[MAA 3.5]

SIN, COS, TAN ON THE UNIT CIRCLE - IDENTITIES

SOLUTIONS

Compiled by: Christos Nikolaidis

O. Practice questions

1.

sin160º	p	sin200º	$-p$
cos160º	$-q$	cos200º	$-q$
tan160º	$-p/q$	tan200º	p/q

sin340º	$-p$	sin(-20º)	$-p$
cos340º	q	cos(-20º)	q
tan340º	$-p/q$	tan(-20º)	$-p/q$

2. (a)

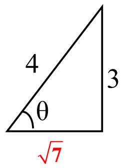
tan 20º =	$\frac{p}{q}$	(in terms of p and q)
sin 40º =	$2pq$	(in terms of p and q)
cos 40º =	q^2-p^2	(in terms of p and q)
	$1-2p^2$	(in terms of p **only**)
	$2q^2-1$	(in terms of q **only**)

3. (a) $\cos^2 x+\sin^2 x=1 \Rightarrow \cos x=\sqrt{1-p^2}$

(b)

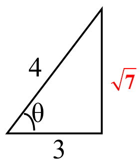
	Formula	**Expression in terms of** p
tan x	$=\frac{\sin x}{\cos x}$	$\frac{p}{\sqrt{1-p^2}}$
cos $2x$	$=1-2\cos^2 x$	$1-2p^2$
sin $2x$	$=2\sin x\cos x$	$2p\sqrt{1-p^2}$
tan $2x$	$=\frac{\sin 2x}{\cos 2x}$	$\frac{2p\sqrt{1-p^2}}{1-2p^2}$
sin $4x$	$=2\sin 2x\cos 2x$	$4p\sqrt{1-p^2}(1-2p^2)$

4. (a)



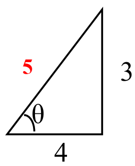
$\cos\theta = \dfrac{\sqrt{7}}{4}, \quad \tan\theta = \dfrac{3}{\sqrt{7}}$

(b)



$\sin\theta = \dfrac{\sqrt{7}}{4}, \quad \tan\theta = \dfrac{\sqrt{7}}{3}$

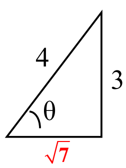
(c)



$\sin\theta = \dfrac{3}{5}, \quad \cos\theta = \dfrac{3}{5}$

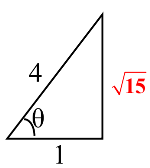
5. In the 2nd quadrant only sin is positive. Hence

(a)



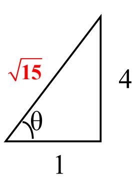
$\cos\theta = -\dfrac{\sqrt{7}}{4}, \quad \tan\theta = -\dfrac{3}{\sqrt{7}}$

(b)



$\sin\theta = \dfrac{\sqrt{15}}{4}, \quad \tan\theta = -\sqrt{15}$

(c)



$\sin\theta = \dfrac{1}{\sqrt{15}}, \quad \cos\theta = -\dfrac{4}{\sqrt{15}}$

6. (a) $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} = \dfrac{2\sin^2\theta}{2\sin\theta\cos\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

(b) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{1+(2\cos^2\theta-1)} = \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

7. (a) $(\cos\theta+\sin\theta)^2 = a^2 \Rightarrow \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = a^2 \Rightarrow 1+\sin 2\theta = a^2$

$\sin 2\theta = a^2 - 1$

(b) $(\cos\theta-\sin\theta)^2 = \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta = 1-\sin 2\theta = 2-a^2$

Hence, $\cos\theta - \sin\theta = \sqrt{2-a^2}$

8. (a) $(\cos\theta+\sin\theta)^2=\left(\frac{4}{3}\right)^2 \Rightarrow \cos^2\theta+2\sin\theta\cos\theta+\sin^2\theta=\frac{16}{9}\Rightarrow 1+\sin 2\theta=\frac{16}{9}$

