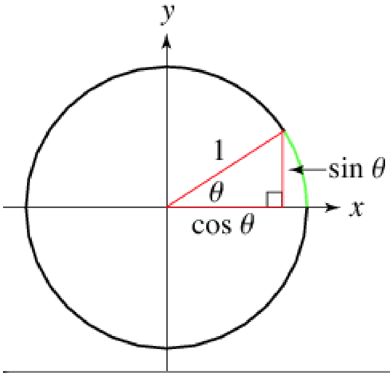
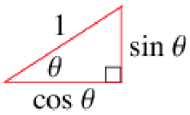
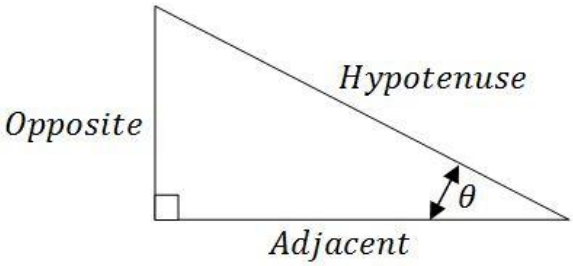
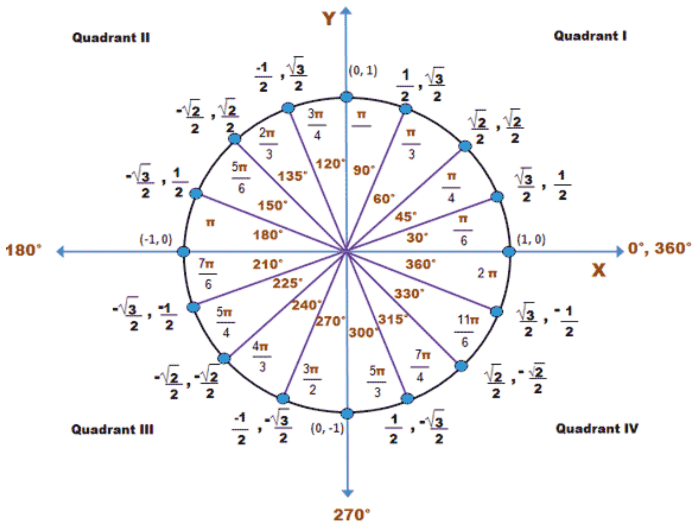
Circular Functions







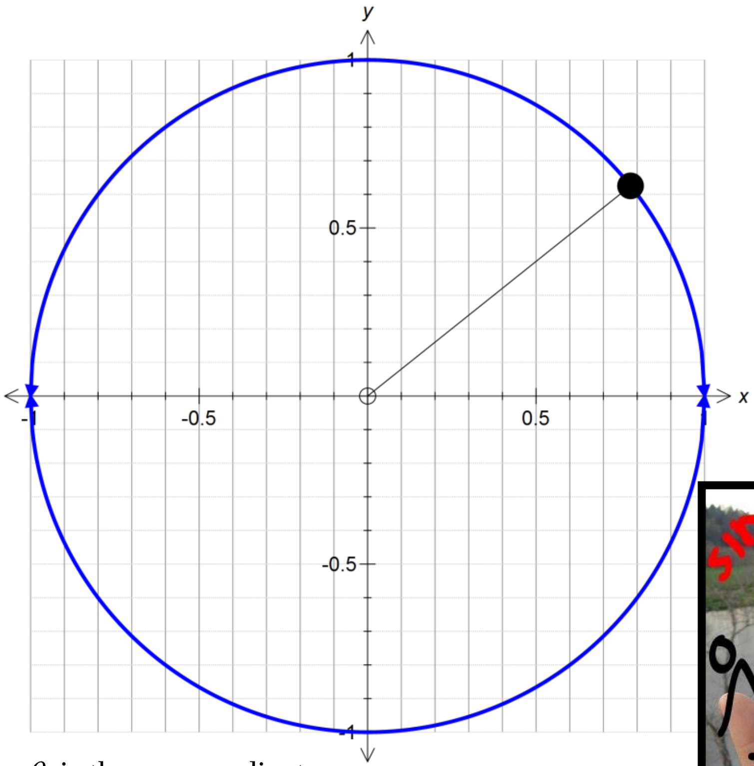


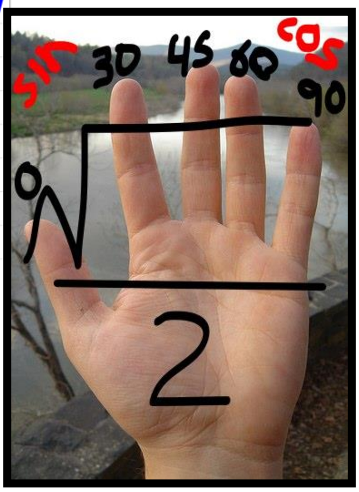
Name:

Circular Functions (Trigonometry)

Circular functions Revision

Where do $\sin\theta, \cos\theta$ and $\tan\theta$ come from?
Unit circle (of radius 1)



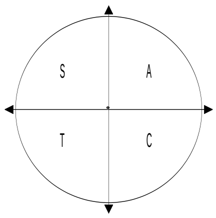


- $\cos\theta$ is the x – coordinate
- $\sin\theta$ is the y – coordinate
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- all 3 are measures of length.
- Remember SOH CAH TOA
- Exact values:

θ°	0	30° $\frac{\pi}{6}$	45° $\frac{\pi}{4}$	60° $\frac{\pi}{3}$	90° $\frac{\pi}{2}$	180° π	270° $\frac{3\pi}{2}$	360° 2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0	undefined	0

- Angle conversions (between radians and degrees).

- Quadrants and symmetry:
    - All Students Talk C.. (ASTC)



Finding Exact values:

Example: What is the exact value of:

(a) $\sin\frac{5\pi}{4}$; (b) $\tan\frac{-2\pi}{3}$.

(a) 1. Sign: 3rd Quadrant $\Rightarrow$ -ve

2. Angle Equivalent (1st Quadrant): $\frac{5\pi}{4} = \pi + \frac{\pi}{4} \Rightarrow \frac{\pi}{4}$

3. So: $\sin\frac{5\pi}{4} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \; or \; \frac{-\sqrt{2}}{2}$

(b) 1. Sign: 2nd "negative" Quadrant $\Rightarrow$ +ve

2. Angle Equivalent (1st Quadrant): $\frac{-2\pi}{3} = -\pi + \frac{\pi}{3} \Rightarrow \frac{\pi}{3}$

3. So: $\tan\frac{-2\pi}{3} = +\tan\frac{\pi}{3} = \sqrt{3}$

Jump Start Holiday Questions

Review: radians, definitions, exact values, symmetry	**Ex6A** Q 1, 2, 3, 4 (ace for all); **Ex6B** Q 1, 2acegik, 3 acegikmoqsu, 4 aceg, 5 abdfgj, 6 **Ex6C** Q 2

CALCULATOR MODE: Always work in radians

Solving equations involving circular functions.

Finding axis intercepts:

1. Y-intercepts:

- $f(0)$ or $x = 0$.
- E.g. what is the Y-intercept of $f(x) = 3\sin 2\left(x - \frac{\pi}{6}\right) + 2$
   - $$f(0) = 3\sin 2\left(0 - \frac{\pi}{6}\right) + 2$$
     $$f(0) = 3\sin\left(2 \times \frac{-\pi}{6}\right) + 2$$
     $$f(0) = 3\sin\left(\frac{-\pi}{3}\right) + 2$$
     $$f(0) = 3 \times \frac{-\sqrt{3}}{2} + 2$$
     $$f(0) = \frac{-3\sqrt{3}}{2} + 2 = \frac{-3\sqrt{3} + 4}{2}$$

2. X-intercepts:

- $f(x) = 0$ or $y = 0$.

