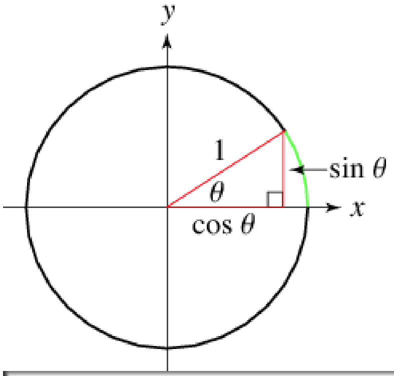
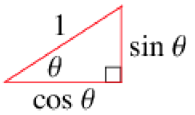
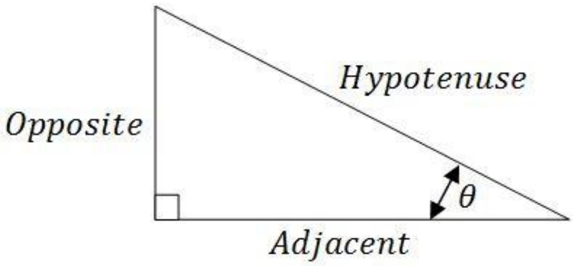
Circular Functions







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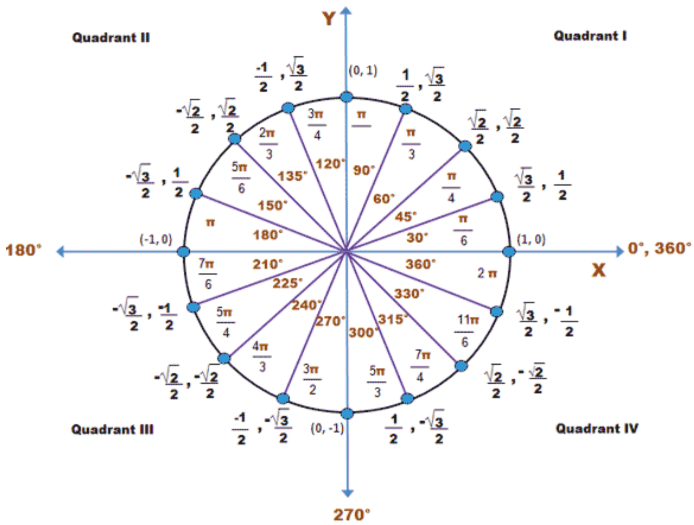
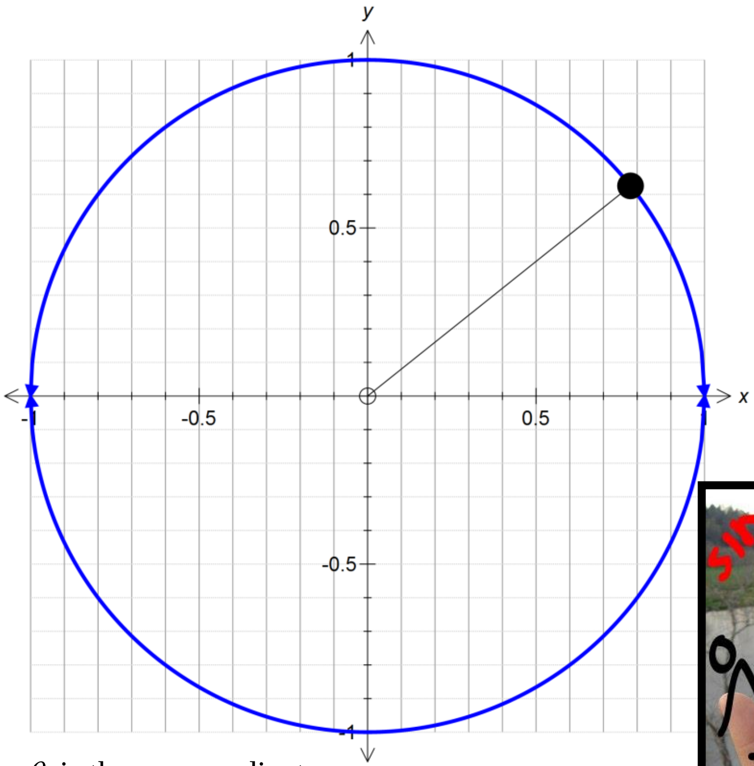
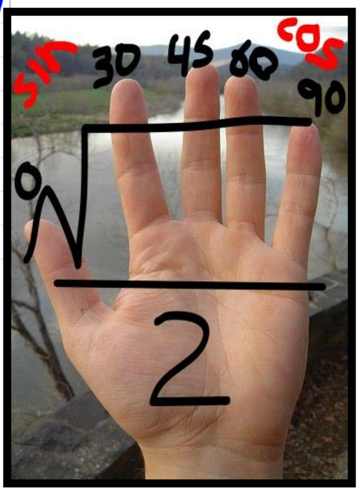


Circular Functions (Trigonometry)

Circular functions Revision

Where do $\sin\theta, \cos\theta$ and $\tan\theta$ come from?
Unit circle (of radius 1)



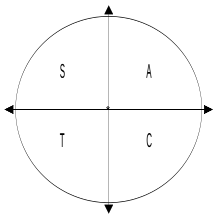


- $\cos\theta$ is the x – coordinate
- $\sin\theta$ is the y – coordinate
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- all 3 are measures of length.
- Remember SOH CAH TOA
- Exact values:

θ°	0	30^o $\frac{\pi}{6}$	45^o $\frac{\pi}{4}$	60^o $\frac{\pi}{3}$	90^o $\frac{\pi}{2}$	180^o π	270^o $\frac{3\pi}{2}$	360^o 2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0	undefined	0

- Angle conversions (between radians and degrees).

- Quadrants and symmetry:
    - All Students Talk C.. (ASTC)



Finding Exact values:

Example: What is the exact value of:

(a) $\sin\frac{5\pi}{4}$; (b) $\tan\frac{-2\pi}{3}$.