$\sin 2\theta=\frac{7}{9}$

(b) $\cos 4\theta=1-2\sin^2 2\theta=1-2\left(\frac{7}{9}\right)^2=1-2\left(\frac{49}{81}\right)=\frac{17}{81}$

9. (a) $(\cos\theta-\sin\theta)^2=\left(\frac{1}{2}\right)^2 \Rightarrow \cos^2\theta-2\sin\theta\cos\theta+\sin^2\theta=\frac{1}{4}\Rightarrow 1-\sin 2\theta=\frac{1}{4}$

$\sin 2\theta=\frac{3}{4}$

(b) $\cos 4\theta=1-2\sin^2 2\theta=1-2\left(\frac{3}{4}\right)^2=1-2\left(\frac{9}{16}\right)=-\frac{2}{16}$

A. Exam style questions (SHORT)

10. (a) Acute angle 30° $\Rightarrow \theta$ = 150° **(**2nd quadrant since sine positive and cosine negative)

(b) $\tan 150° = \dfrac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}=-\frac{1}{\sqrt{3}}$

11. (a) x is an acute angle => cos$x$ is positive.

$\cos^2 x+\sin^2 x=1 \Rightarrow \cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{1}{3}\right)^2}=\sqrt{\frac{8}{9}}$ $\left(=\frac{2\sqrt{2}}{3}\right)$

(b) $\cos 2x = 1-2\sin^2 x = 1-2\left(\frac{1}{3}\right)^2=\frac{7}{9}$

12. (a) $BC=\sqrt{3^2-2^2}=\sqrt{5}$ $\sin\theta=\frac{\sqrt{5}}{3}$

$\sin 2\theta=2\sin\theta\cos\theta=2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)=\frac{4\sqrt{5}}{9}$

(b) $\cos 2\theta=\frac{4}{9}-\frac{5}{9}=-\frac{1}{9}$ **OR** $\cos 2\theta=1-2\times\frac{5}{9}=-\frac{1}{9}$

13. $\sin A=\frac{5}{13}\Rightarrow \cos A=\pm\frac{12}{13}$ But A is obtuse $\Rightarrow \cos A=-\frac{12}{13}$

$\sin 2A=2\sin A\cos A=2\times\frac{5}{13}\times\left(-\frac{12}{13}\right)=-\frac{120}{169}$

14. (a) (i) $\sin 140° = p$ (ii) $\cos 70° = -q$

(b) **METHOD 1**

using $\sin^2\theta+\cos^2\theta=1$

$\cos 140° = -\sqrt{1-p^2}$

METHOD 2

using $\cos^2\theta = 2\cos^2\theta - 1$

$\cos 140° = 2\cos^2 70 - 1 = 2(-q)^2 - 1 = 2q^2 - 1$

(c) **METHOD 1**

$\tan 140° = \frac{\sin 140°}{\cos 140°}=-\frac{p}{\sqrt{1-p^2}}$

METHOD 2

$\tan 140° = \frac{p}{2q^2-1}$

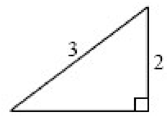
15. (a) $\cos 2A = 2\cos^2 A - 1 = 2 \times \left(\frac{1}{3}\right)^2 - 1! = -\frac{7}{9}$

(b) **METHOD 1**

using $\sin^2 B + \cos^2 B = 1$

$\cos B = \pm\sqrt{\frac{5}{9}} \quad \left(= \pm\frac{\sqrt{5}}{3}\right)$ $\qquad \cos B = = -\frac{\sqrt{5}}{3}$

METHOD 2



Diagram, eg

third side equals $\sqrt{5}$

$\cos B = -\frac{\sqrt{5}}{3}$

16. (a) $\tan\theta = \frac{3}{4}$

(b) (i) by using a right-angles triangle with sides 3,4,5

$\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$

$\sin 2\theta = 2\sin x \cos x = \frac{24}{25}$

(ii) $\cos 2\theta = 1 - 2\left(\frac{3}{5}\right)^2 = \frac{7}{25}$ **OR** $\cos 2\theta = \left(\frac{3}{5}\right)^2, \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