Examples: Find all values of θ for:

(a) $\left\{\theta : \cos\theta = \frac{\sqrt{3}}{2}, \quad \theta \in [0, 2\pi]\right\}$

1. Sign: (+ve) $\Rightarrow$Q1, 4

2. Angle: $\frac{\pi}{6}$

3. $\theta = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$
   $\theta = \frac{\pi}{6}, \frac{11\pi}{6}$

(b) $\{\theta : \sin\theta = -0.7, \quad \theta \in [0, 2\pi]\}$

1. Sign: (-ve) $\Rightarrow$Q3, 4
2. Angle: $\sin^{-1}(0.7) = 0.7754$
3. $\theta = \pi + 0.7754,\ 2\pi - 0.7754$
   $\theta = 3.9170, 5.5078$

(c) $\{\theta : 2\sin\theta + 1 = 0, \quad \theta \in [-2\pi, 2\pi]\}$

Rewrite: $\sin\theta = -\frac{1}{2}$

1. Sign: (-ve) $\Rightarrow$ Q3, 4, -1, -2

2. Angle: $\frac{\pi}{6}$

3. $\theta = -\pi + \frac{\pi}{6}, 0 - \frac{\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(d) $\{4\cos 2\theta + 2 = 0, \quad \theta \in [0, 2\pi]\}$

Rewrite: $\cos\theta = -\frac{1}{2}$

Let $a = 2\theta \qquad a \in [0, 4\pi]$

$\cos a = -\frac{1}{2}$

1. Sign: (-ve) ⇒Q 2, 3, 6, 7

2. Angle: $\frac{\pi}{3}$

$a = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, 3\pi + \frac{\pi}{3}$

3. $2\theta = a = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

- **Ex6E** 1 ace, 2 ac, 3 ac, 4 ab, 5 abc, 6 ace, 7 ace, 8 acegi; **Ex6J** 4, 5, 6
- **2011 Exam1 Question**

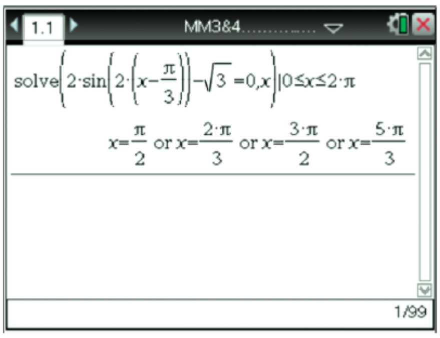
b. Solve the equation

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \text{ for } x \in [0, \pi].$$

-

Using the TI-Nspire

Use **Solve** from the **Algebra** menu as shown.

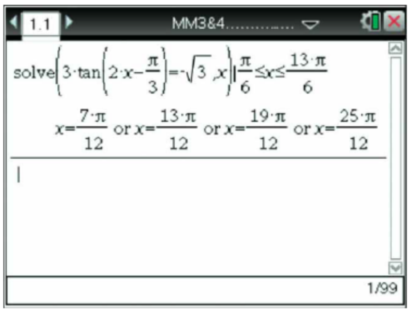


Using the TI-Nspire

To find the x-axis intercepts,

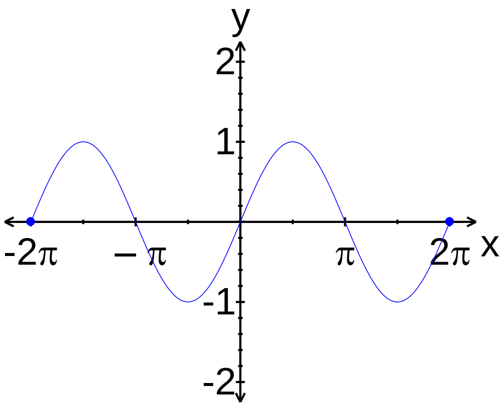
Enter

solve$\left(3\tan\left(2x-\frac{\pi}{3}\right)=-\sqrt{3},x\right)|\frac{\pi}{6}\le x\le\frac{13\pi}{6}$

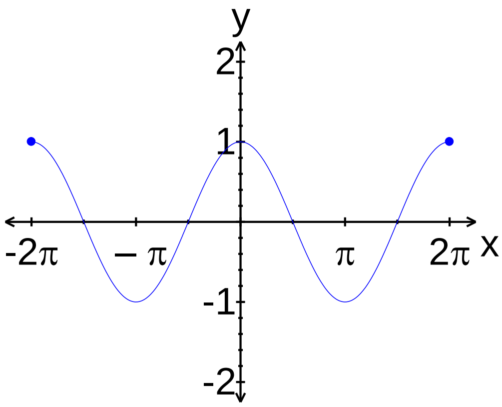


Graphs of Circular Functions

$y = \sin\theta$



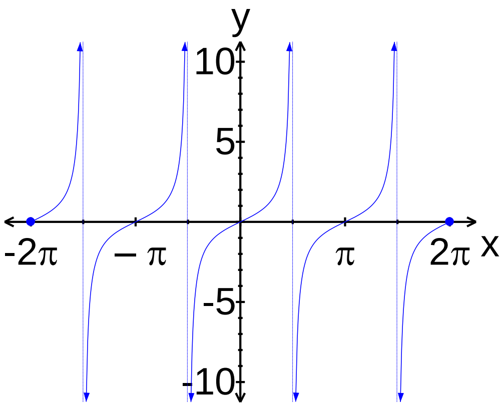
$y = \cos\theta$



- Period = 2π
- Amplitude = 1
- Range: [-1, 1]

$$period = \frac{2\pi}{n}$$

$y = \tan\theta$



- Period = π
- We don't refer to the amplitude for $y = \tan\theta$
- Range: R

$$period = \frac{\pi}{n}$$

Transformations of $y = \sin\theta$ & $y = \cos\theta$

$y = \sin\theta \rightarrow y = a\sin n(\theta - b) + c$ & $y = \cos\theta \rightarrow y = a\cos n(\theta - b) + c$

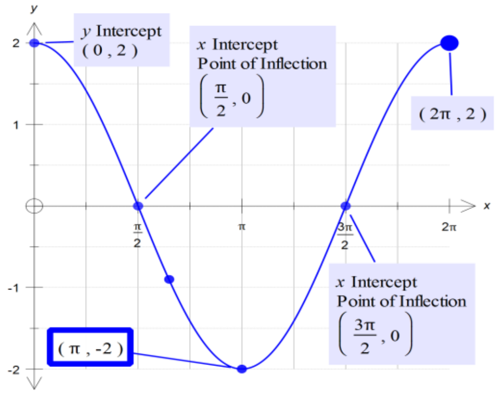
- *a:* a dilation of factor "*a*" from the *x-axis*.
- *n:* a dilation of factor "$\frac{1}{n}$" from the *y-axis*.
- *b:* a translation of *b* units along the *x-axis*.
- *c:* a translation of *c* units along the *y-axis*.

1. Dilations

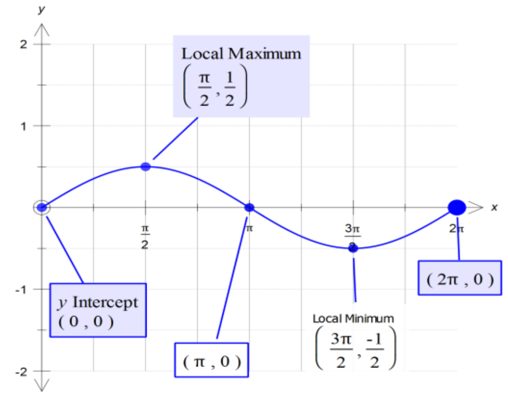
(a) The effect of "a"

Graph the following graphs: (i) $y = 2\cos\theta$; (ii) $y = \frac{\sin\theta}{2}$; where $\theta \in [0, \quad 2\pi]$

(i)



(ii)

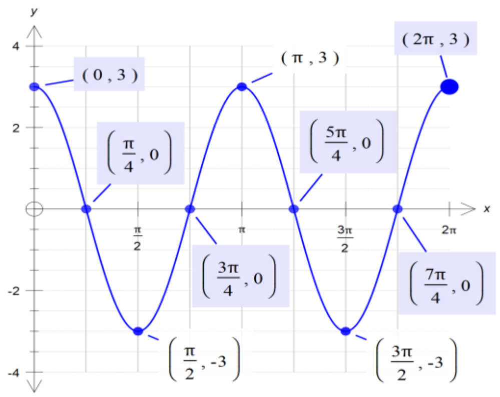


- "*a*" affects the amplitude.