(a) 1. Sign: 3rd Quadrant $\Rightarrow$ -ve

2. Angle Equivalent (1st Quadrant): $\frac{5\pi}{4} = \pi + \frac{\pi}{4} \Rightarrow \frac{\pi}{4}$

3. So: $\sin\frac{5\pi}{4} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \; or \; \frac{-\sqrt{2}}{2}$

(b) 1. Sign: 2nd "negative" Quadrant $\Rightarrow$ +ve

2. Angle Equivalent (1st Quadrant): $\frac{-2\pi}{3} = -\pi + \frac{\pi}{3} \Rightarrow \frac{\pi}{3}$

3. So: $\tan\frac{-2\pi}{3} = +\tan\frac{\pi}{3} = \sqrt{3}$

Jump Start Holiday Questions

Review: radians, definitions, exact values, symmetry	**Ex6A** Q 1, 2, 3, 4 (ace for all); **Ex6B** Q 1, 2acegik, 3 acegikmoqsu, 4 aceg, 5 abdfgj, 6 **Ex6C** Q 2

CALCULATOR MODE: Always work in radians

Solving equations involving circular functions.

Finding axis intercepts:
1. Y-intercepts:

- $f(0)$ or $x = 0$.
- E.g. what is the Y-intercept of $f(x) = 3\sin 2\left(x - \frac{\pi}{6}\right) + 2$

2. X-intercepts:

- $f(x) = 0$ or $y = 0$.

Examples: Find all values of θ for:

(a) $\left\{\theta : \cos\theta = \frac{\sqrt{3}}{2}, \quad \theta \in [0, 2\pi]\right\}$

(b) $\{\theta : \sin\theta = -0.7, \quad \theta \in [0, 2\pi]\}$

(c) $\{\theta : 2\sin\theta + 1 = 0, \quad \theta \in [-2\pi, 2\pi]\}$

(d) $\{4\cos 2\theta + 2 = 0, \quad \theta \in [0, 2\pi]\}$

- **Ex6E** 1 ace, 2 ac, 3 ac, 4 ab, 5 abc, 6 ace, 7 ace, 8 acegi; **Ex6J** 4, 5, 6

b. Solve the equation

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \text{ for } x \in [0, \pi].$$

2011 Exam1

Using the TI-Nspire

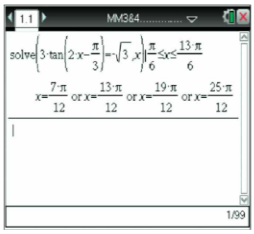
Use **Solve** from the **Algebra** menu as shown.

Using the TI-Nspire

To find the x-axis intercepts,

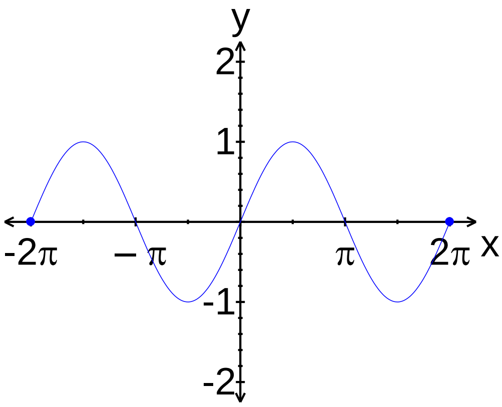
Enter

solve$\left(3\tan\left(2x - \frac{\pi}{3}\right) = -\sqrt{3}, x\right) | \frac{\pi}{6} \le x \le \frac{13\pi}{6}$

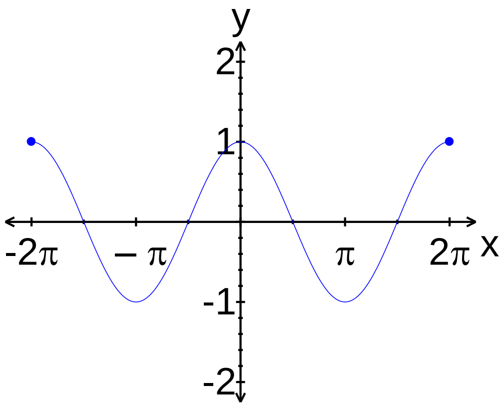


Graphs of Circular Functions

$y = \sin\theta$



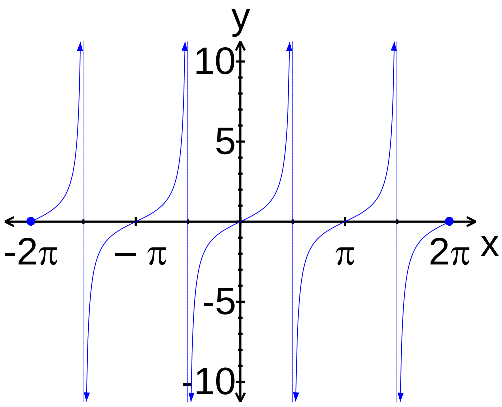
$y = \cos\theta$



- Period = 2π
- Amplitude = 1
- Range: [-1, 1]

$$period = \frac{2\pi}{n}$$

$y = \tan\theta$



- Period = π
- We don't refer to the amplitude for $y = \tan\theta$
- Range: R

$$period = \frac{\pi}{n}$$

Transformations of $y = \sin\theta$ & $y = \cos\theta$

$y = \sin\theta \rightarrow y = a\sin n(\theta - b) + c$ & $y = \cos\theta \rightarrow y = a\cos n(\theta - b) + c$

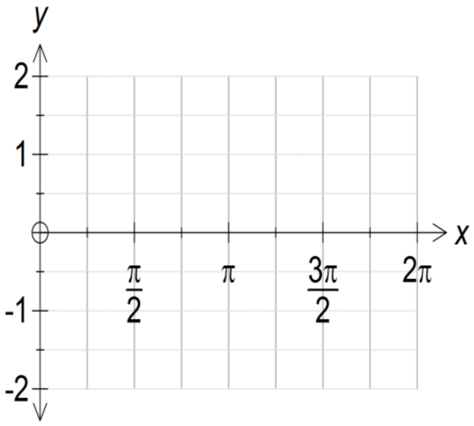
- *a:* a dilation of factor "*a*" from the *x-axis*.
- *n:* a dilation of factor "$\frac{1}{n}$" from the *y-axis*.
- *b:* a translation of *b* units along the *x-axis.*
- *c:* a translation of *c* units along the *y-axis.*

