[MAA 5.5-5.6] MONOTONY AND CONCAVITY

SOLUTIONS

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O. Practice questions

1. (a) $f'(x)=3x^2+1>0$, f increasing (b) $f'(x)=-25x^4\le 0$, f decreasing

(c) $f'(x)=-6e^{2x}<0$, f decreasing (d) $f'(x)=\dfrac{7}{(3x+5)^2}>0$, f increasing

2. (a) $f'(x)=3x^2+6x-9$

$3x^2+6x-9=0 \Leftrightarrow x^2+2x-3=0$

$x=-3$ (max) $x=1$ (min) (either by table of signs or by 2nd derivative test)

(b) $f''(x)=6x+6$

$6x+6=0 \Leftrightarrow x=-1$

$x=-1$ (point of inflexion) (by table of signs)

(c) look at the GDC

3. (a) $f'(x)=3x^2+6x+3$

$3x^2+6x+3=0 \Leftrightarrow x^2+2x+1=0$

$x=-1$ neither max nor min (by table of signs)

(b) $f''(x)=6x+6$

$6x+6=0 \Leftrightarrow x=-1$

$x=-1$ (point of inflexion) (by table of signs)

So at $x=-1$, stationary point of inflexion

(c) look at the GDC

4. by using table of signs

$x=1$ (max) $x=3$ (min) $x=4$ (stationary point of inflexion) ,

5. by using table of signs

$x=1$ and $x=3$ are points of inflexion ($x=4$ is not)

6. (a)

Interval	g'	g''
$a<x<b$	**positive**	**positive**
$b<x<c$	**positive**	**negative**
$c<x<d$	**negative**	**negative**
$d<x<e$	**negative**	**positive**
$e<x<f$	**negative**	**negative**

(b)

	Point	g'	g''
B	$x=b$	**positive**	**zero**
C	$x=c$	**zero**	**negative**
D	$x=d$	**negative**	**zero**
E	$x=e$	**zero**	**zero**

7. $f(x) = ax^3 + bx^2 + cx$

(a) $f'(x) = 3ax^2 + 2bx + c$, $f''(x) = 6ax + 2b$

(b) $f(1) = 4 \Leftrightarrow a + b + c = 4$

$f'(1) = 0 \Leftrightarrow 3a + 2b + c = 0$

$f''(2) = 0 \Leftrightarrow 12a + 2b = 0$

(c) $a = 1, b = -6, c = 9$

(d) $f'(x) = 3ax^2 + 2bx + c = 0 \Leftrightarrow 3x^2 - 12x + 9 = 0 \Leftrightarrow x^2 - 4x + 3 = 0 \Leftrightarrow x = 1,\ x = 3$

minimum at $x = 3$ (by using table or 2nd derivative test)

8. If $f : (x) \mapsto x^2e^x$ then $f'(x) = x^2e^x + 2xe^x = x(x+2)e^x$

stationary points : $x = 0$, $x = -2$

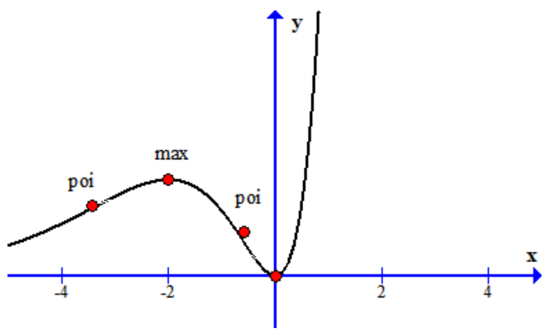
Using table of signs: max at $x = -2$, min at $x = 0$

$f''(x) = x^2e^x + 4xe^x + 2e^x = e^x(x^2 + 4x + 2)$

For a point of inflexion solve $f''(x) = 0$

$x = -2 - \sqrt{2}$, $x = -2 + \sqrt{2}$

Using table of signs: point of inflexion at both points



A. Exam style questions (SHORT)

9. (a)

	A	B	E
$f'(x)$	negative	0	negative

(b)

	A	C	E
$f''(x)$	positive	positive	negative

(c)

$f(0)$	$f'(0)$	$f''(0)$
positive	positive	negative

(d) One point of inflexion

10. (a) $f'(x) = x^2 + 4x - 5$

(b) $f'(x) = 0 \Leftrightarrow x = -5, x = 1$

so $x = -5$

(c) $f''(x) = 2x + 4$, $2x + 4 = 0$

$x = -2$

(d) $f'(3) = 16$

$y - 12 = 16(x - 3) \Rightarrow y = 16x - 36$

OR $12 = 16 \times 3 + b \Rightarrow b = -36$. Hence $y = 16x - 36$

11. (a) $g'(x) = 3x^2 - 6x - 9$

$3x^2 - 6x - 9 = 0 \Leftrightarrow 3(x-3)(x+1) = 0 \Leftrightarrow x = 3 \quad x = -1$

(b) **METHOD 1**
$g'(x < -1)$ is positive, $g'(x > -1)$ is negative
$g'(x < 3)$ is negative, $g'(x > 3)$ is positive

min when $x = 3$, max when $x = -1$

METHOD 2
Evidence of using second derivative
$g''(x) = 6x - 6$
$g''(3) = 12$ (or positive), $g''(-1) = -12$ (or negative)

min when $x = 3$, max when $x = -1$

12. (a) $f''(x) = 0$ **OR** the max and min of f' gives the points of inflexion on f

-0.114, 0.364

(b) **METHOD 1**

graph of g is a quadratic function, so it does not have any points of inflexion

METHOD 2

graph of g is concave down over entire domain therefore no change in concavity

METHOD 3

$g''(x) = -144$, therefore no points of inflexion as $g''(x) \neq 0$

13. (a) (i) $x = -\frac{5}{2}$ (ii) $y = \frac{3}{2}$

(b) By quotient rule: $\frac{dy}{dx} = \frac{(2x+5)(3) - (3x-2)(2)}{(2x+5)^2} = \frac{19}{(2x+5)^2}$

(c) There are no stationary points, since $\frac{dy}{dx} \neq 0$ (or by the graph) (A1)

(d) There are no points of inflexion.