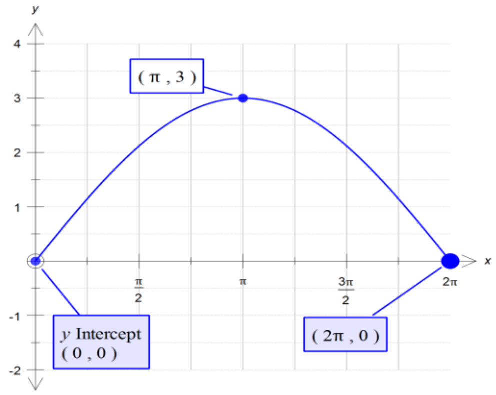
(b) The effect of "n"

Graph the following graphs: (i) $y = 3\cos 2\theta$; (ii) $y = 3\sin\left(\frac{\theta}{2}\right)$; where $\theta \in [0, \quad 2\pi]$

(i)



(ii)



- "n" affects the period.
- $period = \frac{2\pi}{n}$

2. Reflections.

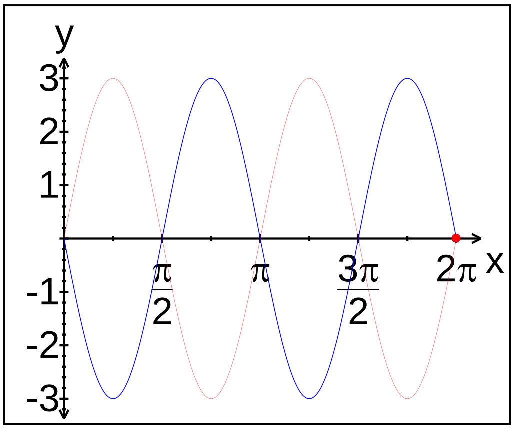
- Two types:
  - Reflection in the *x-axis*: $-f(x)$
  - Reflection in the *y-axis*: $f(-x)$

Examples:

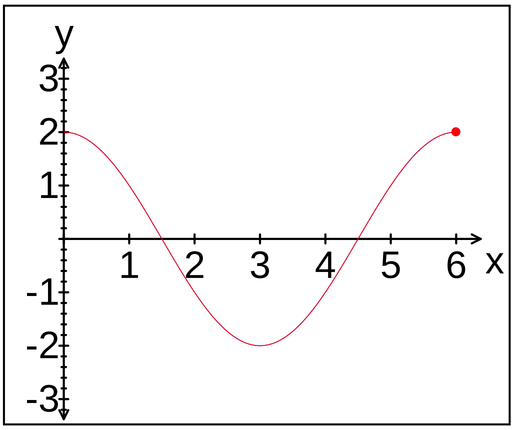
Sketch the graphs of the following:

(a) $y=-3\sin 2\theta$; (b) $y=2\cos\left(-\frac{\pi\theta}{3}\right)$; where $\theta\in[0,\quad 2\pi]$

(a)



(b)

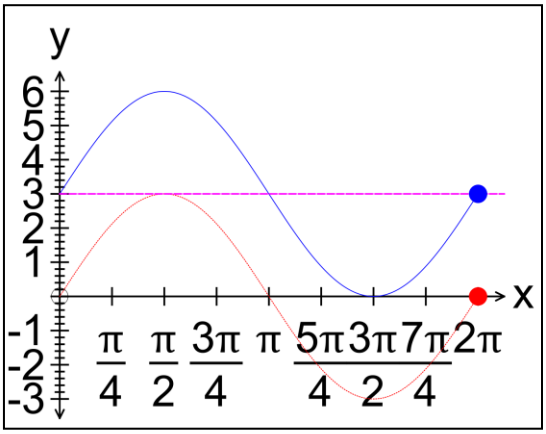


3. Translations

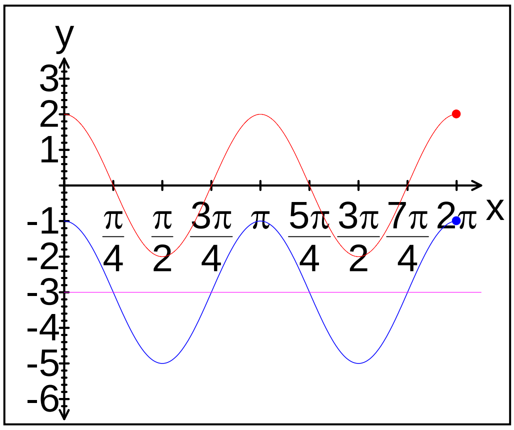
(a) The effect of "*c*"

Sketch the following: (i) $y=3\sin\theta+3$; (ii) $y=2\cos 2\theta-3$; where $\theta\in[0,\quad 2\pi]$

(i)



(ii)

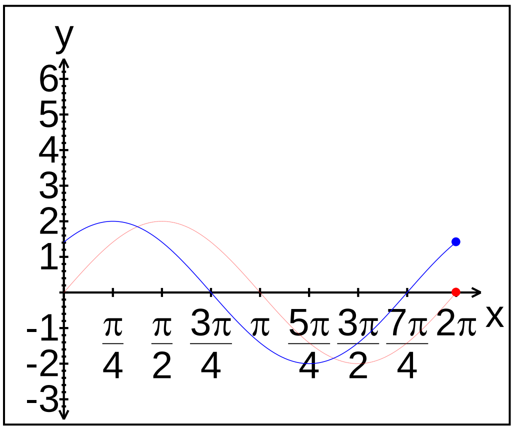


(b) The effect of "*b*"

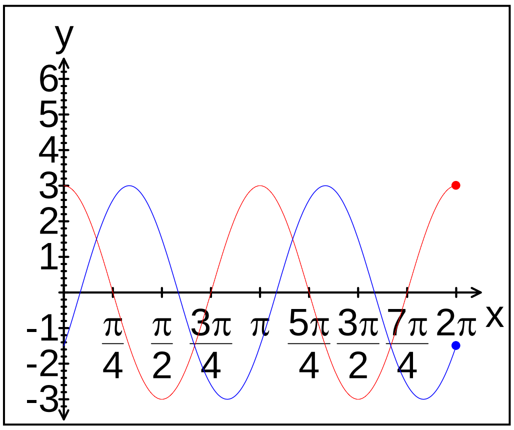
Sketch the following:

(i) $y = 2\sin\left(\theta + \frac{\pi}{4}\right)$; (ii) $y = 3\cos 2\left(\theta - \frac{\pi}{3}\right)$; where $\theta \in [0, \quad 2\pi]$

(i)



(ii)



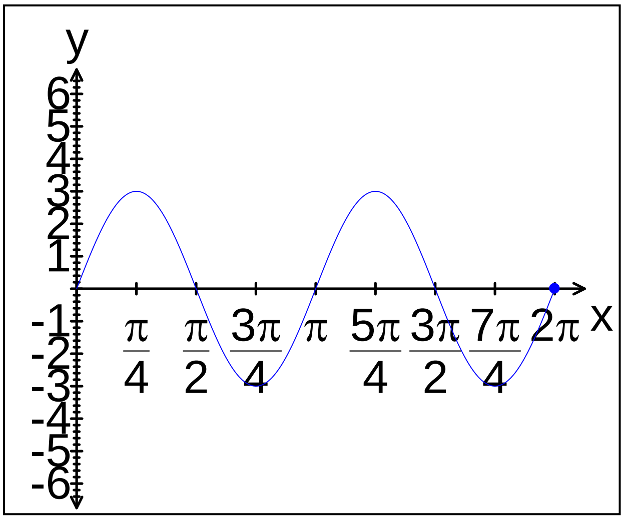
Combining all transformations

Example: Sketch the graph of $f(\theta) = 3\sin\left(2\theta - \frac{\pi}{2}\right) + 2, \quad \theta \in [0, 2\pi]$

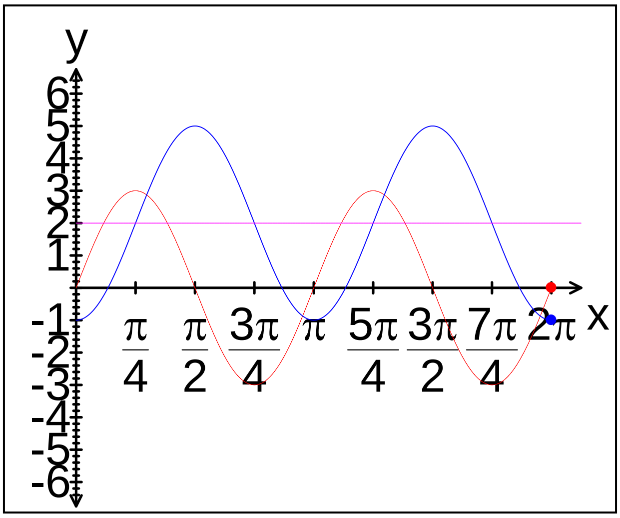
Rewrite: $f(\theta) = 3\sin 2\left(\theta - \frac{\pi}{4}\right) + 2$

$a = 3, b = \frac{\pi}{4}, c = 2 \; and \; n = 2$

Sketch $f(\theta) = 3\sin 2\theta$ first:



Secondly with translations:



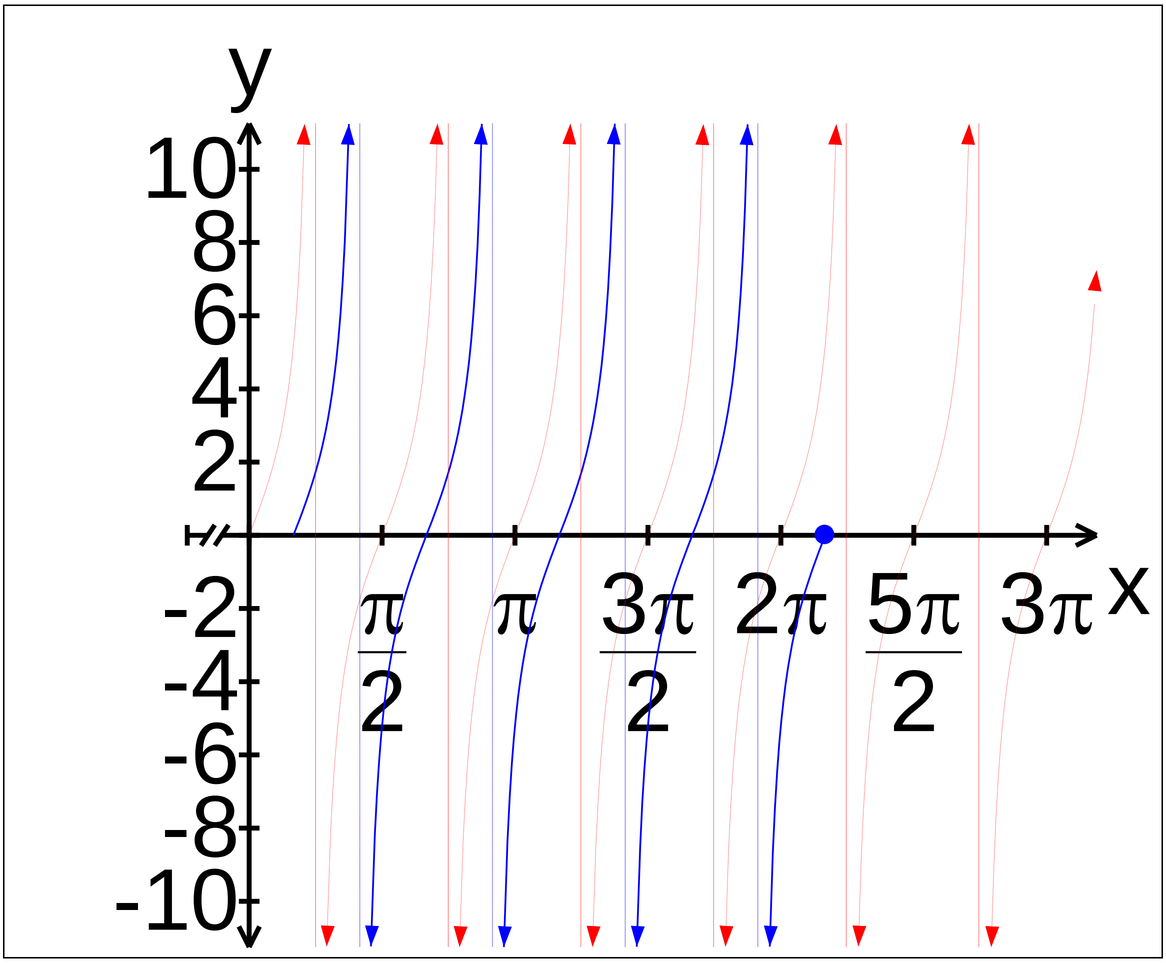
Note: X-intercepts need to be found!!

- **Ex6F** 1 adfhi, 2, 4, 5; **Ex6G** 1, 2 ac, 3 ef, 5 acfgh, 6, 7

Graphs &Transformations of the Tangent function

Example: Sketch $y = 3\tan\left(2x - \frac{\pi}{3}\right)$ *for* $\frac{\pi}{6} \le x \le \frac{13\pi}{6}$

Rewrite: $y = 3\tan 2\left(x - \frac{\pi}{6}\right)$



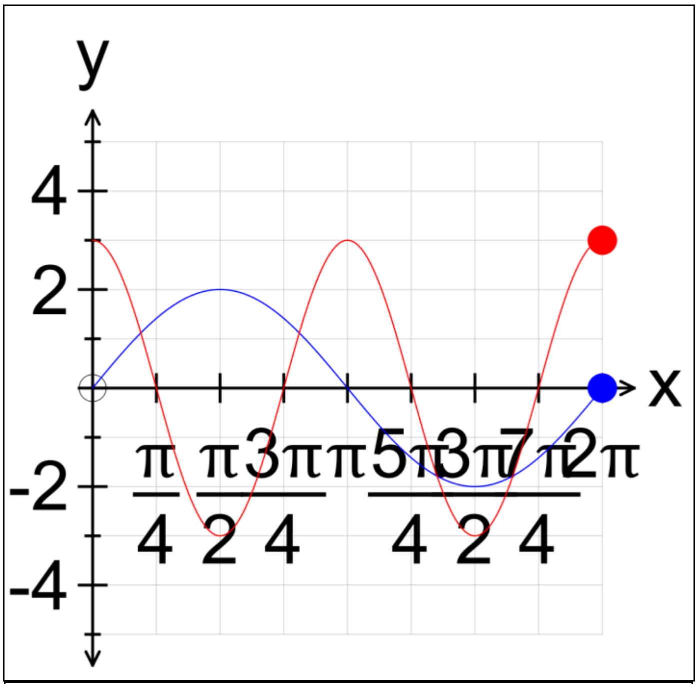
- **Ex6J** 1, 2, 7, 8, 9

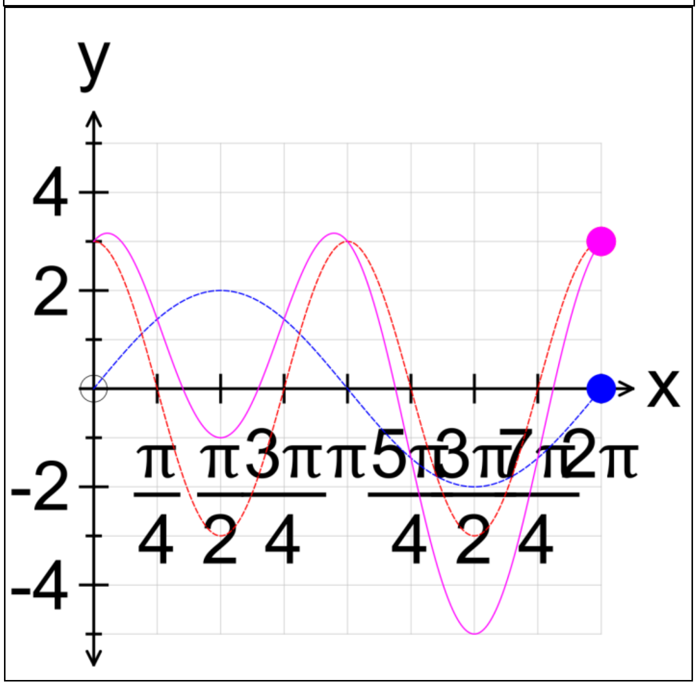
Addition of ordinates (add the '*y*' values)

Example:

(a) On the same set of axes sketch $f(x) = 2\sin x$ and $g(x) = 3\cos 2x$ for $0 \le x \le 2\pi$;

(b) Use addition of ordinates to sketch the graph of $y = 2\sin x + 3\cos 2x$.





Note: For $y = 2\sin(x) - 3\cos(2x)$ it is easier to do $y = 2\sin(x) + (-3\cos(2x))$

- **Ex6H** 1 ace

Solving Equations where both *sin* & *cos* appear

Example: Solve for x, $x \in [0, 2\pi]$:

(i)

$$\sin x = 0.5\cos x$$

divide both sides by $\cos x$

$$\frac{\sin x}{\cos x} = \frac{0.5\cos x}{\cos x}$$

$$\tan x = 0.5$$

1. Sign: (+ve) $\Rightarrow$Q 1, 3
2. Angle: $\tan^{-1}(0.5) = 0.464$
3. $x = 0.464, \pi + 0.464$
   $x = 0.464, 3.605$

(ii) $\sin 3x - \sqrt{3}\cos 3x = 0$

$$\sin 3x - \sqrt{3}\cos 3x = 0$$

$$\tan 3x - \sqrt{3} = 0$$

$$\tan 3x = \sqrt{3}$$

$$3x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}, \frac{16\pi}{3}$$

$$x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$$

- **Ex6J** 10, 11 acegi, 12

General Solutions to Circular Functions

Example: Solve $\cos x = \frac{1}{2}$

Solution:

$\cos x = \frac{1}{2}$

1.*Cos* positive Quad ...- 4,-1, 1, 4, 5, 9, ...

2.*Angle*: $\frac{\pi}{3}$

3. $x = ...-2\pi + \frac{\pi}{3}, \frac{-\pi}{3}, \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}, ...$ or $\frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$ $\frac{\pi(1+6n)}{3}, \frac{\pi(5+6n)}{3}$

$x = ..., \frac{-5\pi}{3}, \frac{-\pi}{3} \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, ...$

generally:

$x = 2n\pi \pm \frac{\pi}{3}$ $\quad$ *Check*: $n = 0, n = 1, n = -1 \quad n \in Z$

So in general terms: $\cos x = a$
$x = 2n\pi \pm \cos^{-1}(a), n \in Z$

Example: Solve $\sin x = \frac{1}{2}$

Solution:

$\sin x = \frac{1}{2}$

1. *Sin* positive Quad ...- 4,-3, 1, 2, 5, 6, ...