1. Dilations

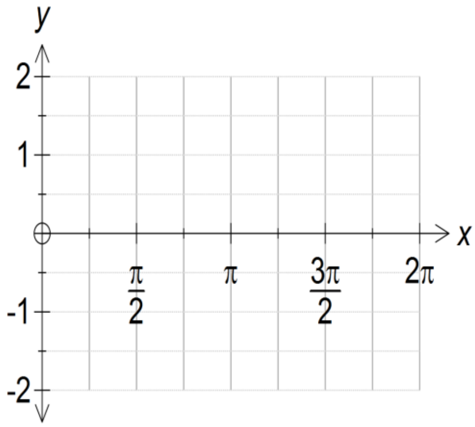
(a) The effect of "a"

Graph the following graphs: (i) $y = 2\cos\theta$; (ii) $y = \frac{\sin\theta}{2}$; where $\theta \in [0, \quad 2\pi]$

(i)



(ii)

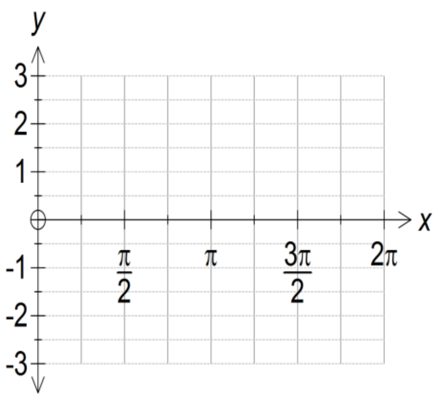


- "*a*" affects the amplitude.

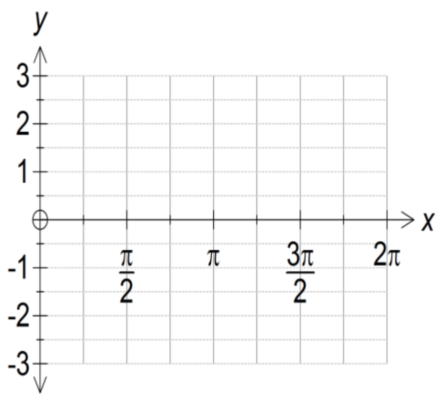
(b) The effect of "n"

Graph the following graphs: (i) $y = 3\cos 2\theta$; (ii) $y = 3\sin\left(\frac{\theta}{2}\right)$; where $\theta \in [0, \quad 2\pi]$

(i)



(ii)



- "n" affects the period.
- $period = \frac{2\pi}{n}$

2. Reflections.

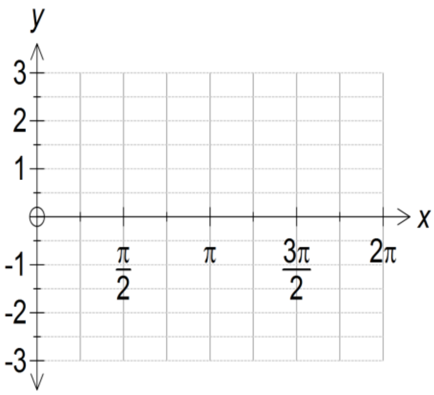
- Two types:
  - Reflection in the *x-axis*: $-f(x)$
  - Reflection in the *y-axis*: $f(-x)$

Examples:

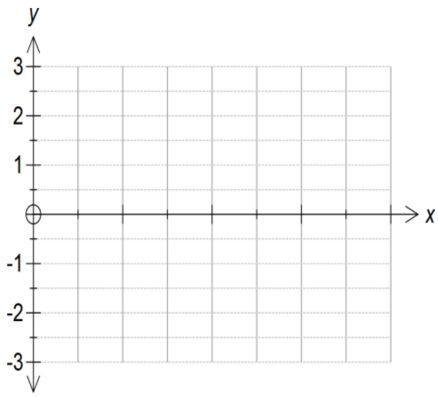
Sketch the graphs of the following:

(a) $y=-3\sin 2\theta$; (b) $y=2\cos\left(-\frac{\pi\theta}{3}\right)$; where $\theta\in[0,\quad 2\pi]$

(a)



(b)

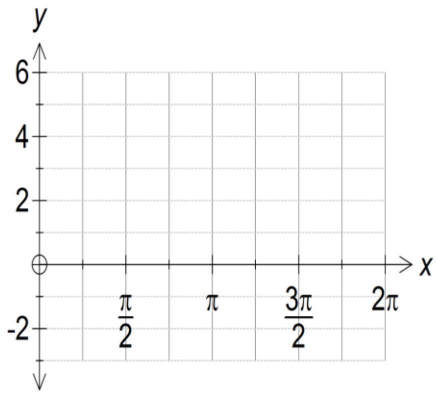


3. Translations

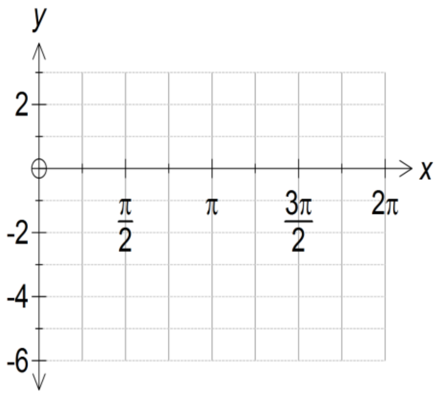
(a) The effect of "*c*"

Sketch the following: (i) $y=3\sin\theta+3$; (ii) $y=2\cos 2\theta-3$; where $\theta\in[0,\quad 2\pi]$

(i)



(ii)

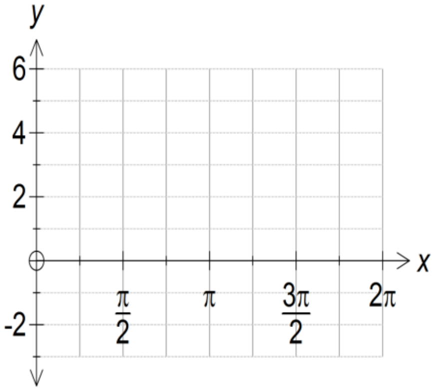


(b) The effect of "*b*"