14. (a) $f''(x) = 3(x-3)^2$

(b) $f'(3) = 0, f''(3) = 0$

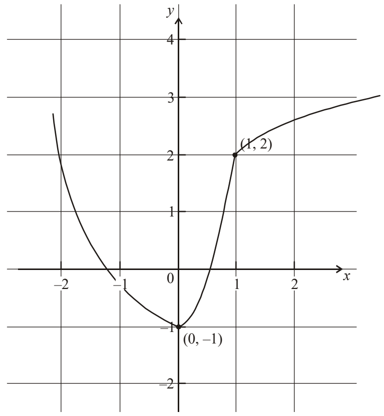
(c) f'' (i.e. concavity) does not change sign at P

15. (a) $f'(x) = 2xe^{-x} - x^2e^{-x} = (2-x)\,x\,e^{-x}$

(b) Maximum occurs at $x = 2$
Exact maximum value $= 4e^{-2}$

(c) $f''(x) = 2e^{-x} + 2xe^{-x} - 2xe^{-x} + x^2e^{-x} = (x^2 - 4x + 2)e^{-x}$
For inflexion, $f''(x) = 0$
$x = \frac{4+\sqrt{8}}{2} \left(= 2 + \sqrt{2}\right)$

16.



Notes: *On [– 2,0], decreasing, concave up. On [0,1], increasing, concave up.*
On [1,2], change of concavity, concave down.

17. (a) $g'(x) = \dfrac{x^2\left(\dfrac{1}{x}\right) - 2x\ln x}{x^4} = \dfrac{x - 2x\ln x}{x^4} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3}$

(b) $g'(x) = 0 \Leftrightarrow 1 - 2\ln x = 0 \Leftrightarrow \ln x = \dfrac{1}{2} \Leftrightarrow x = e^{\frac{1}{2}}$

18. (a) $x = 1$

(b) Using quotient rule

$$h'(x) = \frac{(x-1)^2(1) - (x-2)[2(x-1)]}{(x-1)^4}$$

$$= \frac{(x-1) - (2x-4)}{(x-1)^3} = \frac{3-x}{(x-1)^3}$$

(c) at point of inflexion $g''(x) = 0$

$x = 4$

$y = \dfrac{2}{9} = 0.222$ *ie* $\text{P}\left(4, \dfrac{2}{9}\right)$

19. (a) $x = 1$

EITHER The gradient of $g(x)$ goes from positive to negative

OR when $x = 1$, $g''(x)$ is negative

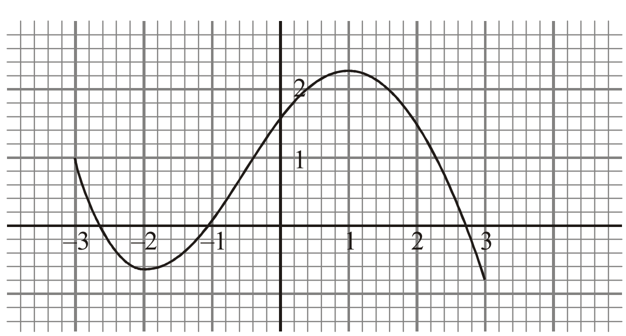
(b) $-3 < x < -2$ **and** $1 < x < 3$

$g'(x)$ is negative

(c) $x = -\dfrac{1}{2}$

$g''(x)$ changes from positive to negative **OR** concavity changes

(d)



20. (a) $f'(x) = \dfrac{e^x(e^x+1) - e^x(e^x-1)}{(e^x+1)^2} = \dfrac{2e^x}{(e^x+1)^2} > 0$, it is increasing.

(b) since the function is increasing it is 1-1 (horizontal line test).

$$\frac{e^x-1}{e^x+1} = y \Leftrightarrow e^x - 1 = ye^x + y \Leftrightarrow (1-y)e^x = y+1 \Leftrightarrow e^x = \frac{1+y}{1-y} \Leftrightarrow x = \ln\left(\frac{1+y}{1-y}\right)$$

$$f^{-1}(x) = \ln\left(\frac{1+x}{1-x}\right)$$

21. $f'(x) = 2(x-1)(x-4)^3 + 3(x-1)^2(x-4)^2 = (x-1)(x-4)^2[2(x-4)+3(x-1)]$

$= (x-1)(x-4)^2(5x-11)$

$f'(x) = 0 \Leftrightarrow x = 1$ or $x = 4$ or $x = 11/5$ ($= 2.2$)

x	**1**	**2.2**	**4**	
f'	+	–	+	+
		max	**min**	

max at $x = 1$, min at $x = 2.2$,

($x = 4$ stationary)