2.*Angle*: $\frac{\pi}{6}$

3. $x = ...-2\pi + \frac{\pi}{6}, -\pi - \frac{\pi}{6}, \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, ...$

$x = ..., \frac{-11\pi}{6}, \frac{-7\pi}{6} \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, ...$

generally:

$x = 2n\pi + \frac{\pi}{6}$ $\quad$ *Check*: $n = 0, n = 1, n = -1 \quad n \in Z$

or

$x = (2n+1)\pi + \frac{\pi}{6} = 2n\pi + \frac{5\pi}{6}$ $\quad$ *Check*: $n = 0, n = 1, n = -1$

So in general terms: $\sin x = a$
$x = 2n\pi + \sin^{-1}(a) \quad or \quad (2n+1)\pi - \sin^{-1}(a), n \in Z$

The above can be simplified to $x = n\pi + (-1)^n \sin^{-1}(a) \quad , n \in Z$

For $\tan x = a$
$x = n\pi + \tan^{-1}(a) \quad , n \in Z$

Example 1: Find the general solution for $2\sin\left(x+\frac{\pi}{3}\right)=-1$

Solution:

$$x=2n\pi+\sin^{-1}(a) \qquad or \qquad x=(2n+1)\pi-\sin^{-1}(a)$$

$$2\sin\left(x+\frac{\pi}{3}\right)=-1$$

$$\sin\left(x+\frac{\pi}{3}\right)=\frac{-1}{2}$$

$$x+\frac{\pi}{3}=2n\pi+\sin^{-1}\left(\frac{-1}{2}\right) \qquad or \qquad x+\frac{\pi}{3}=(2n+1)\pi-\sin^{-1}\left(\frac{-1}{2}\right)$$

$$x+\frac{\pi}{3}=2n\pi+\frac{7\pi}{6} \qquad or \qquad x+\frac{\pi}{3}=(2n+1)\pi-\frac{7\pi}{6}$$

$$x=2n\pi+\frac{7\pi}{6}-\frac{\pi}{3} \qquad or \qquad x=(2n+1)\pi-\frac{7\pi}{6}-\frac{\pi}{3}$$

$$x=2n\pi+\frac{5\pi}{6} \qquad or \qquad x=(2n+1)\pi-\frac{3\pi}{2}=2n\pi-\frac{\pi}{2}$$

Example 2: Find the general solution to $2\cos\left(2x+\frac{\pi}{4}\right)=\sqrt{2}$, and hence find all the solutions from $(-2\pi,2\pi)$.

Solution:

$$x=2n\pi\pm\cos^{-1}(a)$$

$$2\cos\left(2x+\frac{\pi}{4}\right)=\sqrt{2}$$

$$\cos\left(2x+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$2x+\frac{\pi}{4}=2n\pi\pm\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$2x+\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}$$

$$2x=2n\pi \qquad or \qquad 2x=2n\pi-\frac{\pi}{2}$$

$$x=n\pi \qquad or \qquad x=n\pi-\frac{\pi}{4} \qquad general\ solution$$

for solutions of domain $(-2\pi,2\pi)$

$$n=-2 \qquad x=-2\pi\ (not\ in\ domain)\ \&\ x=-2\pi-\frac{\pi}{4}=\frac{-9\pi}{4}\ (not\ in\ domain)$$

$$n=-1 \qquad x=-\pi\ \&\ x=-\pi-\frac{\pi}{4}=\frac{-5\pi}{4}$$

$$n=0 \qquad x=0\ \&\ x=-\frac{\pi}{4}=\frac{-\pi}{4}$$

$$n=1 \qquad x=\pi\ \&\ x=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

$$n=2 \qquad x=2\pi\ (not\ in\ domain)\ \&\ x=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$$

$$x=\frac{-5\pi}{4},-\pi,\frac{-\pi}{4},0,\frac{3\pi}{4},\pi,\frac{7\pi}{4}$$

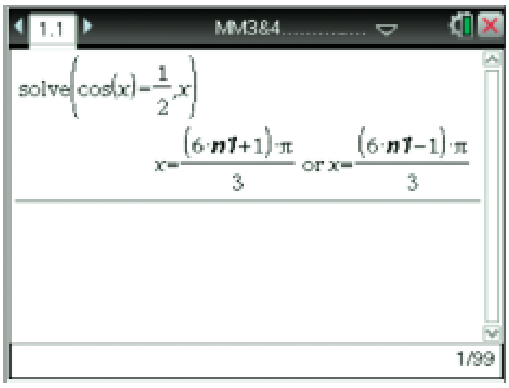
- **Ex6K** 1, 2, 3, 6ab, 8, 9

Using the TI-Nspire

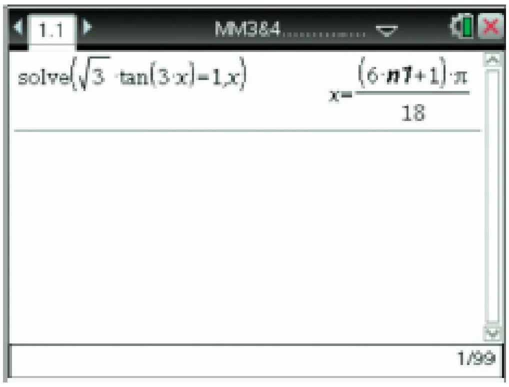
Make sure the calculator is in Radian mode.

a Use **Solve** from the **Algebra** menu and complete as shown.

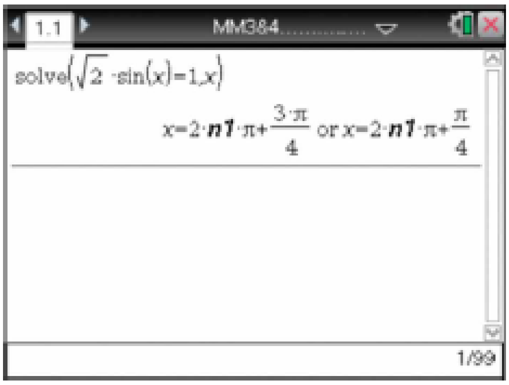
Note the use of $\frac{1}{2}$ rather than 0.5 to ensure that the answer is exact.



b Complete as shown.

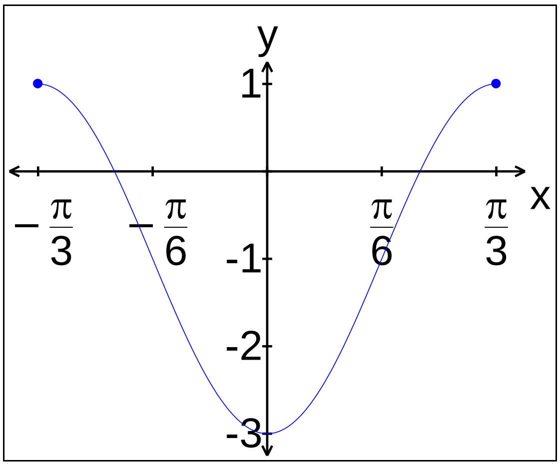


c Complete as shown.



- **Determining Rules for Circular Functions**

Example: The graph shown has the rule of the form: $y = a\cos n(t-b)+c$, find a, b, c & n.



- $a: range = [-3, 1] \Rightarrow a = 2$
- $c: middle\ of\ range\ y = -1 \Rightarrow c = -1$
- $n: period = \frac{2\pi}{n} \quad \therefore \frac{2\pi}{3} = \frac{2\pi}{n} \quad \Rightarrow n = 3$

$b:\ from\ previous\ knowledge: b = \frac{\pi}{3}$

$or\ (0, -3)\ is\ on\ the\ curve:$

$-3 = 2\cos 3(0+b) - 1$

- $-2 = 2\cos 3b$

$-1 = \cos 3b$

$\pi = 3b$

$\frac{\pi}{3} = b$

- **Ex6I** 1, 2, 3, 4, 5, 6, 7, 8, 9; **Ex6J** 14, 15

Applications of Circular Functions

worked example 24

The temperature in degrees Celsius on a day in May at Mt Buller is expected to follow the model

$T = 5 - 7\cos\frac{\pi}{12}(t-4)$

where *t* is the number of hours after midnight. The snow-making machines will only operate efficiently when the temperature is below 5°C. Sketch the graph of the temperature for one full day, and predict the period of time for which the machine will be able to operate.