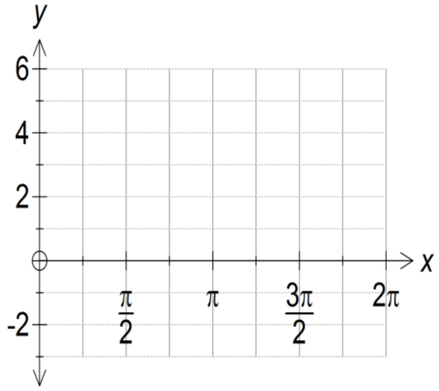
Sketch the following:

(i) $y = 2\sin\left(\theta + \frac{\pi}{4}\right)$; (ii) $y = 3\cos 2\left(\theta - \frac{\pi}{3}\right)$; where $\theta \in [0, \quad 2\pi]$

(i)



(ii)



Combining all transformations

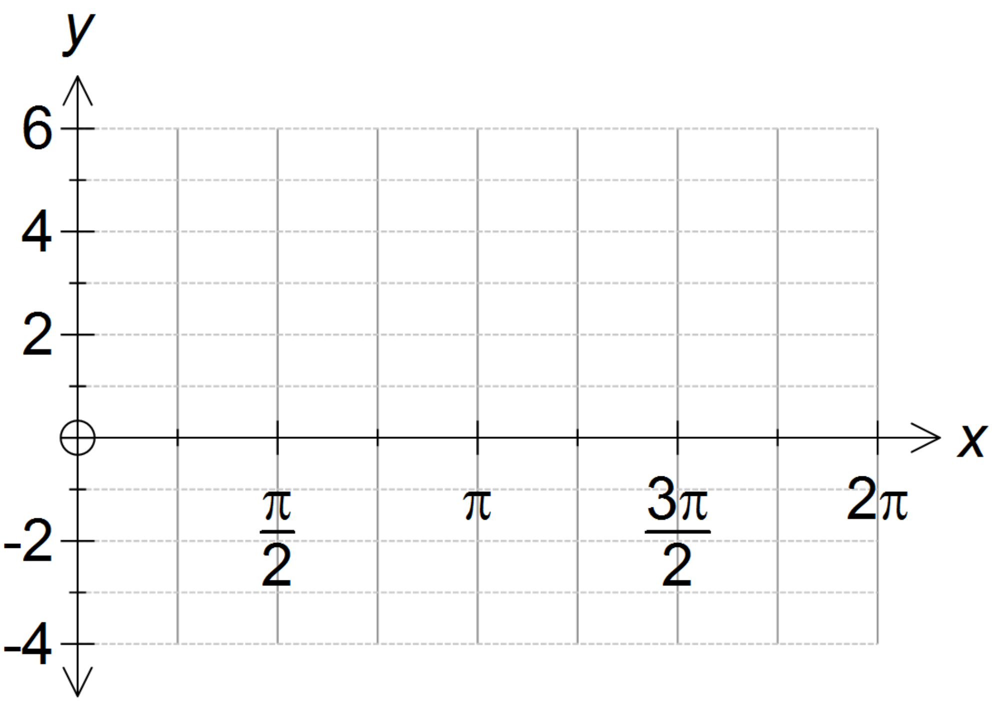
Example: Sketch the graph of $f(\theta) = 3\sin\left(2\theta - \frac{\pi}{2}\right) + 2, \quad \theta \in [0, 2\pi]$

Rewrite: $f(\theta) = 3\sin 2\left(\theta - \frac{\pi}{4}\right) + 2$

$a = 3, b = \frac{\pi}{4}, c = 2 \; and \; n = 2$

Sketch $f(\theta) = 3\sin 2\theta$ first:

Secondly with translations:



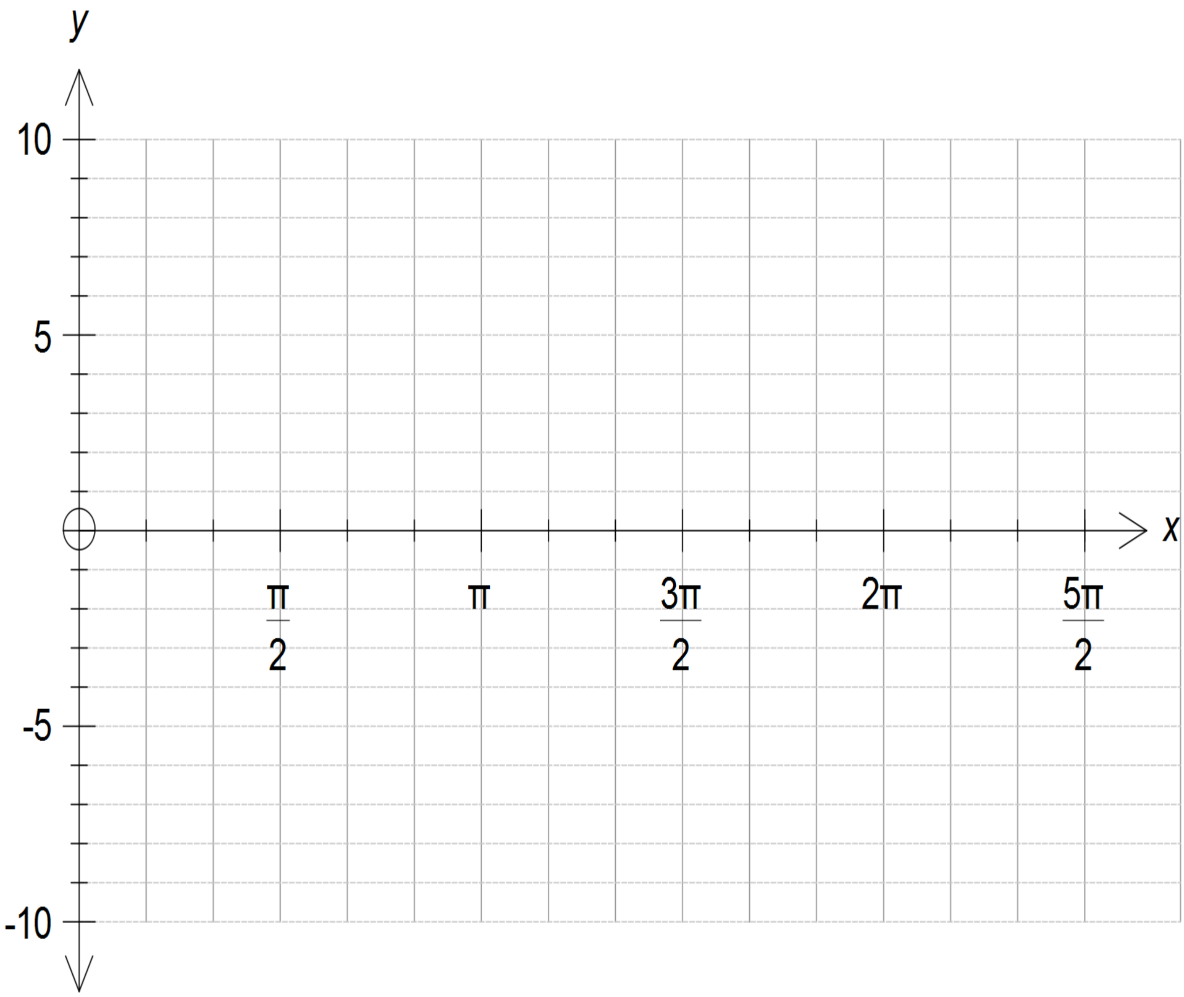
Note: X-intercepts need to be found!!