22. **METHOD A**

$f''(x) = (x-4)^2(5x-11) + 2(x-1)(x-4)(5x-11) + 5(x-1)(x-4)^2$

$= (x-4)[(x-4)(5x-11) + 2(x-1)(5x-11) + 5(x-1)(x-4)]$

$= (x-4)[5x^2 - 31x + 44 + 10x^2 - 32x + 22 + 5x^2 - 25x + 20]$

$= (x-4)(20x^2 - 88x + 86) = 2(x-4)(10x^2 - 44x + 43)$

Or directly $f''(x) = 0$

Roots $x = 1.47$, $x = 2.93$, $x = 4$

They are all points of inflexion (by using a table of signs)

METHOD B

Use graph of $f'(x)$ to find max/ min

METHOD C

Use graph of $f''(x)$ to find roots and then table of signs

23. (a) $f'(x) = 2x - \dfrac{p}{x^2}$

(b) $f'(-2) = 0 \Leftrightarrow -4 - \dfrac{p}{4} = 0 \Leftrightarrow -\dfrac{p}{4} = 4 \Leftrightarrow p = -16$

24.

(a) Use of quotient (or product) rule

$f'(x) = \dfrac{2(x^2+6) - (2x \times 2x)}{(x^2+6)^2}$ $\quad 2x(-1)(x^2+6)^{-2}(2x) + 2(x^2+6)^{-1}$

$= \dfrac{12 - 2x^2}{(x^2+6)^2}$

(b) Solving $f'(x) = 0$ for x

$x = \pm\sqrt{6}$

f has to be $1-1$ for f^{-1} to exist and so the least value of b

is the larger of the two x-coordinates (accept a labelled sketch)

Hence $b = \sqrt{6}$

25.

(a) $f'(x) = 1 - \dfrac{2}{x^{\frac{1}{3}}}$

$\Rightarrow 1 - \dfrac{2}{x^{\frac{1}{3}}} = 0 \Rightarrow x^{\frac{1}{3}} = 2 \Rightarrow x = 8$

(b) $f''(x) = \dfrac{2}{3x^{\frac{4}{3}}}$

$f''(8) > 0 \Rightarrow$ at $x = 8$, $f(x)$ has a minimum.

26. **METHOD 1**

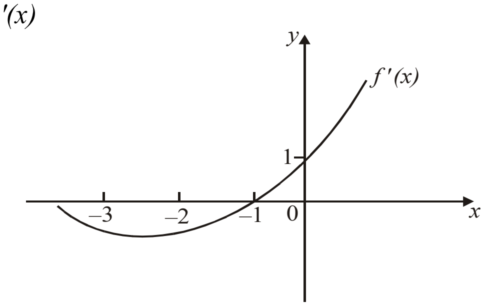
$y = xe^x \qquad \dfrac{dy}{dx} = xe^x + e^x$

$\dfrac{d^2y}{dx^2} = xe^x + 2e^x = e^x(x+2)$

Therefore the x-coordinate of the point of inflexion is $x = -2$

METHOD 2

Sketching $y = f'(x)$



$f'(x)$ has a minimum when $x = -2$.

Thus, $f(x)$ has point of inflexion when $x = -2$

27. (a) Given $f(x) = e^{\sin x}$

Then $f'(x) = \cos x \times e^{\sin x}$

(b) $f''(x) = \cos^2 x \times e^{\sin x} - \sin x \times e^{\sin x} = e^{\sin x}(\cos^2 x - \sin x)$

For the point of inflexion,

$f''(x) = 0 \Rightarrow e^{\sin x}(\cos^2 x - \sin x) = 0 \Rightarrow \cos^2 x - \sin x = 0$

$\Rightarrow 1 - \sin^2 x - \sin x = 0 \Rightarrow \sin^2 x + \sin x - 1 = 0$

$\sin x = \dfrac{-1 \pm \sqrt{5}}{2}$

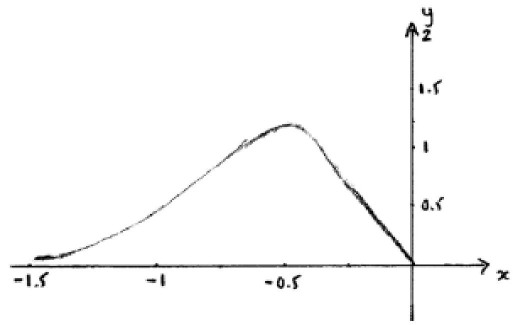
But $\dfrac{-1-\sqrt{5}}{2} < -1$

Hence $\sin x = \dfrac{\sqrt{5}-1}{2}$

28.

EITHER

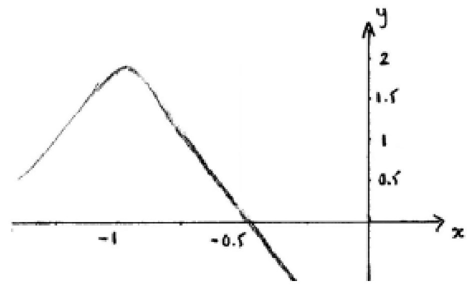
Using the graph of $y = f'(x)$



The maximum of $f'(x)$ occurs at $x = -0.5$.

OR

Using the graph of $y = f''(x)$.



The zero of $f''(x)$ occurs at $x = -0.5$.

THEN

Note: Do not award this ***A1*** for stating $x = \pm 0.5$ as the final answer for x.

