squares constructed inwardly.

As geometry yields a never-ending treasure trove of theorems, one might be inspired by the previous statements to wonder what else the Thébault configuration has to offer us. Indeed, there are a number of other interesting and interrelated results, summarised in our theorem below.

Theorem: Given a parallelogram $ABCD$, construct squares ABQ_1R_2, BCR_1S_2, CDS_1P_2 and DAP_1Q_2 outwardly (or inwardly) on its sides. Let segments P_1P_2, Q_1Q_2, R_1R_2, S_1S_2 have midpoints A', B', C', D' respectively. Then:

(i) Lines P_1P_2, Q_1Q_2, R_1R_2, S_1S_2 enclose a rectangle with the same centre as $ABCD$.

(ii) Segments P_1P_2, Q_1Q_2, R_1R_2, S_1S_2 are of equal length, and the perimeter of the enclosed rectangle is twice that length.

(iii) A', B', C', D' are the vertices of a parallelogram that is congruent to $ABCD$ and inclined 90 degrees to it.

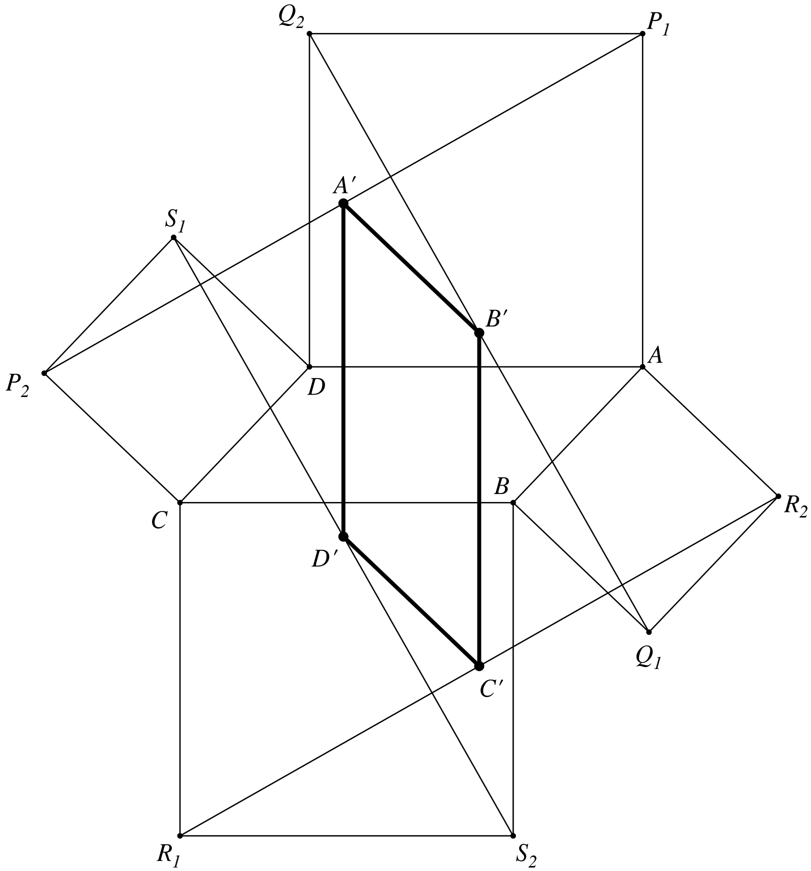

FIGURE 3

In [2] a variant of the Thébault configuration is considered, which we repeat here not only for the sake of completing our brief survey of related results, but also for ease of reference, as we will employ it as a lemma in the proof of the theorem.

Lemma: The centres of the squares drawn on both sides of both diagonals of a parallelogram form another parallelogram congruent to the original but rotated 90° about its centre.