- **Ex6F** 1 adfhi, 2, 4, 5; **Ex6G** 1, 2 ac, 3 ef, 5 acfgh, 6, 7

Graphs &Transformations of the Tangent function

Example: Sketch $y = 3\tan\left(2x - \frac{\pi}{3}\right)$ *for* $\frac{\pi}{6} \le x \le \frac{13\pi}{6}$

Rewrite: $y = 3\tan 2\left(x - \frac{\pi}{6}\right)$



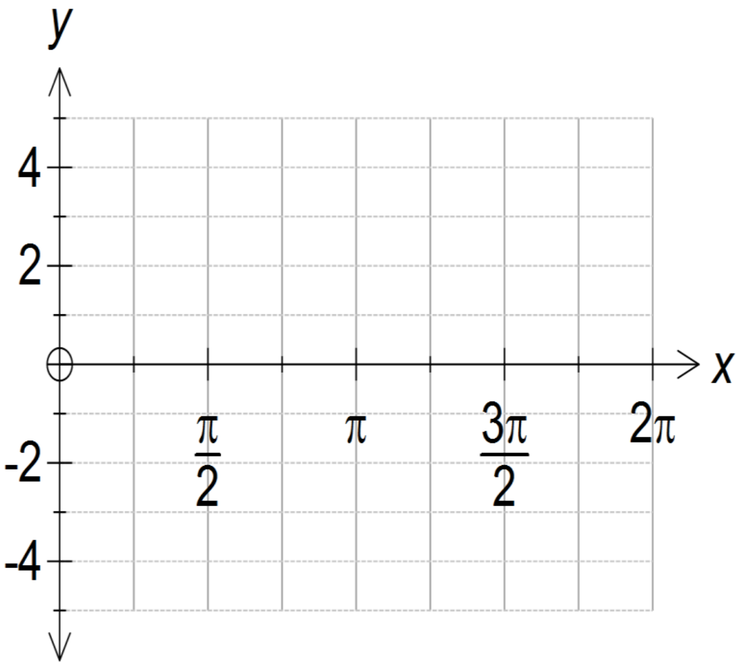
- **Ex6J** 1, 2, 7, 8, 9

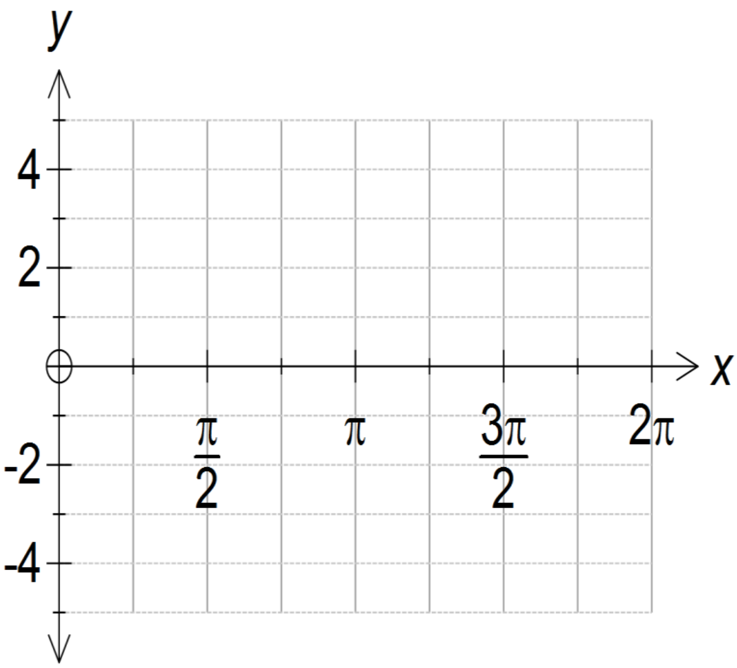
Addition of ordinates (add the 'y' values)

Example:

(a) On the same set of axes sketch $f(x) = 2\sin x$ and $g(x) = 3\cos 2x$ for $0 \le x \le 2\pi$;

(b) Use addition of ordinates to sketch the graph of $y = 2\sin x + 3\cos 2x$.





