PROJECTILE MOTION *Range*

The range is the maximum x-displacement of the projectile. This occurs at the time of flight, so that, starting with the kinematic equation for the x-motion,

$$x(t) = x_0 + v_0\, t \cos(\theta) + \frac{1}{2} a_x t^2$$

and with the initial x defined to be zero (we can always shift the coordinate system to make this so), and having zero acceleration in the x direction, we have

$$x(T) = R = v_0\, T \cos(\theta)$$

The TOF has been developed elsewhere, and is

$$T = \frac{1}{g}\left[v_0 \sin(\theta) + \sqrt{(v_0 \sin(\theta))^2 + 2\, g\, y_0}\right]$$

Thus we can write a general expression for the range:

$$\boxed{R = \frac{v_0}{g} \cos(\theta)\left[v_0 \sin(\theta) + \sqrt{(v_0 \sin(\theta))^2 + 2\, g\, y_0}\right]} \tag{1}$$

This can also be written:

$$R = \frac{v_0^2 \sin(2\,\theta)}{2\, g}\left(1 + \sqrt{1 - \frac{2\, g\, y_0}{v_0^2 \sin(\theta)^2}}\right)$$

In the special case of a zero initial height we have the usual textbook result

$$R = \frac{2\, v_0^2 \sin(\theta) \cos(\theta)}{g} = \frac{v_0^2}{g} \sin(2\,\theta) \qquad y_0 = 0$$

This is the range for a given angle; to get a given range X, the angle needed is found by inverting, so that

$$\theta_X = \frac{1}{2} \operatorname{asin}\left(\frac{X\, g}{v_0^2}\right)$$

If the initial angle is zero, then

$$R = v_0 \sqrt{\frac{2\, y_0}{g}} \qquad \theta = 0$$

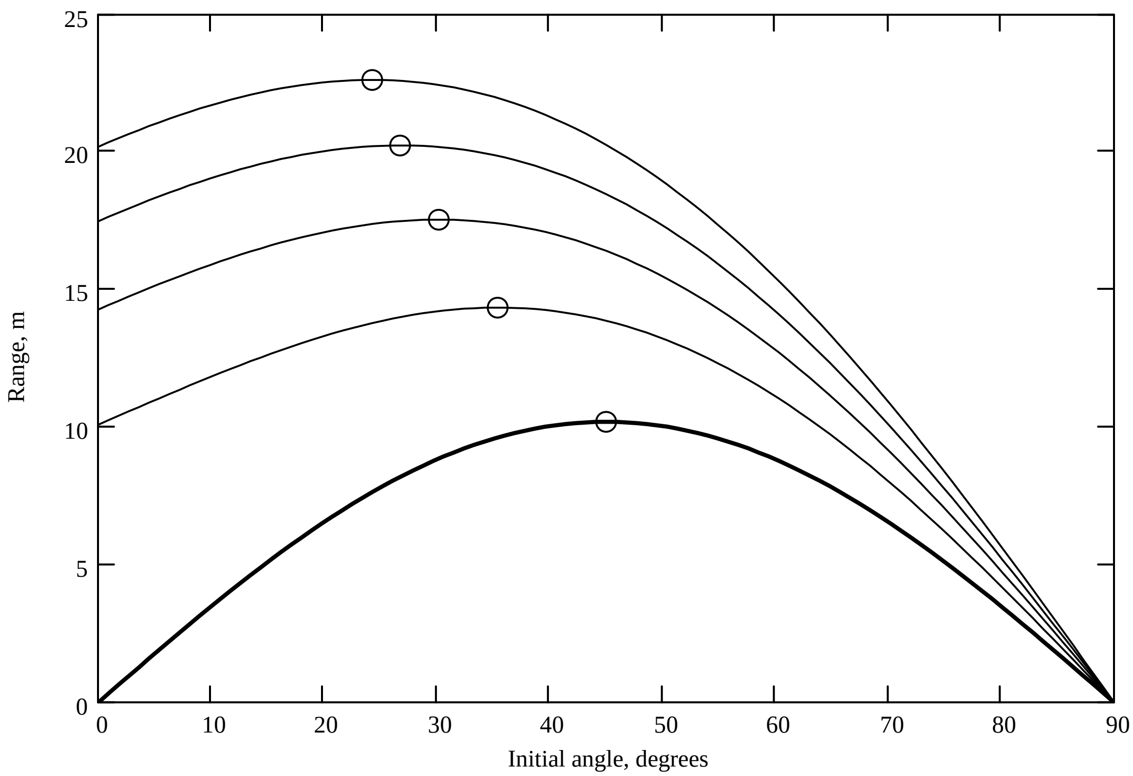
which is just the initial velocity times the TOF for this case.

Much more can be done with this (see plots), in terms of finding the maximum range for the general case. This is treated in a separate paper in this series. For now, just observe that the maximum range for the zero initial height case is obtained at an angle of 45 degrees, since the sine of 2θ is maximized at unity when 2θ is 90 degrees, so θ is 45 degrees. We can also find this by differentiating the zero-height range equation with respect to the angle, set this to zero, and find the angle at the critical point:

$$\frac{dR}{d\theta} = \frac{d}{d\theta}\frac{v_0^2}{g}\sin(\theta) = \frac{2\,v_0^2}{g}\cos(2\,\theta_{opt}) = 0 \qquad \theta_{opt} = \frac{\pi}{4}$$

$$\frac{d^2}{d\theta^2}R = \frac{-4\,v_0^2}{g}\sin(2\,\theta)$$ since this is negative for θ in $(0,\pi/2)$, R is a maximum

One interesting plot is to show the range as a function of the launch angle for various initial heights, for some given initial velocity. Note that only the zero initial height case (thick line) is symmetric. This says that there are two angles that will produce the same range. See the paper on "Galileo angles". Note also that the zero-height line has its maximum at 45 degrees, as we just proved.



Range vs. initial angle, for initial height of 0 (thick line), 0:5:20 m; initial velocity 10 m/s.

From the "Optimum angle" paper we will find that the angle which maximizes the range is

$$\theta_{opt} = \operatorname{atan}\left(\frac{v_0}{\sqrt{v_0^2 + 2\,g\,y_0}}\right)$$

and that this maximum range will be

$$R_{max} = \frac{v_0}{g}\sqrt{v_0^2 + 2\,g\,y_0}$$

These points are illustrated on the plot above as circles.