$f(-0.5) = 0.607 \ (= e^{-0.5})$

EITHER

Correctly labelled graph of $f'(x)$ for $x < 0$ denoting the maximum $f'(x)$

(*e.g.* $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated)

OR

Correctly labelled graph of $f''(x)$ for $x < 0$ denoting the maximum $f'(x)$

(*e.g.* $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated)

OR

$f'(0.5) \approx 1.21$. $f'(x) < 1.21$ just to the left of $x = -\dfrac{1}{2}$

and $f'(x) < 1.21$ just to the right of $x = -\dfrac{1}{2}$

(*e.g.* $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated)

OR

$f''(x) > 0$ just to the left of $x = -\dfrac{1}{2}$ and $f''(x) < 0$ just to the right of $x = -\dfrac{1}{2}$

(*e.g.* $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated)

29. $f'(x) = 4x^3 - \dfrac{2}{x^2}$

$f''(x) = 12x^2 + \dfrac{4}{x^3}$

$f''(x) = 0 \Rightarrow x = -\dfrac{1}{\sqrt[5]{3}} = -0.803$ and $y = -2.08$ (accept -2.07)

The point of inflexion is $(-0.803, -2.08)$ $\left(\text{or } \left(-\dfrac{1}{\sqrt[5]{3}}, -\dfrac{5}{3}\sqrt[5]{3}\right)\right)$

30.

$$\frac{dy}{dx} = x^2 - 2x - 3$$

at $\frac{dy}{dx} = 0$, $(x-3)(x+1) = 0$

$x = 3, -1$; $y = -5, \frac{17}{3}$

So $P(3, -5)$ and $Q\left(-1, \frac{17}{3}\right)$

Equation of (PQ) is $\frac{y+5}{\left(\frac{17}{3}+5\right)} = \frac{x-3}{-1-3}$

$$\frac{3y+15}{32} = \frac{x-3}{-4}$$

$$\frac{3y+15}{8} = \frac{x-3}{-1}$$

$$-3y - 15 = 8x - 24$$

$$8x + 3y - 9 = 0$$

31.

$f(x) = ax^3 + bx^2 + 30x + c$

$f'(x) = 3ax^2 + 2bx + 30$, $f'(1) = 0 \Rightarrow 3a + 2b + 30 = 0$

$f''(x) = 6ax + 2b$, $f''(3) = 0 \Rightarrow 18a + 2b = 0$

$a = 2$

$b = -18$

$f(1) = 7 \Rightarrow 2 - 18 + 30 + c = 7$

$c = -7$

32. (a) $f(x) = 2x^3 - 6x^2 + 5$

(b) min at (2,-3)

B. Exam style questions (LONG)

33. (a) (i) $f'(x) = \dfrac{\left(x \times \frac{1}{2x} \times 2\right) - (\ln 2x \times 1)}{x^2} = \dfrac{1 - \ln 2x}{x^2}$

(ii) $f'(x) = 0 \Leftrightarrow \dfrac{1 - \ln 2x}{x^2} = 0$ only at 1 point, when $x = \dfrac{e}{2}$

(iii) Maximum point when $f'(x) = 0$.

$f'(x) = 0$ for $x = \frac{e}{2}$ (= 1.36)

$y = f\left(\frac{e}{2}\right) = \frac{2}{e}$ (= 0.736)

(b) $f''(x) = \dfrac{-\frac{1}{2x} \times 2 \times x^2 - (1 - \ln 2x)2x}{x^4} = \dfrac{2\ln 2x - 3}{x^3}$

Inflexion point $\Rightarrow f''(x) = 0 \Rightarrow 2\ln 2x = 3 \Rightarrow x = \dfrac{e^{1.5}}{2}$ (= 2.24)

34.

(a) $f'(x)=(1+2x)e^{2x}$

$f'(x)=0$

$\Rightarrow (1+2x)e^{2x}=0 \Rightarrow x=-\dfrac{1}{2}$

$f''(x)=(2^2x+2\times 2^{2-1})e^{2x}=(4x+4)e^{2x}$

$f''\left(-\dfrac{1}{2}\right)=\dfrac{2}{e}$

$\dfrac{2}{e}>0 \Rightarrow$ at $x=-\dfrac{1}{2}$, $f(x)$ has a minimum.

$P\left(-\dfrac{1}{2}, -\dfrac{1}{2e}\right)$

(b) $f''(x)=0 \Rightarrow 4x+4=0 \Rightarrow x=-1$

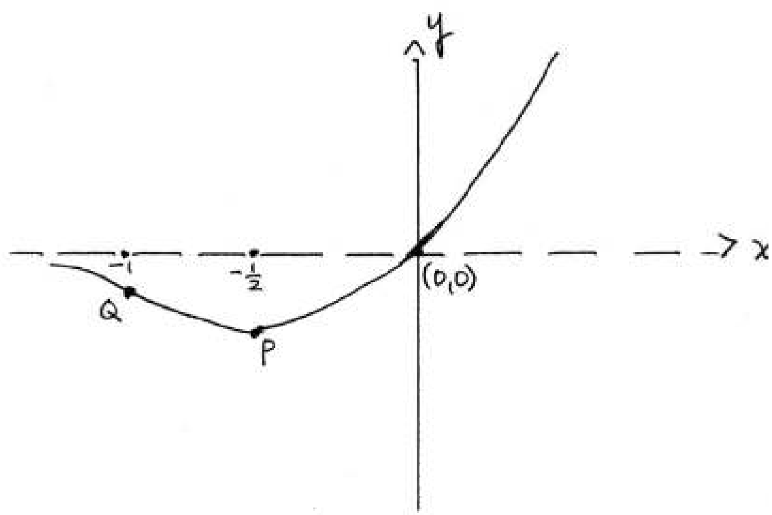
Using the 2nd derivative $f''\left(-\dfrac{1}{2}\right)=\dfrac{2}{e}$ and $f''(-2)=-\dfrac{4}{e^4}$,

the sign change indicates a point of inflexion.