17. (a) $f(x) = \sin^3 x + \cos^3 x \frac{\sin x}{\cos x} = \sin x(\sin^2 x + \cos^2 x) = \sin x$

(b) $f(2x) = \sin 2x = 2\sin x \cos x$

$\sin^2 x + \cos^2 x = 1 \Rightarrow \cos x = -\frac{\sqrt{5}}{3}$

$f(2x) = 2\left(\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) = -\frac{4\sqrt{5}}{9}$ $f(2x)$

18. (a) $\cos 30^o = 1 - 2\sin^2 15^o \Rightarrow \frac{\sqrt{3}}{2} = 1 - 2\sin^2 15^o \Rightarrow \sqrt{3} = 2 - 4\sin^2 15^o$

$\sin^2 15^o = \frac{2-\sqrt{3}}{4} \Rightarrow \sin 15^o = \frac{\sqrt{2-\sqrt{3}}}{2}$

(b) $\cos 30^o = 2\cos^2 15^o - 1 \Rightarrow \frac{\sqrt{3}}{2} = 2\cos^2 15^o - 1 \Rightarrow \sqrt{3} = 4\cos^2 15^o - 2$

$\cos^2 15^o = \frac{2+\sqrt{3}}{4} \Rightarrow \cos 15^o = \frac{\sqrt{2+\sqrt{3}}}{2}$

19. $\frac{\sin 4\theta(1-\cos 2\theta)}{\cos 2\theta(1-\cos 4\theta)} = \frac{2\sin 2\theta\cos 2\theta(1-\cos 2\theta)}{\cos 2\theta(1-1+2\cos^2 2\theta)} = \frac{2\sin 2\theta(1-\cos 2\theta)}{2\sin^2 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta}$

$= \frac{1-1+2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

20. $2\sin 4x - 3\sin 2x = 0 \Leftrightarrow 4\sin 2x\cos 2x - 3\sin 2x = 0$

$\sin 2x(4\cos 2x - 3) = 0$

$4\cos 2x - 3 = 0 \;\Leftrightarrow\; \cos 2x = \dfrac{3}{4}$

$2\cos^2 x - 1 = \dfrac{3}{4} \Leftrightarrow \cos^2 x = \dfrac{7}{8}$

21. $2a\sin 2x\cos 2x + b\sin 2x = 0$

$\sin 2x(2a\cos 2x + b) = 0$

$\cos 2x = -\dfrac{b}{2a}$

$2\cos^2 x - 1 = -\dfrac{b}{2a}$

$\Rightarrow \cos^2 x = \left(1 - \dfrac{b}{2a}\right)\dfrac{1}{2} = \dfrac{1}{2} - \dfrac{b}{4a} \quad \left(= \dfrac{2a-b}{4a}\right)$

22.

$\dfrac{9}{\sin C} = \dfrac{12}{\sin 2C}$

Using double angle formula $\dfrac{9}{\sin C} = \dfrac{12}{2\sin C\cos C}$

$\Rightarrow 9(2\sin C\cos C) = 12\sin C$

$\Rightarrow 6\sin C(3\cos C - 2) = 0$ or equivalent

$(\sin C \neq 0)$

$\Rightarrow \cos C = \dfrac{2}{3}$

23. (a) $\dfrac{\sin x}{10} = \dfrac{\sin 2x}{\text{AC}} \Leftrightarrow \dfrac{\sin x}{10} = \dfrac{2\sin x\cos x}{\text{AC}} \Leftrightarrow \text{AC} = 20\cos x$

(b) Area $\text{ABC} = \dfrac{1}{2}\text{AC}\times\text{BC}\sin C \Leftrightarrow 50\cos x = \dfrac{1}{2}10\times 20\cos x\sin C$