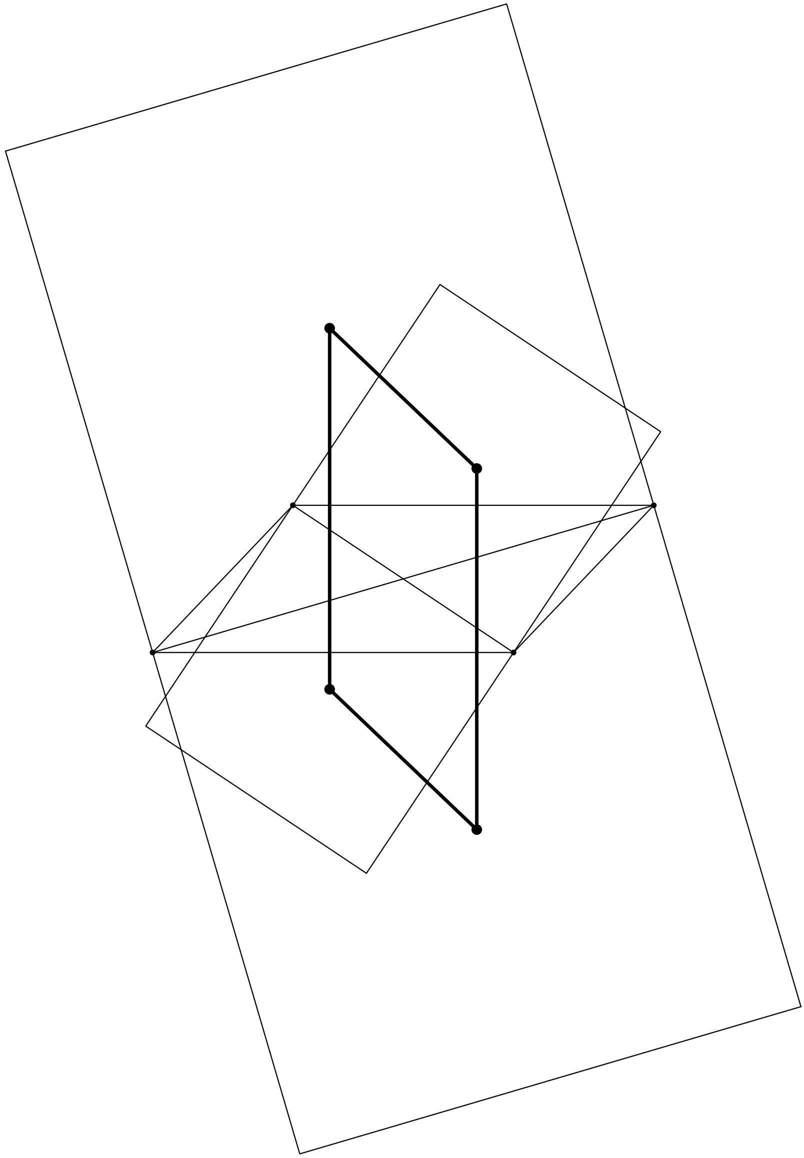

FIGURE 4

Proof of the Theorem: We begin by assuming the squares are constructed outwardly; if appropriate segments are extended where necessary, the proof is similar for inwardly constructed squares.

(i) From symmetry, it suffices to consider one angle in question, say, the one made by Q_1Q_2 and R_1R_2. Let Q_1Q_2 intersect DA and AB at A_1 and A_2 respectively, and R_1R_2 intersect squares at B_1 and B_2, as shown. Triangles AA_1A_2 and BB_1B_2 are congruent by their placement in identical 'hinged square' configurations (where two squares share a common vertex). The desired right angle follows from elementary angle chasing, hinted at in the diagram.

We postpone the centre argument until the conclusion of part (iii), where it will follow quickly.

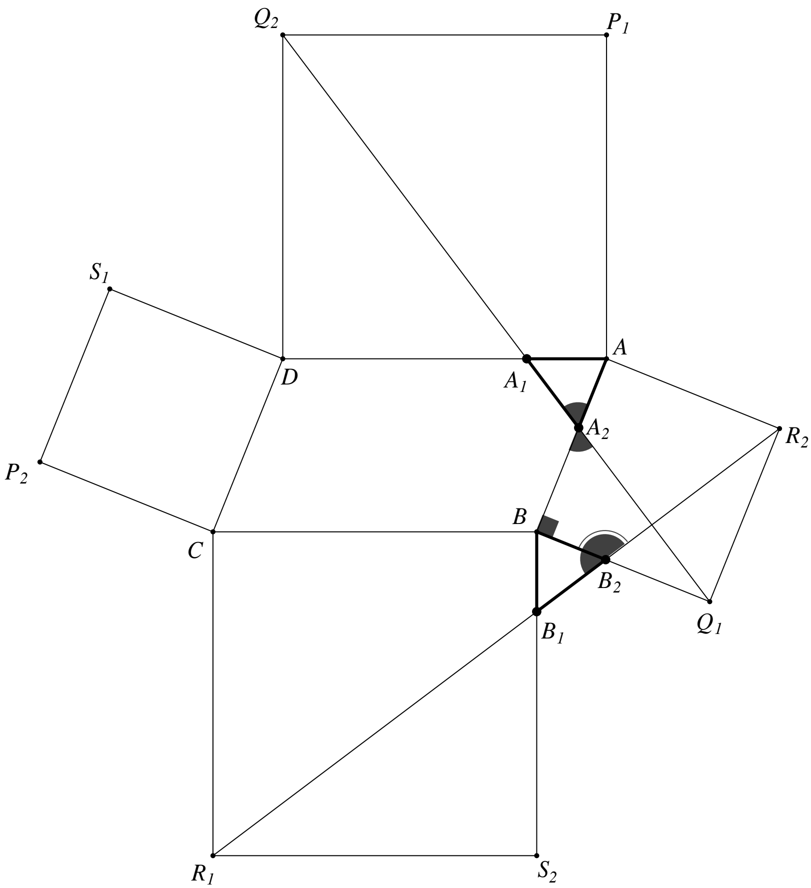

FIGURE 5

(ii) That the segments are congruent is evident from the diagram, as they join 'opposite corners' of identical configurations of hinged squares.