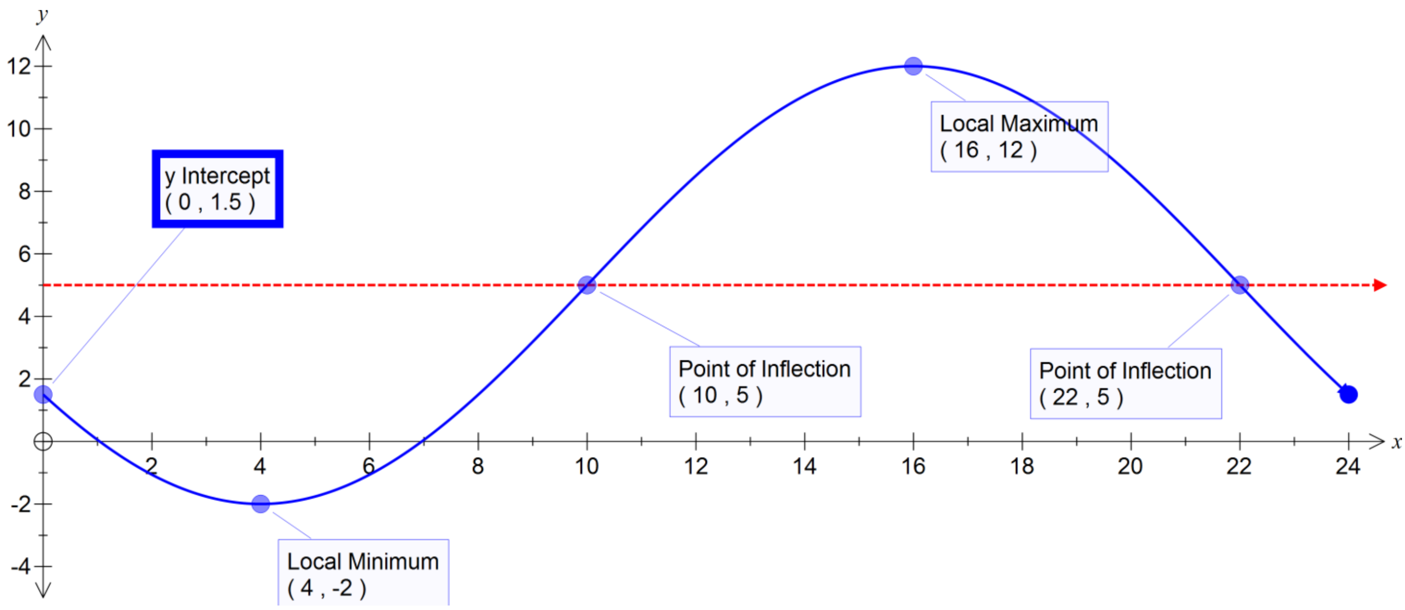
Rewrite:

$$T = -7\cos\left(\frac{\pi(t-4)}{12}\right) + 5$$

$$a = -7,\ b = 4,\ c = 5,\ n = \frac{\pi}{12}$$

$$period = \frac{2\pi}{\frac{\pi}{12}} = 24\ hours$$

$$at\ t = 0,\quad T = -7\cos\left(\frac{\pi(0-4)}{12}\right) + 5 = -7\cos\left(\frac{-\pi}{3}\right) + 5 = -\frac{7}{2} + 5 = \frac{3}{2}$$



The machine will not be able to operate for 12 hours, i.e $10 \le t \le 22$. (i.e from 10a.m. to 10 p.m.)

- **Ex6L** 1, 2, 4, 6 **Ex 6N**

Past Exam Questions

2008

Question 3

Solve the equation $\cos\left(\frac{3x}{2}\right) = \frac{1}{2}$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Question 3

Marks	0	1	2	Average
%	31	28	41	1.2

$\cos\left(\frac{3x}{2}\right) = \frac{1}{2},\ x\ \varepsilon \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\frac{3x}{2} = \frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3} \ \ldots$

$x = \frac{-2\pi}{9}, \frac{2\pi}{9}, \frac{10\pi}{9} \ldots$

$x = \frac{-2\pi}{9}$ or $\frac{2\pi}{9}$

2 marks

Question 18

Let $f: \left[0, \frac{\pi}{2}\right] \to R,\ f(x) = \sin(4x) + 1$. The graph of f is transformed by a reflection in the x-axis followed by a dilation of factor 4 from the y-axis.

The resulting graph is defined by

A. $g: \left[0, \frac{\pi}{2}\right] \to R \qquad g(x) = -1 - 4\sin(4x)$

B. $g: [0, 2\pi] \to R \qquad g(x) = -1 - \sin(16x)$

C. $g: \left[0, \frac{\pi}{2}\right] \to R \qquad g(x) = 1 - \sin(x)$

D. $g: [0, 2\pi] \to R \qquad g(x) = 1 - \sin(4x)$

E. $g: [0, 2\pi] \to R \qquad g(x) = -1 - \sin(x)$

18	13	10	31	9	36	0	$f(x) = \sin(4x) + 1$ Reflection in the x-axis $h(x) = -\sin(4x) - 1$ Dilation by a factor of 4 from the y-axis $g(x) = -\sin(x) - 1$ **Note** the dilation by a factor of 4 from the y-axis affects the domain, $\left[0 \times 4, \frac{\pi}{2} \times 4\right] = [0, 2\pi]$

Question 4

Solve the equation $\tan(2x) = \sqrt{3}$ for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

Q4 $\tan(2x) = \sqrt{3},\ x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right),$

$2x = \frac{\pi}{3}, \frac{4\pi}{3}$

$x = \frac{\pi}{6}, \frac{2\pi}{3}$

3 marks

2009

Question 4

The general solution to the equation $\sin(2x) = -1$ is

A. $x = n\pi - \frac{\pi}{4},\ n \in Z$

B. $x = 2n\pi + \frac{\pi}{4}$ or $x = 2n\pi - \frac{\pi}{4},\ n \in Z$

C. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{2},\ n \in Z$

D. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{4},\ n \in Z$

E. $x = n\pi + \frac{\pi}{4}$ or $x = 2n\pi + \frac{\pi}{4},\ n \in Z$

Q4 $\sin 2x = -1$, $2x = 2n\pi + \frac{3\pi}{2}$ or $2n\pi - \frac{\pi}{2}$, where $n \in Z$.

$\therefore x = n\pi + \frac{3\pi}{4}$ or $n\pi - \frac{\pi}{4}$.

Question 12

A transformation $T: R^2 \to R^2$ that maps the curve with equation $y = \sin(x)$ onto the curve with equation $y = 1 - 3\sin(2x + \pi)$ is given by

A. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \pi \\ 1 \end{bmatrix}$

B. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{2} \\ 1 \end{bmatrix}$

C. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \pi \\ 1 \end{bmatrix}$

D. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ 1 \end{bmatrix}$

E. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ -1 \end{bmatrix}$

Q12 $y' = 1 - 3\sin(2x' + \pi)$, $\frac{y' - 1}{-3} = \sin(2x' + \pi)$,

$\therefore y = \frac{y' - 1}{-3}$, i.e. $y' = -3y + 1$, and $x = 2x' + \pi$, i.e. $x' = \frac{x}{2} - \frac{\pi}{2}$.

$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ 1 \end{bmatrix}$ D

2010

Question 4

a. Write down the amplitude and period of the function

$$f: R \to R, f(x) = 4\sin\left(\frac{x+\pi}{3}\right).$$

2 marks

b. Solve the equation $\sqrt{3}\sin(x) = \cos(x)$ for $x \in [-\pi, \pi]$.

Q4a $amplitude = 4$, $period = \dfrac{2\pi}{\frac{1}{3}} = 6\pi$

Q4b $\dfrac{\sin(x)}{\cos(x)} = \dfrac{1}{\sqrt{3}}$, $\tan(x) = \dfrac{1}{\sqrt{3}}$, $x = -\dfrac{5\pi}{6}, \dfrac{\pi}{6} \in [-\pi, \pi]$

2 marks

Question 3

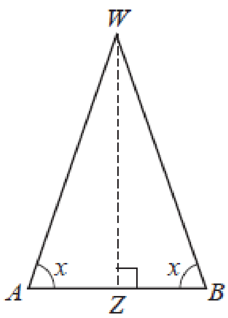
An ancient civilisation buried its kings and queens in tombs in the shape of a square-based pyramid, *WABCD*.

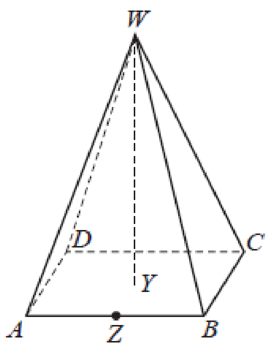
The kings and queens were each buried in a pyramid with $WA = WB = WC = WD = 10$ m.

Each of the isosceles triangle faces is congruent to each of the other triangular faces.

The base angle of each of these triangles is x, where $\frac{\pi}{4} < x < \frac{\pi}{2}$.