(c) (i) $f(x)$ is concave up for $x>-1$.

(ii) $f(x)$ is concave down for $x<-1$.

(d)



35. (a) B, D

(b) (i) $f'(x) = -2xe^{-x^2}$

(ii) product rule

$f''(x) = -2e^{-x^2} - 2x \times -2xe^{-x^2} = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2-2)e^{-x^2}$

(c) $f''(x)=0 \Leftrightarrow (4x^2-2)=0$

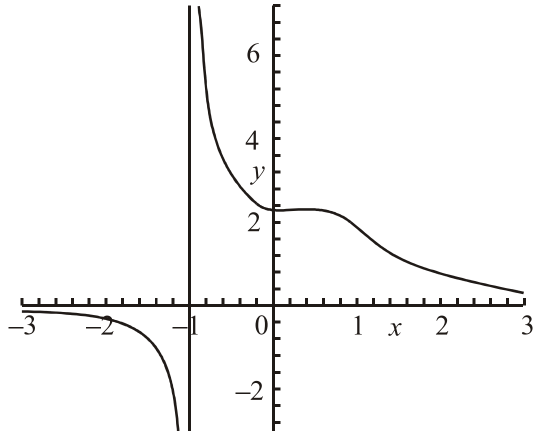
$p=0.707\left(=\dfrac{1}{\sqrt{2}}\right)$, $q=-0.707 \left(=-\dfrac{1}{\sqrt{2}}\right)$

(d) checking sign of f'' on either side of POI

sign change of $f''(x)$

36. (a) (i) Vertical asymptote $x = -1$ (ii) Horizontal asymptote $y = 0$

(iii)



(b) (i) $f'(x) = \dfrac{-6x^2}{(1+x^3)^2}$

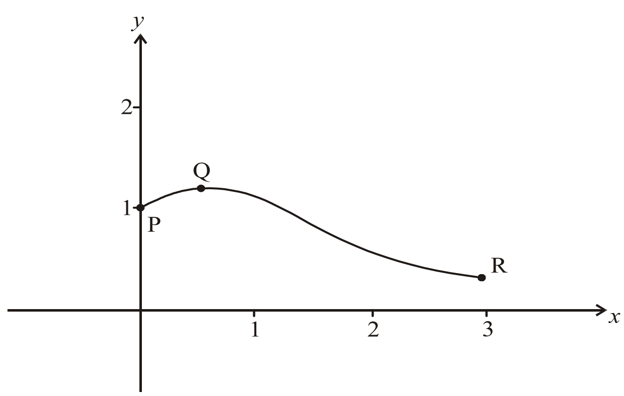
$$f''(x) = \frac{(1+x^3)^2(-12x) + 6x^2(2)(1+x^3)^1(3x^2)}{(1+x^3)^4}$$

$$= \frac{(1+x^3)(-12x) + 36x^4}{(1+x^3)^3} = \frac{-12 - 12x^4 + 36x^4}{(1+x^3)^3} = \frac{12x(2x^3-1)}{(1+x^3)^3}$$

(ii) Point of inflexion => $f''(x) = 0$ => $x = 0$ or $x = \sqrt[3]{\frac{1}{2}}$

$x = 0$ or $x = 0.794$ (3 sf)

37. (a)



(b) (i) $f'(x) = 2e^{-x} + (2x+1)(-e^{-x}) = (1-2x)e^{-x}$

(ii) At **Q**, $f'(x) = 0$

$x = 0.5$, $y = 2e^{-0.5}$ **Q** is $(0.5, 2e^{-0.5})$

(c) $1 \le k < 2e^{-0.5}$

(d) $f''(x) = 0 \Leftrightarrow e^{-x}(-3 + 2x) = 0$

This equation has only one root. So f has only one point of inflexion.

38. (a) $x = 1$

(b) $y = 2$

(c) $f'(x) = \dfrac{(x-1)^2(4x-13) - 2(x-1)(2x^2 - 13x + 20)}{(x-1)^4}$

$= \dfrac{(4x^2 - 17x + 13) - (4x^2 - 26x + 40)}{(x-1)^3} = \dfrac{9x - 27}{(x-1)^3}$

(d) $f'(3) = 0$ $\Rightarrow$ stationary point

$f''(3) = \dfrac{18}{16} > 0$ $\Rightarrow$ minimum

(e) Point of inflexion $\Rightarrow f''(x) = 0 \Rightarrow x = 4$

$x = 4 \Rightarrow y = 0$ $\Rightarrow$ Point of inflexion = (4, 0)

39. (a) (i) –1.15, 1.15

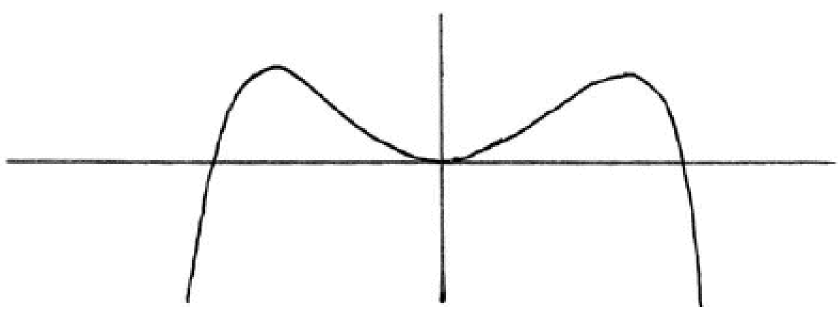
(ii) it occurs at P and Q (when $x = -1.15$, $x = 1.15$)

$k = -1.13$, $k = 1.13$

(b) product rule

$g'(x) = x^3 \times \dfrac{-2x}{4 - x^2} + \ln(4 - x^2) \times 3x^2 = \dfrac{-2x^4}{4 - x^2} + 3x^2 \ln(4 - x^2)$

(c)



(d) $w = 2.69$, $w < 0$

40.