Note: For $y = 2\sin(x) - 3\cos(2x)$ it is easier to do $y = 2\sin(x) + (-3\cos(2x))$

- **Ex6H** 1 ace

Solving Equations where both *sin* & *cos* appear

Example: Solve for x, $x \in [0, 2\pi]$:

(i) $\sin x = 0.5\cos x$

(ii) $\sin 3x - \sqrt{3}\cos 3x = 0$

- **Ex6J** 10, 11 acegi, 12

General Solutions to Circular Functions

Example: Solve $\cos x = \frac{1}{2}$

Solution:

$\cos x = \frac{1}{2}$
1.*Cos positive Quad ...*-
2.*Angle* :

3. $x = ...$

$x = ...,$

generally :
$x =$ $\quad\quad$ $Check : n = 0, n = 1, n = -1$

So in general terms:

Example: Solve $\sin x = \frac{1}{2}$

Solution:

$\sin x = \frac{1}{2}$
1.*Sin positive Quad ...*

2.*Angle* :

3. $x = ...$

$x = ...,$

generally :
$x =$
$Check : n = 0, n = 1, n = -1$
or
$x =$
$Check : n = 0, n = 1, n = -1$

So in general terms:

The above can be simplified to

For $\tan x = a$
$x = n\pi + \tan^{-1}(a) \quad , n \in Z$

Example 1: Find the general solution for $2\sin\left(x+\frac{\pi}{3}\right)=-1$

Solution:

Example 2: Find the general solution to $2\cos\left(2x+\frac{\pi}{4}\right)=\sqrt{2}$, and hence find all the solutions from $(-2\pi, 2\pi)$.

Solution:

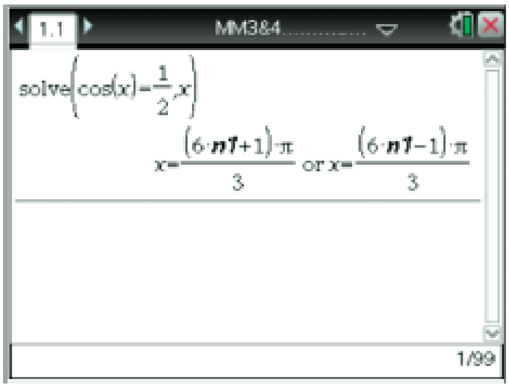
- **Ex6K** 1, 2, 3, 6ab, 8, 9

Using the TI-Nspire

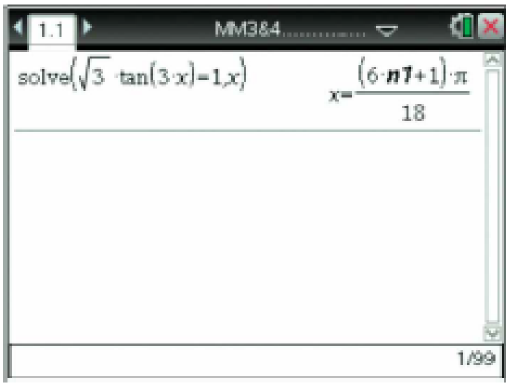
Make sure the calculator is in Radian mode.

a Use **Solve** from the **Algebra** menu and complete as shown.

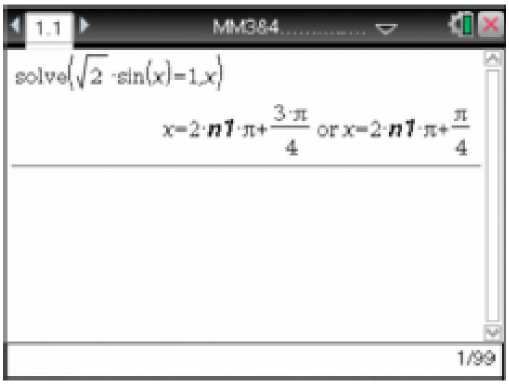
Note the use of $\frac{1}{2}$ rather than 0.5 to ensure that the answer is exact.



b Complete as shown.

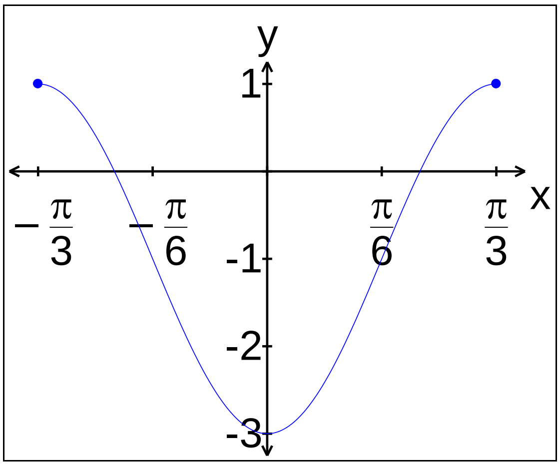


c Complete as shown.



- **Determining Rules for Circular Functions**

Example: The graph shown has the rule of the form: $y = a\cos n(t-b)+c$, find a, b, c & n.



- **Ex6I** 1, 2, 3, 4, 5, 6, 7, 8, 9; **Ex6J** 14, 15

Applications of Circular Functions

worked example 24

The temperature in degrees Celsius on a day in May at Mt Buller is expected to follow the model

$$T = 5 - 7\cos\frac{\pi}{12}(t - 4)$$

where t is the number of hours after midnight. The snow-making machines will only operate efficiently when the temperature is below 5°C. Sketch the graph of the temperature for one full day, and predict the period of time for which the machine will be able to operate.

- **Ex6L** 1, 2, 4, 6 **Ex 6N**

Past Exam Questions

2008

Question 3

Solve the equation $\cos\left(\frac{3x}{2}\right) = \frac{1}{2}$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2 marks

Question 18

Let $f: \left[0, \frac{\pi}{2}\right] \to R$, $f(x) = \sin(4x) + 1$. The graph of f is transformed by a reflection in the x-axis followed by a dilation of factor 4 from the y-axis.