$\sin C = \dfrac{1}{2} \;\Rightarrow\; \hat{C} = 30^\circ$

B. Exam style questions (LONG)

24. (a) $\cos \text{D}\hat{\text{A}}\text{C} = \cos x = \dfrac{3}{5}$

(b) $\cos \text{B}\hat{\text{A}}\text{C} = \cos 2x = 2\cos^2 x - 1 = 2\left(\dfrac{5}{6}\right)^2 - 1 = \dfrac{50}{36} - 1 = \dfrac{14}{36} = \dfrac{7}{18}$

(c) $\cos \text{B}\hat{\text{A}}\text{C} = \dfrac{5}{\text{AB}} \Leftrightarrow \dfrac{7}{18} = \dfrac{5}{\text{AB}} \Leftrightarrow \text{AB} = \dfrac{5\times 18}{7} \Leftrightarrow \text{AB} = \dfrac{90}{7}$

(d) $\sin B = \dfrac{7}{18}$

(d) $\text{CD}^2 + 5^2 = 6^2 \Rightarrow \text{CD} = \sqrt{11}$

$\tan \text{BAD} = \tan x = \dfrac{\text{CD}}{5} = \dfrac{\sqrt{11}}{5}$

25. (a) $A = \frac{1}{2} x\ 3x \sin\theta$ so $\sin\theta = \frac{4.42}{3x^2}$

(b) Cosine rule gives $\cos\theta = \frac{x^2 + (3x)^2 - (x+3)^2}{2 \times x \times 3x} = \frac{3x^2 - 2x - 3}{2x^2}$

(c) (i) Substituting the answers from (a) and (b) into the identity $\cos^2\theta = 1 - \sin^2\theta$ gives

$$\left(\frac{3x^2 - 2x - 3}{2x^2}\right)^2 = 1 - \left(\frac{4.42}{3x^2}\right)^2$$

(ii) (a) $x = 1.24, 2.94$

(b) $\theta = \arccos\left(\frac{3x^2 - 2x - 3}{2x^2}\right)$

$\theta = 1.86$ radians or $\theta = 0.171$

26. (a) (i) $x = 5$ (ii) $y_{\max} = 144$

(b) (i) $z = 10 - x$ (since $x + z = 10$)

(ii) $z^2 = x^2 + 6^2 - 2(x)(6)\cos Z$

(iii) $100 - 20x + x^2 = x^2 + 36 - 12x \cos Z$

$\Leftrightarrow 12x \cos Z = 20x - 64 \Leftrightarrow \cos Z = \frac{20x - 64}{12x} = \frac{5x - 16}{3x}$

(c) $A = \frac{1}{2} \times 6 \times x \times \sin Z = 3x \sin Z \Rightarrow A^2 = 9x^2 \sin^2 Z$

(d) Using $\sin^2 Z = 1 - \cos^2 Z$, Substituting $\frac{5x-16}{3x}$ for $\cos Z$

and expanding $\left(\frac{5x-16}{3x}\right)^2$ to $\left(\frac{25x^2 - 160x + 256}{9x^2}\right)$

$A^2 = 9x^2 - (25x^2 - 160x + 256) = -16x^2 + 160x - 256$

(e) (i) 144 (is maximum value of A^2, from part (a))

$A_{\max} = 12$

(ii) Isosceles

27. (a) For the height h, $\sin\theta = \frac{h}{2} \Leftrightarrow h = 2\sin\theta$

For the base of triangle b, $\cos\theta = \frac{b}{2} \Leftrightarrow b = 2\cos\theta$

Area $y = 2\left(\frac{1}{2} \times 2\cos\theta \times 2\sin\theta\right) + 2 \times 2\sin\theta = 4\sin\theta\cos\theta + 4\sin\theta$

$y = 4\sin\theta + 2\sin 2\theta$

(b) $4\sin\theta + 2\sin 2\theta = 5$

$\theta = 0.856\ (49.0^\circ)$, $\theta = 1.25\ (71.4^\circ)$

(c) By graph GDC $4 < A < 5.20$