Pyramid *WABCD* and a face of the pyramid, *WAB*, are shown here.





Z is the midpoint of *AB*.

a. i. Find *AB* in terms of *x*.

Q3ai $\dfrac{\overline{AZ}}{10} = \cos(x)$, $\overline{AZ} = 10\cos(x)$, $\therefore\ \overline{AB} = 20\cos(x)$

Q3aii $\dfrac{\overline{WZ}}{10} = \sin(x)$, $\overline{WZ} = 10\sin(x)$

c. Find *WY*, the height of the pyramid *WABCD*, in terms of x.

Q3b

$$\textit{Total surface area } S = (20\cos(x))^2 + 4\left(\frac{1}{2}(20\cos(x))(10\sin(x))\right)$$

$$= 400(\cos^2(x) + \cos(x)\sin(x))$$

Q3c $\overline{WY} = \sqrt{\overline{WZ}^2 + \overline{ZY}^2} = \sqrt{100\sin^2(x) - 100\cos^2(x)}$

$$= 10\sqrt{\sin^2(x) - \cos^2(x)} = 10\sqrt{1 - 2\cos^2(x)}$$

2 marks

d. The volume of any pyramid is given by the formula Volume = $\frac{1}{3}$ × area of base × vertical height.

Show that the volume, T m^3, of the pyramid *WABCD* is $\frac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}$.

Q3d $\textit{Volume } T = \frac{1}{3} \times (20\cos(x))^2 \times 10\sqrt{1 - 2\cos^2(x)}$

$$= \frac{4000}{3}\cos^2(x)\sqrt{1 - 2\cos^2(x)} = \frac{4000}{3}\sqrt{\cos^4(x)(1 - 2\cos^2(x))}$$

$$= \frac{4000}{3}\sqrt{(\cos^4(x) - 2\cos^6(x))}$$

2011

Question 3

a. State the range and period of the function

$$h: R \rightarrow R,\ h(x) = 4 + 3\cos\left(\frac{\pi x}{2}\right).$$

2 marks

b. Solve the equation

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \text{ for } x \in [0,\ \pi].$$

Q3a Range is $[1,7]$, period $= \frac{2\pi}{\frac{\pi}{2}} = 4$

Q3b $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ and $0 \le x \le \pi$, i.e. $\frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{7\pi}{3}$,

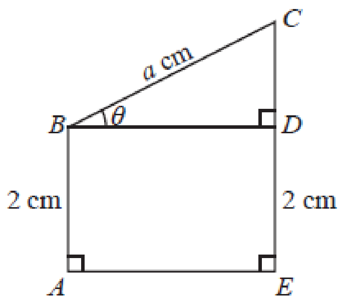
$\therefore\ 2x + \frac{\pi}{3} = \frac{5\pi}{6}$ or $\frac{13\pi}{6}$, $2x = \frac{\pi}{2}$ or $\frac{11\pi}{6}$, $\therefore\ x = \frac{\pi}{4}$ or $\frac{11\pi}{12}$

Question 10

The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. The point D is on line segment EC with $AB = ED = 2$ cm and $BC = a$ cm, where a is a positive constant.

$$\angle BAE = \angle CEA = \frac{\pi}{2}$$

Let $\angle CBD = \theta$ where $0 < \theta < \frac{\pi}{2}$.

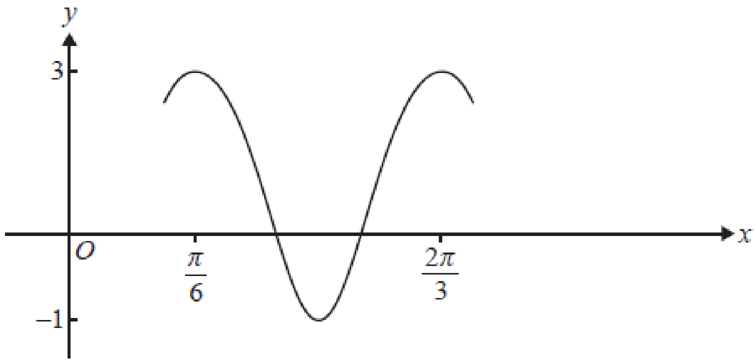


a. Find BD and CD in terms of a and θ.

Q10a $BD = a\cos\theta$, $CD = a\sin\theta$

Q10b $L = 4 + a + a\sin\theta + 2a\cos\theta$

Question 15



The graph shown could have equation

A. $y = 2\cos\left(x + \frac{\pi}{6}\right) + 1$

B. $y = 2\cos 4\left(x - \frac{\pi}{6}\right) + 1$

C. $y = 4\sin 2\left(x - \frac{\pi}{12}\right) - 1$

D. $y = 3\cos\left(2x + \frac{\pi}{6}\right) - 1$

E. $y = 2\sin\left(4x + \frac{2\pi}{3}\right) - 1$

Q15 Amplitude $= 2$, period $= \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2} = \frac{2\pi}{n}$, $\therefore\ n = 4$.

Translated upwards by 1 unit and to the right by $\frac{\pi}{6}$. B

2012

Question 6

The graphs of $y = \cos(x)$ and $y = a\sin(x)$, where a is a real constant, have a point of intersection at $x = \frac{\pi}{3}$.

a. Find the value of a.

Question 6a.

Marks	0	1	2	Average
%	12	23	65	1.5

$\cos\left(\frac{\pi}{3}\right) = a\sin\left(\frac{\pi}{3}\right) \Rightarrow \frac{1}{2} = \frac{a\sqrt{3}}{2}$ or $\tan(x) = \frac{1}{a} \Rightarrow \tan\left(\frac{\pi}{3}\right) = \sqrt{3} = \frac{1}{a}$

$a = \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$

This question was generally well answered.

Question 6b.

Marks	0	1	Average
%	48	52	0.5

$\frac{4\pi}{3}$

Many students also identified that the next solution was $\frac{7\pi}{3}$, either by sketch or the link to $\tan(x)$.

2 marks

b. If $x \in [0, 2\pi]$, find the x-coordinate of the other point of intersection of the two graphs.

1 mark

Question 1

The function with rule $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$ has period

A. 3

B. 5

C. 10

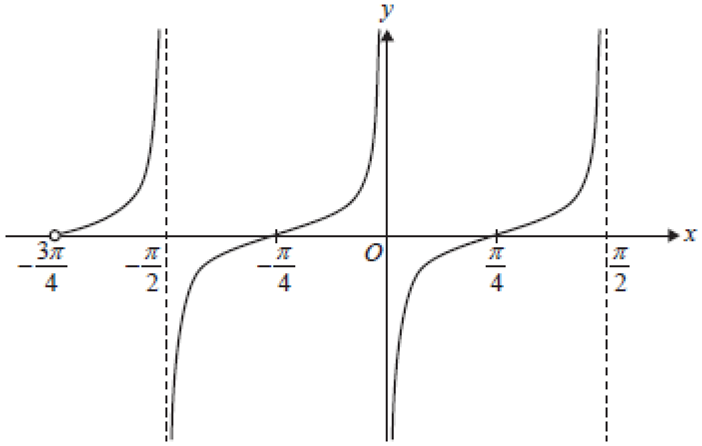
D. $\frac{\pi}{5}$

E. $\frac{\pi}{10}$

Question	% A	% B	% C	% D	% E	% No answer	Comments
1	1	3	91	4	1	0	

Question 6

A section of the graph of f is shown below.



The rule of f could be

A. $f(x) = \tan(x)$

B. $f(x) = \tan\left(x - \frac{\pi}{4}\right)$

C. $f(x) = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$

D. $f(x) = \tan\left(2\left(x - \frac{\pi}{2}\right)\right)$

E. $f(x) = \tan\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right)$

6	2	10	72	9	7	0	

Question 7

The temperature, T °C, inside a building t hours after midnight is given by the function

$$f:[0, 24] \to R,\ T(t) = 22 - 10\cos\left(\frac{\pi}{12}(t-2)\right)$$

The average temperature inside the building between 2 am and 2 pm is

A. 10 °C

B. 12 °C

C. 20 °C

D. 22 °C

E. 32 °C

7	5	11	13	67	3	0	

Question 19

A function f has the following two properties for all real values of θ.

$$f(\pi - \theta) = -f(\theta) \text{ and } f(\pi - \theta) = -f(-\theta)$$

A possible rule for f is

A. $f(x) = \sin(x)$

B. $f(x) = \cos(x)$

C. $f(x) = \tan(x)$

D. $f(x) = \sin\left(\frac{x}{2}\right)$

E. $f(x) = \tan(2x)$

19	13	45	19	12	10	1	As $f(x) = \cos(x)$ is positive in the first and fourth quadrants and negative in the second quadrant, then $f(\pi-\theta) = -f(\theta)$ and $f(\pi-\theta) = -f(-\theta)$. Alternatively, the given conditions imply the function is even; $\cos(x)$ is the only even function provided in the options.

