Elipse as unit circle with respect to some distance

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Abstract

In this geogebra activity we compare a distance induced by an inner product with the usual distance on the Euclidean plane (also induced by the standard dot product on $\mathbb{R}^2$).
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1 Introduction

In this geogebra activity we consider the distance introduced by the following inner products on $\mathbb{R}^2$. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. We define $\langle (x_1, y_1), (x_2, y_2) \rangle := x_1y_1 + x_2y_2$ and $\langle (x_1, y_1), (x_2, y_2) \rangle_A := x_1y_1 + x_2y_2 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1$. Note that the $\langle \cdot, \cdot \rangle$ is the usual dot product and it satisfies the axioms of an inner product. The inner product $\langle \cdot, \cdot \rangle_A$ is defined by the positive definite matrix $A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ as $\langle u, v \rangle_A = u^T Av$ for all $u, v \in \mathbb{R}^2$.

Let us consider the norms $\|(x, y)\| = \sqrt{x^2 + y^2}$ and $\|(x, y)\|_A = \sqrt{x^2 + y^2 + xy}$. Therefore the unit circle defined by the these distances are $C_1 = \{(x, y) \in \mathbb{R}^2 | \|(x, y)\| = 1\}$ and $C_2 = \{(x, y) \in \mathbb{R}^2 | \|(x, y)\|_A = 1\}$. That is

$$C_1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\} \text{ and } C_2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 + xy = 1\}.$$

Now look at the geogebra activity and observe that the green circle is C_1 and the purple elipse is the circle C_2. The red coloured ellipsoid is the proof that $\langle \cdot, \cdot \rangle_A$ is an inner product. That is $\langle (x, y), (x, y) \rangle_A = 0$ if and only if $(x, y) = (0, 0)$. This can also be seen analytically. That is $\langle (x, y), (x, y) \rangle_A = x^2 + y^2 + xy = x^2 + \frac{y^2}{4} + xy + \frac{3y^2}{4} = (x + \frac{y}{2})^2 + \frac{3y^2}{4} \geq 0$. Further $(x + \frac{y}{2})^2 + \frac{3y^2}{4} = 0$ if and only if $(x, y) = (0, 0)$.