(a) Using quotient rule

$f'(x) = \dfrac{x^3 \times \dfrac{1}{x} - 3x^2 \ln x}{x^6}$

$= \dfrac{1 - 3\ln x}{x^4}$

$f''(x) = \dfrac{-\dfrac{3}{x} \times x^4 - 4x^3(1 - 3\ln x)}{x^8}$

$= \dfrac{-7 + 12\ln x}{x^5}$

(b) (i) For a maximum, $f'(x)=0$ giving

$\ln x = \frac{1}{3}$

$x = e^{\frac{1}{3}}$

EITHER

$f''\left(e^{\frac{1}{3}}\right) = \frac{12\times\frac{1}{3}-7}{e^{\frac{5}{3}}} < 0$

$\therefore$ maximum

OR

for $x < e^{\frac{1}{3}}$, $f'(x) > 0$

for $x > e^{\frac{1}{3}}$, $f'(x) < 0$

$\therefore$ maximum

(ii) $f''(0)=0 \Rightarrow \ln(x) = \frac{7}{12}$

$x = e^{\frac{7}{12}}$ (1.79)

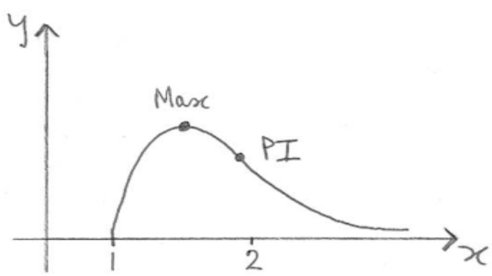
$f''(1.5) = -0.281$

$f''(2) = 0.0412$

Note: Accept any two sensible values either side of 1.79.

$\therefore$ Change of sign $\Rightarrow$ point of inflexion

(iii)



41. (a) $f'(x) = 3x^2 - 4$

(b) $f'(1) = -1$

$3x^2 - 4 = -1 \Leftrightarrow x = \pm 1$

at Q, $x = -1$, $y = 4$ (Q is $(-1, 4)$)

(c) f is decreasing when $f'(x) < 0$

$p = -1.15$, $q = 1.15$; (OR $\pm\frac{2}{\sqrt{3}}$)

(d) $f'(x) \geq -4$, $y \geq -4$, OR $[-4, \infty[$

(e) $f''(x) = 6x$

$6x = 0 \Leftrightarrow x = 0$

The point of inflexion is (0,1)

42. (a) (i) coordinates of A are (0, –2)

(ii) $f(x) = 3 + 20 \times (x^2 - 4)^{-1}$

$f'(x) = 20 \times (-1) \times (x^2 - 4)^{-2} \times (2x) = -40x(x^2 - 4)^{-2}$

OR $\dfrac{(x^2-4)(0)-(20)(2x)}{(x^2-4)^2}$

substituting $x = 0$ into $f'(x)$ gives $f'(x) = 0$

(b) (i) $f'(0) = 0$ (stationary)

$f''(0) = \dfrac{40 \times 4}{(-4)^3}\left(= \dfrac{-5}{2}\right)$ negative

then the graph must have a local maximum

(ii) $f''(x) = 0$ at point of inflexion,

but the second derivative is never 0 (the numerator is always positive)

(c) getting closer to the line $y = 3$, horizontal asymptote at $y = 3$

(d) $y \le -2, y > 3$

43. (a) $f'(x) = e^x(1 - x^2) + e^x(-2x) = e^x(1 - 2x - x^2)$

(b) $y = 0$

(c) at the local maximum or minimum point

$f'(x) = 0 \Leftrightarrow e^x(1 - 2x - x^2) = 0 \Rightarrow 1 - 2x - x^2 = 0$

$r = -2.41$ $s = 0.414$ (**OR** directly by GDC graph)

(d) $f'(0) = 1 \Rightarrow$ gradient of the normal $= -1$

$y - 1 = -1(x - 0) \Leftrightarrow x + y = 1$

(e) (i) intersection points at (0,1) and (1,0)

44. (a) $f'(x) = x^2 - 2x - 3$

$x^2 - 2x - 3 = 0 \Leftrightarrow x = \dfrac{2 \pm \sqrt{16}}{2} \Leftrightarrow x = -1$ or $x = 3$

$x = -1$ (ignore $x = 3$) $y = -\dfrac{1}{3} - 1 + 3 = \dfrac{5}{3}$

coordinates are $\left(-1, \dfrac{5}{3}\right)$

(b) (i) (–3, –9)

(ii) (1, –4)