Now consider the hinged squares at D; Bottema's theorem implies that $\Delta A'Q_2S_1$ is right isosceles with vertex A' (see [3]). Then simple angle chasing (using the right angles from part (i)) implies we have a pair of congruent right triangles, giving rise to congruent segments, represented by the single and double ticks below.

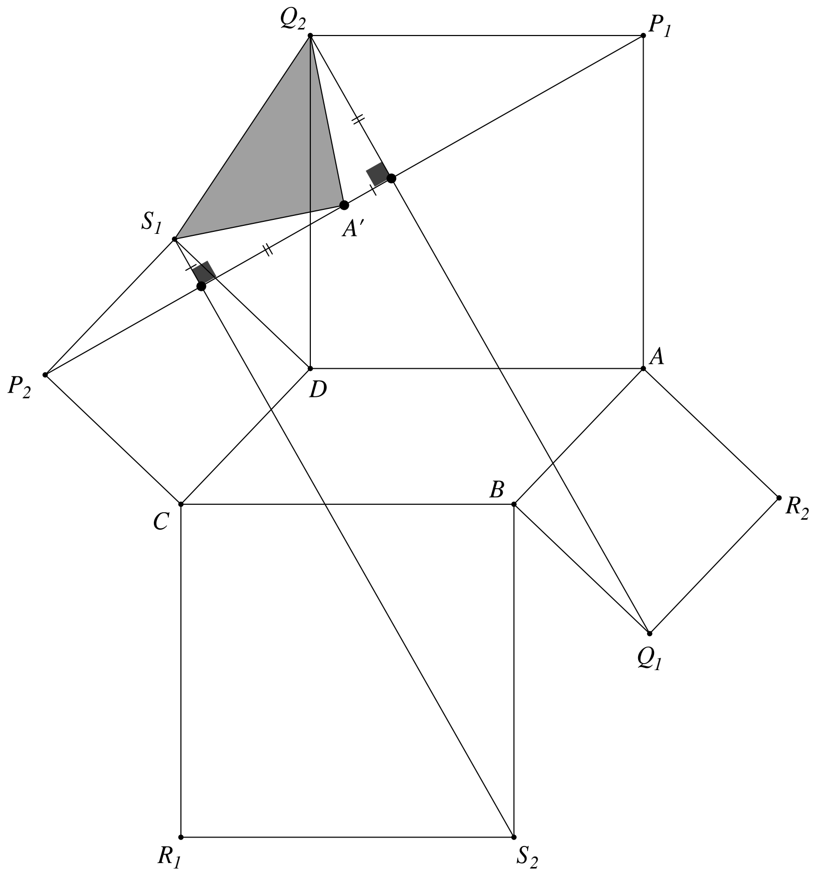

FIGURE 6

The same argument applied to the remaining hinged square pairs (and quickly considering symmetry minimises the required number of ticks) yields all the segment congruences below, and by inspection the rectangle has perimeter equal to $2P_1P_2$.

(iii) Each midpoint lies on the segment connecting the opposite corners of a hinged square configuration. As before, Bottema's theorem implies that each midpoint lies at the vertex of a right isosceles triangle, but with base a diagonal of parallelogram *ABCD*. Thus each midpoint is the centre of a square constructed on the appropriate diagonal of *ABCD*. (For example, A' below is the centre of the square constructed 'above' diagonal *CA*.)

