THE GRAVITY TUNNEL

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Consider the gravitational force on an object inside a perfectly spherical mass ("planet") of uniform density. At a given distance from the center of the planet, we can separate the mass of the planet into two parts: (1) that mass farther from the center than is the object, which makes a shell, and (2) the mass at the same distance from the center as the object, or closer. The latter makes a smaller sphere.

For the shell outside our object's position the *net* gravitational force on the object is zero; any inverse-square pull from one side of the shell is exactly balanced by a pull in the opposite direction from the other side of the shell. (This is oversimplified: see any calculus-based physics text, e.g., Fishbane, et. al., *Physics for Scientists and Engineers*, pp.364-5, for a more complete explanation.)

Newton showed that any sphere acts gravitationally as though all its mass was at its center; thus our object is attracted toward the planet center with an inverse-square force, due only to the spherical mass from the center to the object's distance from the center. This "inside" force can be expressed as

$$F_{in} = -\frac{G\,M_r\,m}{r^2} \tag{1}$$

where G is the universal gravitational constant, M_r is the mass of the planet from the center to the object's position r, and m is the object's mass. The negative sign indicates that the force is one of attraction. The mass of the planet M_r out to radius r can be found using

$$M_r = \frac{4}{3}\,\pi\,r^3\rho \tag{2}$$

where ρ is the uniform density of the planet; this in turn is found from

$$\rho = \frac{M}{\frac{4}{3}\pi\,R^3}$$

where R is the planet radius and M is the planet mass. Using this in Eq(2), and then Eq(1) gives

$$F_{in} = -\frac{G\,M}{R^3}\;m\,r$$

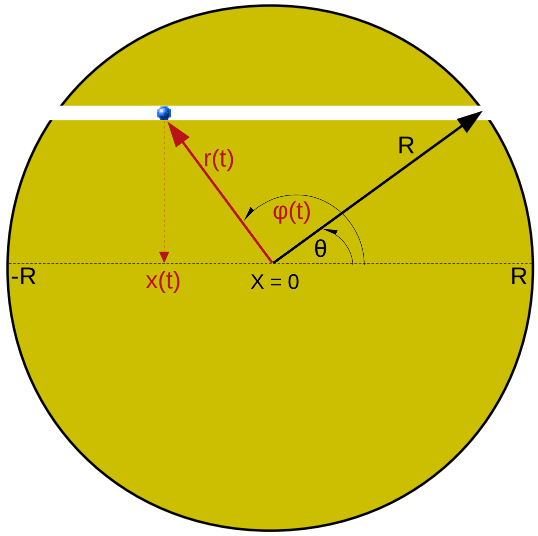
and it will be convenient to define

$$k \equiv \frac{G\,M}{R^3}$$

so that

$$F_{in} = -k\,m\,r$$

with the understanding that r, and thus F_{in}, is directed toward the planet center. Thus we see that the gravitational force within a spherical planet of uniform density varies directly (linearly) with the distance from the planet center. This also means that the net force exactly at the center is zero.



Next, consider an airless, frictionless tunnel drilled completely through the planet, not necessarily passing through the planet center; see the figure above. Since the sphere is symmetric from any point of view, we may just as well place the sphere and the tunnel in a configuration like the standard Cartesian coordinate system, with x the horizontal axis, and the sphere centered at the origin. *We can then rotate the sphere as we wish, and nothing in the physics we are about to discuss will change*.

The tunnel start point on the circle representing the sphere is defined with the angle θ, counter-clockwise in the usual manner from the positive x-axis. This angle ranges from zero, when the tunnel is a diameter of the sphere, to just less than 90 degrees, when the tunnel will be very short. (At exactly 90 degrees the tunnel length is zero.) The distance from the origin, the center of the planet, to the tunnel opening is of course just the radius R of the planet. The tunnel is parallel to the x-axis, and its end point is where this horizontal line intersects the circle. Thus, in 2D, the tunnel is a chord of the circle.

We will be placing an object in the tunnel, to observe its motion, if any. The current position of this object is defined by the time-dependent angle φ, also measured in the usual manner. The object's initial position will often be at the tunnel opening, but it may be elsewhere in the tunnel. The extrema of the object's x-coordinate are defined by

$$\pm\, R\cos(\theta)$$

while its time-dependent position can be seen from the figure to be

$$x_t = r_t \cos\left(\varphi_t\right)$$

To keep the object in the tunnel, φ must be greater than or equal to θ, and less than or equal to π - θ.

Above, we found that the gravitational force F_{in} on the object inside the sphere ("planet") varies linearly with the distance r from the center, and this force is directed toward the center. But what will affect the motion of the object is *the component of this force directed along the tunnel's longitudinal axis*. The component of F_{in} directed *into* the tunnel, perpendicular to the longitudinal axis, will have no effect since the tunnel is frictionless. For the component of the gravitational force acting *along* the tunnel axis, examination of the figure will show that we have

$$F_x = F_{in} \cos\left(\varphi_t\right) = -\, k\, m\, r_t \cos\left(\varphi_t\right)$$

Note that when the object has a positive *x*-coordinate, φ is in the first quadrant, the cosine is positive, and the force is negative (directed to the left, toward the center of the tunnel). When the object has a negative *x*-coordinate, φ is in the second quadrant, the cosine is negative, and the force is positive (directed to the right, toward the center of the tunnel).

Next we recognize that this last expression is just

$$F_x = -k\, m\, x_t$$

and, given this force acting along the tunnel axis, we can use Newton's Second Law to write

$$F_x = m\, a_x = m \frac{d^2 x_t}{dt^2} = -k\, m\, x_t$$

or

$$\frac{d^2 x_t}{dt^2} + k\, x_t = 0; \qquad k \equiv \frac{GM}{R^3}$$

But this is just the ordinary differential equation for simple harmonic motion (SHM); see the PDF on that topic for details. There, it is shown that the time-dependent position of the object will be

$$x_t = x_0 \cos(\omega t) \qquad \omega = \sqrt{k}$$

where x_0 is the initial position of the object; the motion is between $\pm x_0$ and has a period *T*, which is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k}} = \frac{2\pi}{\sqrt{\dfrac{GM}{R^3}}} = \frac{2\pi}{\sqrt{GM}} R^{\frac{3}{2}} \tag{3}$$

and this is, for Earth, 84.3 minutes. Note that *this period of motion does not depend on the initial position of the object, nor on the length of the tunnel.* This surprising fact means that the object's oscillation will take the same time whether the motion covers the tunnel's maximum length (a diameter), or is very short (e.g., when θ is near 90 degrees, or when x_0 is near $x = 0$).

Finally, consider the motion of a satellite in a circular, unrealistically low orbit around this planet. We will take the orbital altitude to be (essentially) zero, that is, just a bit above the radial distance *R* from the planet center. Then we have the condition for orbit, equating the magnitudes of the centripetal and gravitational forces:

$$\frac{m v^2}{R} = \frac{GM}{R^2} m \quad \Rightarrow \quad v^2 = \frac{GM}{R}$$

It is also the case that

$$v = \frac{2\pi R}{T} \quad \Rightarrow \quad v^2 = \frac{4\pi^2 R^2}{T^2} = \frac{GM}{R}$$

from which we find

$$T = \frac{2\pi}{\sqrt{G M}} R^{\frac{3}{2}} \tag{4}$$

which is a way of stating Kepler's Third Law, and which is *exactly the same period of motion* that we found in Eq(3) for the SHM oscillating object inside the planet!