The resulting graph is defined by

A. $g: \left[0, \frac{\pi}{2}\right] \to R \qquad g(x) = -1 - 4\sin(4x)$

B. $g: [0, 2\pi] \to R \qquad g(x) = -1 - \sin(16x)$

C. $g: \left[0, \frac{\pi}{2}\right] \to R \qquad g(x) = 1 - \sin(x)$

D. $g: [0, 2\pi] \to R \qquad g(x) = 1 - \sin(4x)$

E. $g: [0, 2\pi] \to R \qquad g(x) = -1 - \sin(x)$

2009

Question 4

Solve the equation $\tan(2x) = \sqrt{3}$ for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

3 marks

Question 4

The general solution to the equation $\sin(2x) = -1$ is

A. $x = n\pi - \frac{\pi}{4},\ n \in Z$

B. $x = 2n\pi + \frac{\pi}{4}$ or $x = 2n\pi - \frac{\pi}{4},\ n \in Z$

C. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{2},\ n \in Z$

D. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{4},\ n \in Z$

E. $x = n\pi + \frac{\pi}{4}$ or $x = 2n\pi + \frac{\pi}{4},\ n \in Z$

Question 12

A transformation $T: R^2 \rightarrow R^2$ that maps the curve with equation $y = \sin(x)$ onto the curve with equation $y = 1 - 3\sin(2x + \pi)$ is given by

A. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \pi \\ 1 \end{bmatrix}$

B. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{2} \\ 1 \end{bmatrix}$

C. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \pi \\ 1 \end{bmatrix}$

D. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ 1 \end{bmatrix}$

E. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ -1 \end{bmatrix}$

2010

Question 4

a. Write down the amplitude and period of the function

$$f: R \rightarrow R, f(x) = 4\sin\left(\frac{x+\pi}{3}\right).$$

2 marks

b. Solve the equation $\sqrt{3}\sin(x) = \cos(x)$ for $x \in [-\pi, \pi]$.

2 marks

Question 3

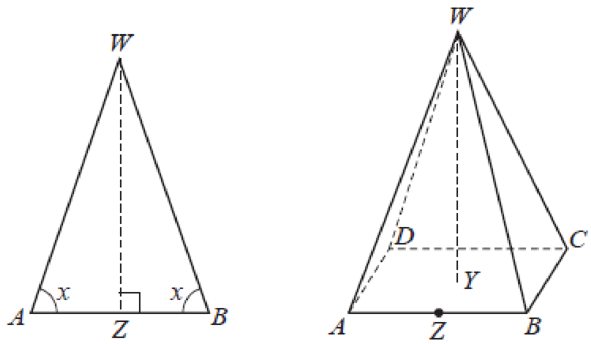
An ancient civilisation buried its kings and queens in tombs in the shape of a square-based pyramid, *WABCD*.

The kings and queens were each buried in a pyramid with $WA = WB = WC = WD = 10$ m.

Each of the isosceles triangle faces is congruent to each of the other triangular faces.

The base angle of each of these triangles is x, where $\frac{\pi}{4} < x < \frac{\pi}{2}$.

Pyramid *WABCD* and a face of the pyramid, *WAB*, are shown here.



Z is the midpoint of AB.

a. **i.** Find AB in terms of x.

b. Show that the total surface area (including the base), S m^2, of the pyramid, $WABCD$, is given by $S = 400(\cos^2(x) + \cos(x)\sin(x))$.

2 marks

c. Find WY, the height of the pyramid $WABCD$, in terms of x.

2 marks

d. The volume of any pyramid is given by the formula Volume $= \frac{1}{3} \times$ area of base $\times$ vertical height.

Show that the volume, T m^3, of the pyramid $WABCD$ is $\frac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}$.

2011

Question 3

a. State the range and period of the function

$$h: R \rightarrow R,\ h(x) = 4 + 3\cos\left(\frac{\pi x}{2}\right).$$

2 marks

b. Solve the equation

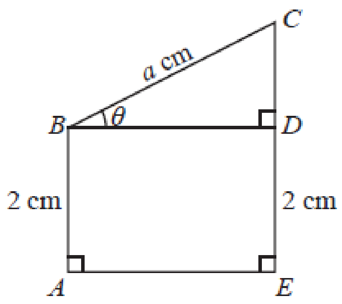
$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \text{ for } x \in [0,\ \pi].$$

Question 10

The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. The point D is on line segment EC with $AB = ED = 2$ cm and $BC = a$ cm, where a is a positive constant.

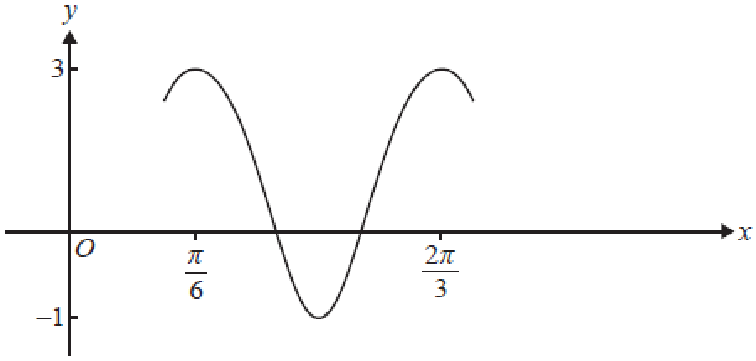
$$\angle BAE = \angle CEA = \frac{\pi}{2}$$

Let $\angle CBD = \theta$ where $0 < \theta < \frac{\pi}{2}$.



a. Find BD and CD in terms of a and θ.

Question 15



The graph shown could have equation

A. $y = 2\cos\left(x + \frac{\pi}{6}\right) + 1$

B. $y = 2\cos 4\left(x - \frac{\pi}{6}\right) + 1$

C. $y = 4\sin 2\left(x - \frac{\pi}{12}\right) - 1$

D. $y = 3\cos\left(2x + \frac{\pi}{6}\right) - 1$

E. $y = 2\sin\left(4x + \frac{2\pi}{3}\right) - 1$

2012

Question 6

The graphs of $y = \cos(x)$ and $y = a\sin(x)$, where a is a real constant, have a point of intersection at $x = \frac{\pi}{3}$.

a. Find the value of a.

2 marks

b. If $x \in [0, 2\pi]$, find the x-coordinate of the other point of intersection of the two graphs.