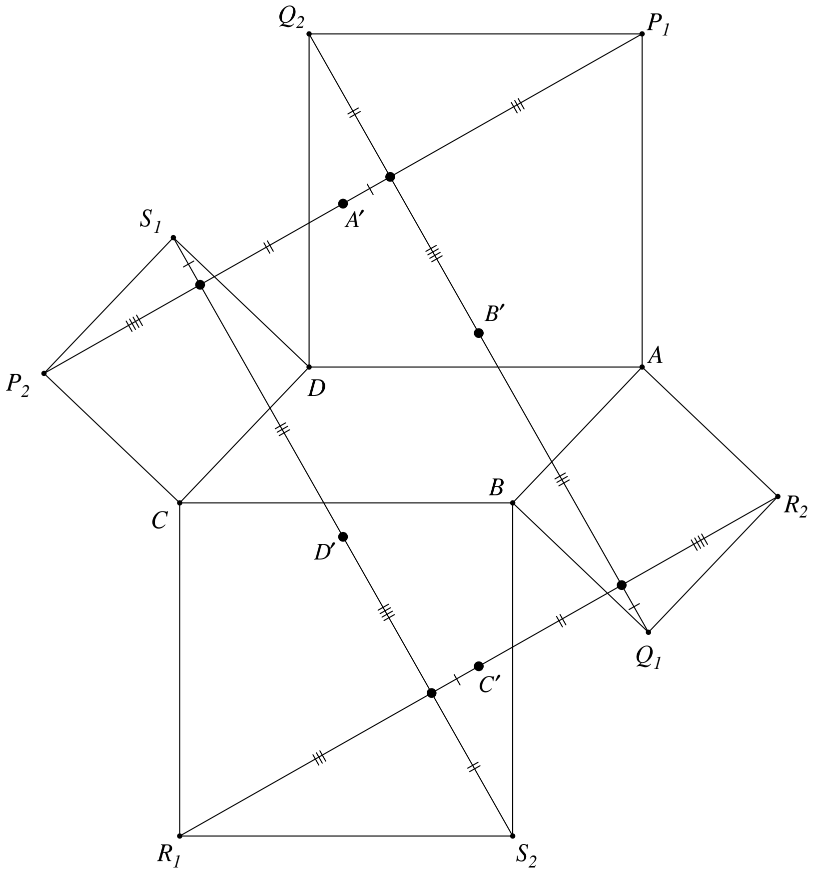

FIGURE 7

Thus A', B', C', D' are the centres of the squares drawn on both sides of both diagonals of parallelogram $ABCD$; the result then follows immediately from the lemma.

Now to finish part (i). Due to symmetry, it is evident that the rectangle and $A'B'C'D'$ have the same centre. But from our lemma $ABCD$ and $A'B'C'D'$ also have the same centre, and we are finished.

References

1. B. Grivoyannis, R. Viglione, Proof without words: the birth of a square (or two), *Math. Gaz.* **101** (November 2017) pp. 497-498.
2. P. Purna, R. Viglione, Proof without words: a variation on Thébault's first problem, *College Math. J.* **44** (2) (2013) p. 135.
3. A. Bogomolny, Bottema's Theorem, accessed August 2019 https://www.cut-the-knot.org/Curriculum/Geometry/Bottema.shtml.

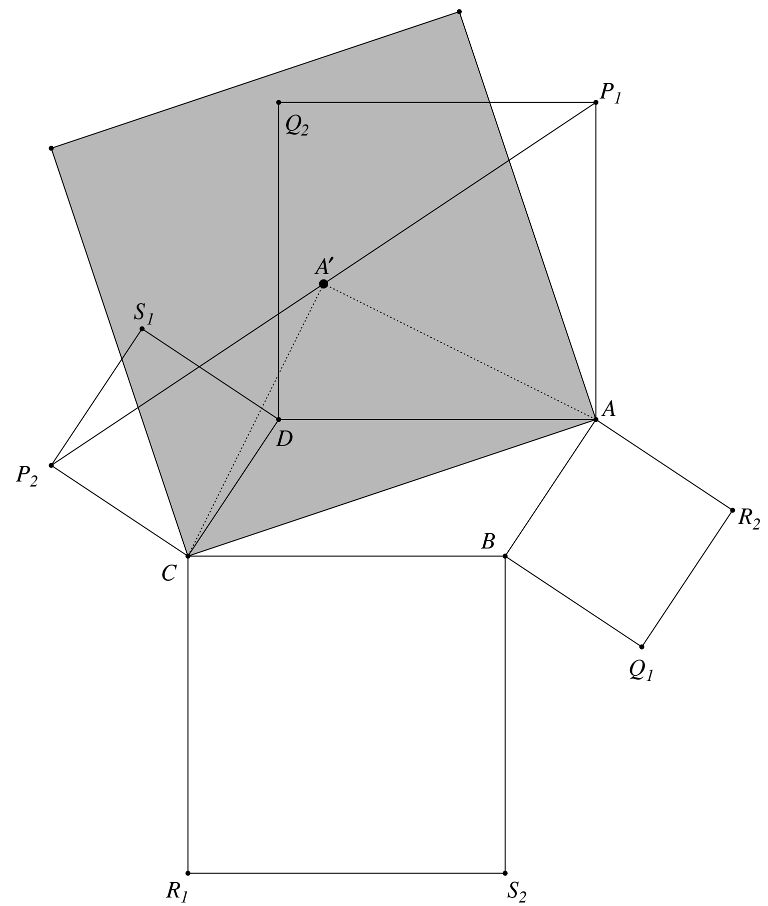

FIGURE 8

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