2013

Question 4 (2 marks)

Solve the equation $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 4\pi]$.

Question 4

Marks	0	1	2	Average
%	23	31	47	1.3

$\frac{x}{2} = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\Rightarrow x = \frac{7\pi}{3}, \frac{11\pi}{3}$

This question was generally well done. Many students identified a base angle of $\frac{\pi}{6}$ but many could not identify the correct quadrants and domain restriction.

Question 1

The function with rule $f(x) = -3\tan(2\pi x)$ has period

A. $\frac{2}{\pi}$

B. 2

C. $\frac{1}{2}$

D. $\frac{1}{4}$

E. 2π

Question	% A	% B	% C	% D	% E	% No Answer	Comments
1	2	8	85	2	3	0	

Question 7

The function $g: [-a, a] \to R$, $g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.

The maximum possible value of a is

A. $\frac{\pi}{12}$

B. 1

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

E. $\frac{\pi}{2}$

7	37	19	18	14	11	1	The function g must be one to one for the inverse function to exist. The period of g is π. Hence, the domain for g could be $\left[-\frac{\pi}{4}+\frac{\pi}{6}, \frac{\pi}{4}+\frac{\pi}{6}\right]$, $\left[-\frac{\pi}{12}, \frac{5\pi}{12}\right]$. However, the domain for g needed to be in the form $[-a, a]$. Thus, the maximum possible value for a was $\frac{\pi}{12}$.

Question 1 (12 marks)

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature (T °C) is given by $T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right)$, $0 \le t \le 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a. State the maximum temperature in the greenhouse and the values of t when this occurs. 2 marks

Question 1

1a.

Marks	0	1	2	Average
%	15	26	58	1.5

Maximum 27 °C, when $t = 0$ or $t = 16$ h

Some students gave the maximum temperature but not the values of t. Some students did not answer both parts of the question.

b. State the period of the function T. 1 mark

1b.

Marks	0	1	Average
%	12	88	0.9

$$\text{Period} = \frac{2\pi}{\frac{\pi}{8}} = \frac{8}{\pi} \times 2\pi = 16 \text{ h}$$

Most students answered this question well.

c. Find the smallest value of t for which $T = 26$. 2 marks

1c.

Marks	0	1	2	Average
%	12	24	64	1.5

Solve $25 + 2\cos\left(\frac{\pi t}{8}\right) = 26$ for t, $t = \frac{8}{3}$ h

d. For how many hours during the 24-hour time interval is $T \ge 26$? 2 marks

1d.

Marks	0	1	2	Average
%	37	18	45	1.1

Solve $T(t) \ge 26$ for $0 \le t \le 24$, $\frac{8}{3} + \frac{56}{3} - \frac{40}{3} = 8$ h

Some students just gave the values of t and did not find the difference between them. Others used approximate values in their calculations.

2014

Question 3 (2 marks)

Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \le x \le \pi$.

Q3 $2\cos(2x) = -\sqrt{3}$ and $0 \le x \le \pi$, $\cos(2x) = -\frac{\sqrt{3}}{2}$

$\therefore\ 2x = \frac{5\pi}{6}, \frac{7\pi}{6}$, $x = \frac{5\pi}{12}, \frac{7\pi}{12}$

Question 1 (7 marks)

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n. 2 marks

Q1a $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$

Period $= \frac{2\pi}{\frac{\pi}{3}} = 6$ months; amplitude $= 400$

b. Find the maximum and minimum populations of wombats in this location. 2 marks

Q1b Maximum population $= 1200 + 400 = 1600$;
minimum population $= 1200 - 400 = 800$

c. Find $n(10)$. 1 mark

Q1c $n(10) = 1200 + 400\cos\left(\frac{\pi \times 10}{3}\right) = 1000$

d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$. 2 marks

Q1d $n < n(10)$ for $2 < t < 4$ and $8 < t < 10$

$\therefore$ 4 months out of 12, i.e. $\frac{1}{3}$

2015

Question 5 (3 marks)

On any given day, the depth of water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right),\ 0 \le t \le 24$$

where h is the depth of water, in metres, and t is the time, in hours, after 6 am.

a. Find the minimum depth of the water in the river. 1 mark

Question 5a.

Marks	0	1	Average
%	31	69	0.7

$h_{min} = 14 - 8 = 6$ metres

This question was generally well handled. Common errors included finding the maximum height rather than the minimum, negative heights (8 – 14) and evaluating $h(10)$. Students who used calculus to obtain a minimum value tended to make careless errors in the differentiation.

b. Find the values of t for which $h(t) = 10$. 2 marks

Question 5b.

Marks	0	1	2	Average
%	25	23	53	1.3

$14 + 8\sin\left(\frac{\pi t}{12}\right) = 10$

$\sin\left(\frac{\pi t}{12}\right) = -\frac{1}{2}$

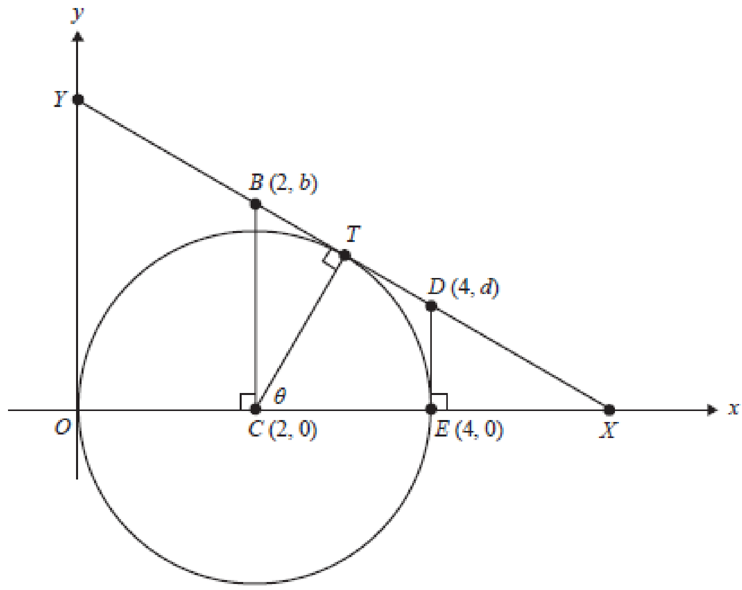
$\frac{\pi t}{12} = \frac{7\pi}{6}, \frac{11\pi}{6}$

$t = 14, 22$

This question was well handled. Most students set up an equation that when solved would yield the two correct answers for the restricted domain. Some students did not recognise the base angle of $\frac{\pi}{6}$.

Question 10 (7 marks)

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates (2, 0). The angle ECT is θ, where $0 < \theta \le \frac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.

a. Find the coordinates of T in terms of θ.

Question 10a.

Marks	0	1	Average
%	80	20	0.2

$(2 + 2\cos(\theta), 2\sin(\theta))$

The most common error in responses to this question was the oversight of + 2 for the x coordinate.

b. Find the gradient of the tangent to the circle at T in terms of θ. 1 mark

Question 10b.

Marks	0	1	Average
%	84	16	0.2

$m_{CT} = \tan(\theta)$

$m_{XT} = -\frac{1}{\tan(\theta)}$

Equivalent expressions such as $m_{XT} = -\frac{\cos(\theta)}{\sin(\theta)}$ were accepted. Some students included the variables of b or d in their final answer. Many students found the gradient of the radius CT rather than the gradient of the line segment XT.

c. The equation of the tangent to the circle at T can be expressed as

$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XY.

Find b in terms of θ. 1 mark

Question 10ci.

Marks	0	1	Average
%	46	54	0.6

$b\sin(\theta) + 2\cos(\theta) = 2 + 2\cos(\theta)$

$b = \frac{2}{\sin(\theta)}$

ii. Point D, with coordinates $(4, d)$, is on the line segment XY.

Find d in terms of θ. 1 mark

Question 10cii.

Marks	0	1	Average
%	53	47	0.5

$d\sin(\theta) + 4\cos(\theta) = 2 + 2\cos(\theta)$

$d = \frac{2 - 2\cos(\theta)}{\sin(\theta)}$

Question 1

Let $f: R \to R$, $f(x) = 2\sin(3x) - 3$.

The period and range of this function are respectively

A. period $= \frac{2\pi}{3}$ and range $= [-5, -1]$

B. period $= \frac{2\pi}{3}$ and range $= [-2, 2]$

C. period $= \frac{\pi}{3}$ and range $= [-1, 5]$

D. period $= 3\pi$ and range $= [-1, 5]$

E. period $= 3\pi$ and range $= [-2, 2]$

Question	% A	% B	% C	% D	% E	% No answer	Comments
1	95	4	1	0	0	0	