(iii) reflection gives (3, 9) stretch gives $\left(\dfrac{3}{2}, 9\right)$

45. (a) quotient rule

$$f'(x) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

(b) **METHOD 1**

$f'(x) = -(\sin x)^{-2}$

$f''(x) = 2(\sin x)^{-3}(\cos x)\left(= \frac{2\cos x}{\sin^3 x}\right)$

METHOD 2

quotient rule: $f''(x) = \frac{\sin^2 x \times 0 - (-1)2\sin x\cos x}{(\sin^2 x)^2} = \frac{2\sin x\cos x}{(\sin^2 x)^2}\left(= \frac{2\cos x}{\sin^3 x}\right)$

(c) substituting $\frac{\pi}{2} \Rightarrow p = -1,\ q = 0$

(d) second derivative is zero, second derivative changes sign

46. (a) $\frac{dy}{dx} = e^x(\cos x + \sin x) + e^x(-\sin x + \cos x) = 2e^x \cos x$

(b) $\frac{dy}{dx} = 0 \Rightarrow 2e^x\cos x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2} \Rightarrow a = \frac{\pi}{2}$

$y = e^{\frac{\pi}{2}}(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}) = e^{\frac{\pi}{2}} \Rightarrow b = e^{\frac{\pi}{2}}$

(c) At D, $\frac{d^2y}{dx^2} = 0 \Rightarrow 2e^x\cos x - 2e^x\sin x = 0 \Rightarrow 2e^x(\cos x - \sin x) = 0$

$\Rightarrow \cos x - \sin x = 0 \Rightarrow x = \frac{\pi}{4}$

$y = e^{\frac{\pi}{4}}(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}) = \sqrt{2}\, e^{\frac{\pi}{4}}$

47. (a) $y = 0$

(b) $f'(x) = \frac{-2x}{(1+x^2)^2}$

(c) $f'(x) = -2x(1+x^2)^{-2}$,

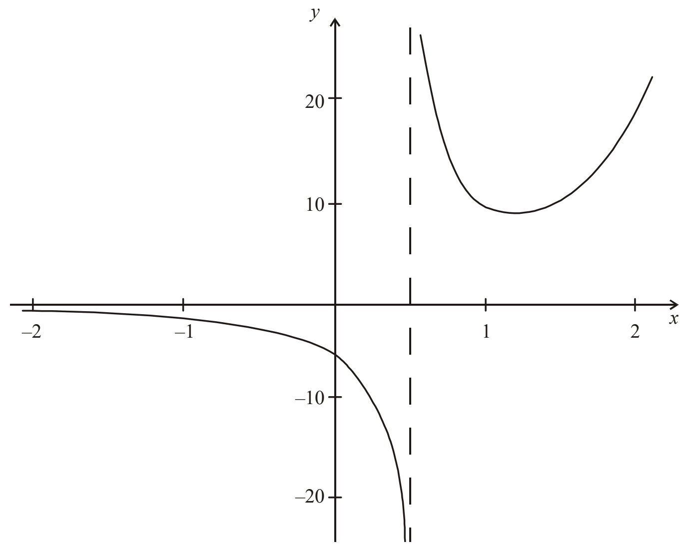
$$f''(x) = -2(1+x^2)^{-2} + 4x(1+x^2)^{-3}2x = \frac{-2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}$$

$$= \frac{-2(1+x^2)}{(1+x^2)^3} + \frac{8x^2}{(1+x^2)^3} = \frac{6x^2-2}{(1+x^2)^3}$$

(d) $f''(x) = 0 \Leftrightarrow 6x^2 - 2 = 0 \Leftrightarrow x = \pm\sqrt{\frac{1}{3}}$

The maximum gradient is at $x = \frac{-1}{\sqrt{3}}$

48. (a)



(b) $x = \frac{1}{2}$ (must be an equation)

(c) $f'(x) = 2e^{2x-1} - 10(2x-1)^{-2}$

(e) (i) $x = 1.11$ (accept (1.11, 7.49)) (ii) $p = 0, q = 7.49$ $(0 \le k < 7.49)$

49. (a) π

(b) (i) $f'(x) = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x)$

(ii) At B, $f'(x) = 0$

(c) $f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x = 2e^x \cos x$

(d) (i) At A, $f''(x) = 0$

(ii) $2e^x \cos x = 0 \Leftrightarrow \cos x = 0$

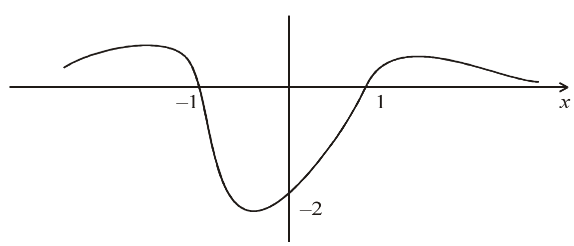
$x = \frac{\pi}{2},\ y = e^{\frac{\pi}{2}}$ Coordinates are $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$

50. (a) (i) $f'(x) = \dfrac{(2x-1)(x^2+x+1) - (2x+1)(x^2-x+1)}{(x^2+x+1)^2} = \dfrac{2(x^2-1)}{(x^2+x+1)^2}$

(ii) $f'(x) = 0 \Rightarrow x = \pm 1$

$A\left(1, \frac{1}{3}\right)$ B(–1, 3) $\left(\text{or A }(-1, 3)\ \ B\left(1, \frac{1}{3}\right)\right)$