1 mark

Question 1

The function with rule $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$ has period

A. 3

B. 5

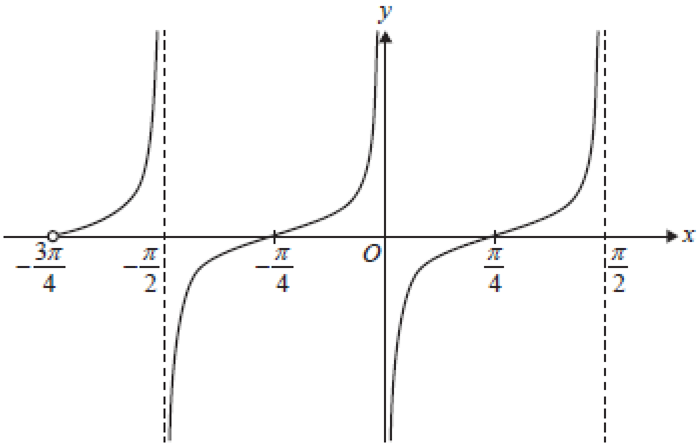
C. 10

D. $\frac{\pi}{5}$

E. $\frac{\pi}{10}$

Question 6

A section of the graph of f is shown below.



The rule of f could be

A. $f(x) = \tan(x)$

B. $f(x) = \tan\left(x - \frac{\pi}{4}\right)$

C. $f(x) = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$

D. $f(x) = \tan\left(2\left(x - \frac{\pi}{2}\right)\right)$

E. $f(x) = \tan\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right)$

Question 7

The temperature, T °C, inside a building t hours after midnight is given by the function

$$f:[0, 24] \to R, T(t) = 22 - 10\cos\left(\frac{\pi}{12}(t-2)\right)$$

The average temperature inside the building between 2 am and 2 pm is

A. 10 °C

B. 12 °C

C. 20 °C

D. 22 °C

E. 32 °C

Question 19

A function f has the following two properties for all real values of θ.

$$f(\pi - \theta) = -f(\theta) \text{ and } f(\pi - \theta) = -f(-\theta)$$

A possible rule for f is

A. $f(x) = \sin(x)$

B. $f(x) = \cos(x)$

C. $f(x) = \tan(x)$

D. $f(x) = \sin\left(\frac{x}{2}\right)$

E. $f(x) = \tan(2x)$

2013

Question 4 (2 marks)

Solve the equation $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 4\pi]$.

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Question 1

The function with rule $f(x) = -3\tan(2\pi x)$ has period

A. $\frac{2}{\pi}$

B. 2

C. $\frac{1}{2}$

D. $\frac{1}{4}$

E. 2π

Question 7

The function $g: [-a, a] \rightarrow R,\ g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.
The maximum possible value of a is

A. $\frac{\pi}{12}$

B. 1

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

E. $\frac{\pi}{2}$

Question 1 (12 marks)

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature (T °C) is given by $T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right)$, $0 \le t \le 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a. State the maximum temperature in the greenhouse and the values of t when this occurs. 2 marks

b. State the period of the function T. 1 mark

c. Find the smallest value of t for which $T = 26$. 2 marks

d. For how many hours during the 24-hour time interval is $T \ge 26$? 2 marks

2014

Question 3 (2 marks)

Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \le x \le \pi$.

Question 1 (7 marks)

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n. 2 marks

b. Find the maximum and minimum populations of wombats in this location. 2 marks

c. Find $n(10)$. 1 mark

d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$. 2 marks

2015

Question 5 (3 marks)

On any given day, the depth of water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right),\ 0 \le t \le 24$$

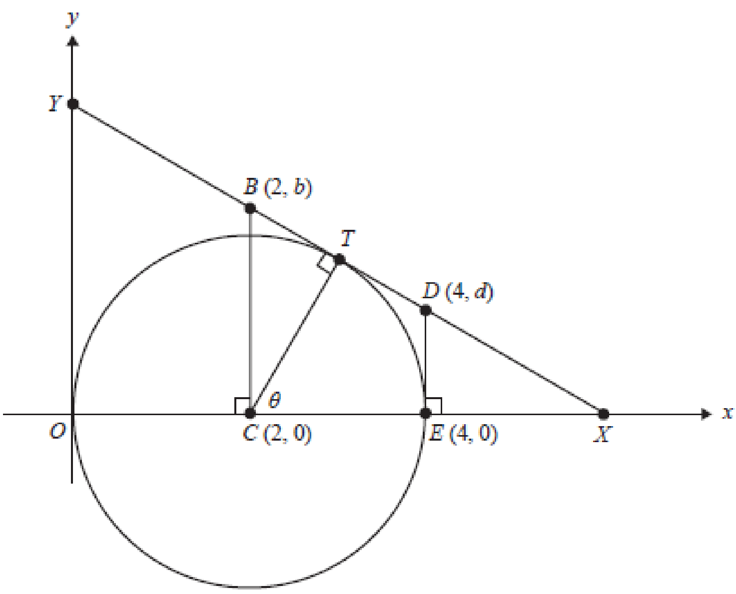
where h is the depth of water, in metres, and t is the time, in hours, after 6 am.

a. Find the minimum depth of the water in the river. 1 mark

b. Find the values of t for which $h(t) = 10$. 2 marks

Question 10 (7 marks)

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates (2, 0). The angle ECT is θ, where $0 < \theta \le \frac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.

a. Find the coordinates of T in terms of θ. 1 mark

b. Find the gradient of the tangent to the circle at T in terms of θ. 1 mark

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c. The equation of the tangent to the circle at T can be expressed as

$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XY.

Find b in terms of θ. 1 mark

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ii. Point D, with coordinates $(4, d)$, is on the line segment XY.

Find d in terms of θ. 1 mark

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Question 1

Let $f: R \to R$, $f(x) = 2\sin(3x) - 3$.

The period and range of this function are respectively

A. period $= \frac{2\pi}{3}$ and range $= [-5, -1]$

B. period $= \frac{2\pi}{3}$ and range $= [-2, 2]$

C. period $= \frac{\pi}{3}$ and range $= [-1, 5]$

D. period $= 3\pi$ and range $= [-1, 5]$

E. period $= 3\pi$ and range $= [-2, 2]$