DERIVATION OF UNIFORM CIRCULAR MOTION EQUATIONS

This approach uses the position vector and its derivatives, along with the dot and cross products.

We begin with the equation of a circle in cartesian coordinates, expressed parametrically using the position vector. The position vector for a point moving around the circle at any time t is

$$\mathbf{r}(t) = \left[R\cos(\omega t)\right]\mathbf{i} + \left[R\sin(\omega t)\right]\mathbf{j} \tag{1}$$

where R is the radius of the circle and ω is the angular velocity. Differentiating this gives the velocity vector:

$$\mathbf{v}(t) = \left[-\omega R\sin(\omega t)\right]\mathbf{i} + \left[\omega R\cos(\omega t)\right]\mathbf{j} \tag{2}$$

The magnitude of the velocity then will be

$$\|\mathbf{v}(t)\| = \sqrt{\left[-\omega R\sin(\omega t)\right]^2 + \left[\omega R\cos(\omega t)\right]^2} = v = \omega R \tag{3}$$

This is our first significant result, a formula for the **tangential speed**.

Let us next prove that this velocity vector is in fact tangent to the circle. If this is so, then the velocity vector must be normal to the position vector, since that latter vector points outward from the center of the circle. For the velocity vector to be normal to the position vector, the dot product of these vectors must vanish:

$$\mathbf{v}\bullet\mathbf{r} = \left\{\left[-\omega R\sin(\omega t)\right]\mathbf{i} + \left[\omega R\cos\omega t\right]\mathbf{j}\right\} \bullet \left\{R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j}\right\}$$

Multiplying out the components and adding them up can be seen to yield zero. So the velocity in Eq(2) and (3) is tangent to the circle.

The acceleration in x-y components is obtained by differentiating the components in Eq(2) w.r.t. time:

$$\mathbf{a}(t) = \left[-\omega^2 R\cos(\omega t)\right]\mathbf{i} + \left[-\omega^2 R\sin(\omega t)\right]\mathbf{j} \tag{4}$$

The magnitude of this acceleration will then be

$$\|\mathbf{a}(t)\| = \sqrt{\left[-\omega^2 R\cos(\omega t)\right]^2 + \left[-\omega^2 R\sin(\omega t)\right]^2} = \omega^2 R = \frac{v^2}{R} \tag{5}$$

This is another significant result, the **centripetal acceleration**. The second way of writing it used v from Eq(3).

The term *centripetal* means "toward the center." This implies that this acceleration is directed toward the center of the circle, so there must be a nonzero normal component of acceleration. We also know that, since the speed of the object in uniform circular motion is constant, the tangential component of acceleration must be zero.

We can find these components of the acceleration using formulas provided in an earlier document. For the tangential component of acceleration:

$$a_{\text{tangent}}(t) = \frac{\mathbf{v} \bullet \mathbf{a}}{\| \mathbf{v} \|} \Rightarrow \mathbf{v} \bullet \mathbf{a} = \left[-\omega R \sin(\omega t)\right]\left[-\omega^2 R \cos(\omega t)\right] + \left[\omega R \cos(\omega t)\right]\left[-\omega^2 R \sin(\omega t)\right]$$

If we expand this, it is seen that the result is zero, so that the **tangential acceleration is zero**. For the radial (or normal) component of acceleration we have

$$a_{\text{normal}}(t) = \frac{\| \mathbf{v} \times \mathbf{a} \|}{\| \mathbf{v} \|} \Rightarrow$$

$$\| \mathbf{v} \times \mathbf{a} \| = \left[-\omega R \sin(\omega t)\right]\left[-\omega^2 R \sin(\omega t)\right] - \left[\omega R \cos(\omega t)\right]\left[-\omega^2 R \cos(\omega t)\right]$$

Expanding this and dividing by the magnitude of the velocity that we found above in Eq(3):

$$\frac{\omega^3 R^2 \sin^2(\omega t) + \omega^3 R^2 \cos^2(\omega t)}{\omega R} = \omega^2 R$$

which is the same result we found above for the magnitude of the acceleration. That magnitude and this normal component are equal, and this makes sense, since we already showed that the tangential component is zero.

We know that this acceleration is directed normal to the circle. But in which direction? It is shown in more advanced courses that "this [normal] component is always directed toward the center of curvature on the concave side of the path of motion." In the present case, this of course means toward the center of the circle.

EQUATION SUMMARY

$$2\pi \text{ radians} = 360 \text{ degrees} \qquad \text{radians} \times \frac{180}{\pi} = \text{degrees} \qquad \text{degrees} \times \frac{\pi}{180} = \text{radians}$$

$$\omega = \text{angular speed} = \frac{d\theta}{dt} \quad \text{rpm (revolutions per minute)} \qquad \theta(t) = \omega t$$

$$T = \text{period (time for one cycle)} \qquad f = \frac{1}{T} = \text{frequency (cycles per second, Hz)}$$

$$\omega = \frac{2\pi}{T} = 2\pi f \qquad v = \omega R = \frac{2\pi R}{T} \qquad a = \frac{v^2}{R} = \omega^2 R$$

One revolution or one "cycle" = 2π radians. SI units for angular velocity are radians per second, but it is frequently given in rpm. The tangential or "linear" velocity is, as usual, in m/s. Another way to write Hz that helps in dimensional analysis is s^{-1}.