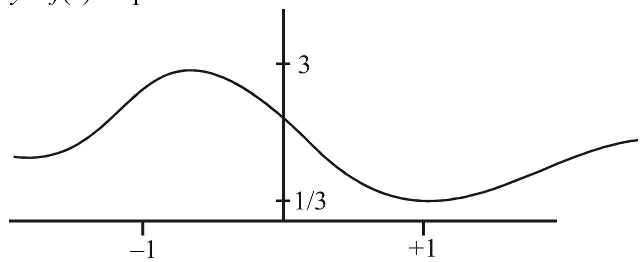
(b) (i)



(ii) The points of inflexion can be found by locating the max/min on the graph of f'.
This gives $x = -1.53, -0.347, 1.88$.
OR

$f''(x) = \dfrac{-4(x^3 - 3x - 1)}{(x^2+x+1)^3}$ $f''(x) = 0 \Rightarrow x^3 - 3x - 1 = 0 \Rightarrow x = 1.53, -0.347, 1.88$

(c) The graph of $y = f(x)$ helps:



(i) Range of f is $\left[\frac{1}{3}, 3\right]$.

(ii) We require the image set of $\left[\frac{1}{3}, 3\right]$.

$$f\left(\frac{1}{3}\right) = \frac{\frac{1}{9} - \frac{1}{3} + 1}{\frac{1}{9} + \frac{1}{3} + 1} = \frac{7}{13}, f(3) = \frac{9 - 3 + 1}{9 + 3 + 1} = \frac{7}{13}$$

Range of g is $\left[\frac{1}{3}, \frac{7}{13}\right]$.

51. (a) $y = e^{2x} \cos x$

$$\frac{dy}{dx} = e^{2x} (-\sin x) + \cos x\, (2e^{2x}) = e^{2x} (2 \cos x - \sin x)$$

(b) $$\frac{d^2y}{dx^2} = 2e^{2x} (2 \cos x - \sin x) + e^{2x} (-2 \sin x - \cos x)$$

$$= e^{2x} (4 \cos x - 2 \sin x - 2 \sin x - \cos x) = e^{2x} (3 \cos x - 4 \sin x)$$

(c) (i) At P, $\frac{d^2y}{dx^2} = 0 \Rightarrow 3 \cos x = 4 \sin x \Rightarrow \tan x = \frac{3}{4}$

At P, $x = a$, $\tan a = \frac{3}{4}$

(ii) The gradient at any point is $e^{2x} (2 \cos x - \sin x)$

Therefore, the gradient at P = $e^{2a} (2 \cos a - \sin a)$

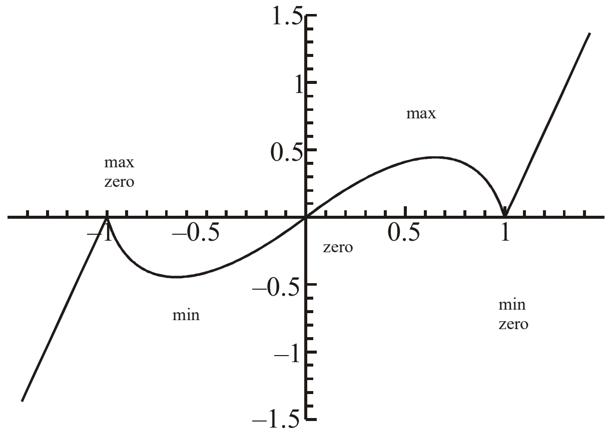
When $\tan a = \frac{3}{4}$, by using a right angle triangle:

$\cos a = \frac{4}{5}$, $\sin a = \frac{3}{5}$

(by drawing a right triangle, or by calculator)

Therefore, the gradient at P = $e^{2a}\left(\frac{8}{5} - \frac{3}{5}\right) = e^{2a}$

52. (a) $f(x) = x\left(\sqrt[3]{(x^2-1)^2}\right)$



Notes: *(sharp points) at x = ±1. zeros at x = ±1 and x = 0.*
maximum at x = –1 and minimum at x = 1.
max at approx. x = 0.65,and min at approx. x = –0.65. There are no asymptotes.

(b) (i) Let $f(x) = x\left(x^2-1\right)^{\frac{2}{3}}$

Then $f'(x) = \frac{4}{3}x^2(x^2-1)^{-\frac{1}{3}} + (x^2-1)^{\frac{2}{3}}$

$f'(x) = (x^2-1)^{-\frac{1}{3}}\left[\frac{4}{3}x^2 + (x^2-1)\right]$

$f'(x) = (x^2-1)^{-\frac{1}{3}}\left(\frac{7}{3}x^2 - 1\right)$ (or equivalent)

$f'(x) = \dfrac{7x^2-3}{3(x^2-1)^{-\frac{1}{3}}}$ (or equivalent)

The domain is $-1.4 \le x \le 1.4$, $x \ne \pm 1$ (accept $-1.4 < x < 1.4$, $x \ne \pm 1$)

(ii) For the maximum or minimum points let $f'(x) = 0$
i.e. $(7x^2 - 3) = 0$ or use the graph.
Therefore, the x-coordinate of the maximum point is
$x = \sqrt{\frac{3}{7}}$ (or 0.655) and

the x-coordinate of the minimum point is $x = \sqrt{\frac{3}{7}}$ (or –0.655).

(c) The x-coordinate of the point of inflexion is $x = \pm 1.1339$
OR
$f''(x) = \dfrac{4x(7x^2-9)}{9\sqrt[3]{(x^2-1)^4}}$, $x \ne \pm 1$

For the points of inflexion let $f''(x) = 0$ and use the graph,
i.e. $x = \sqrt{\frac{9}{7}} = 1.1339$.