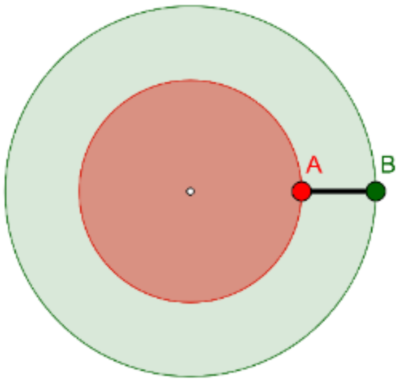
THE TWO LOVERS RACE….

To introduce the topic of conjunctions, oppositions and retrograde motion of planets, i really like starting by telling my students a love story.

Consider two lovers: Alice and Bob. They are running along two circles with the same center but different radius.

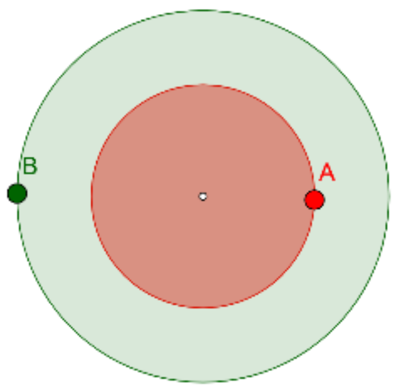


Alice is running along the inside circle and her frequency is twice that of Bob.

At time t=0 Alice and Bob are aligned with the circles center, and they will run counterclockwise.

Alice, running twice frequency, reaches the end of her first-round while Bob, at the same time, runs only a half of his round.

Thus, Alice does not find Bob where he was…



Alice's path

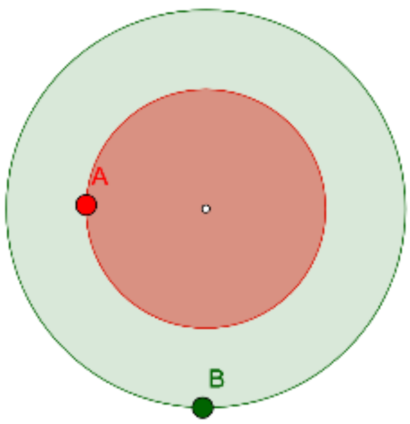
$$\textbf{length} = 2\pi r_A$$

$$\textbf{duration} = T_A$$

Alice realizes that Bob is half a round ahead of her.

Going on with the race, Alice reaches the end of her next half round, while Bob, at the same time, runs a quarter of his round.

Once again, Alice does not find Bob where he was…



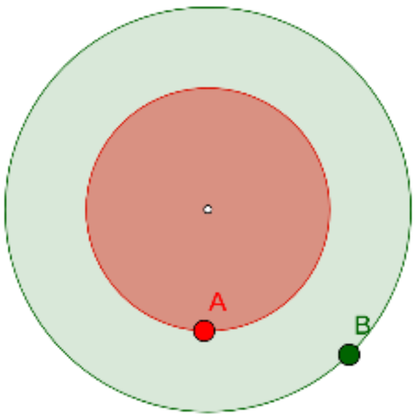
Alice's path

$$\textbf{length} = 2\pi r_A\left(1+\frac{1}{2}\right)$$

$$\textbf{duration} = T_A\left(1+\frac{1}{2}\right)$$

Alice realizes that Bob is 1/4 of a round ahead of her.

Now, I think you already figured out how the race is going on… poor Alice.

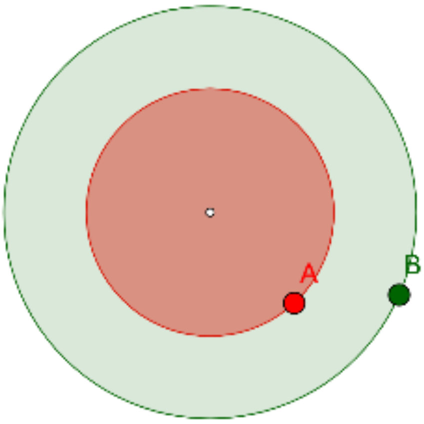


Alice's path

$$\textbf{length} = 2\pi r_A\left(1+\frac{1}{2}+\frac{1}{4}\right)$$

$$\textbf{duration} = T_A\left(1+\frac{1}{2}+\frac{1}{4}\right)$$

Alice realizes that Bob is no longer where he was, now he is 1/8 of a round ahead of her.



Alice's path

$$\text{length} = 2\pi r_A \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)$$

$$\text{duration} = T_A \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)$$

Alice realizes that Bob is no longer where he was, now he is 1/16 of round ahead of her.

Alice can now see Bob at a very short distant ahead of her….

At each race step, Alice is getting closer to Bob…. But, Her path length and her time duration needed to reach Bob are increasing more and more… with no end ….

$$\text{length} = 2\pi r_A \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$$

$$\text{duration} = T_A \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$$

If we were ancient Greek philosophers, because of the infinite number of the race steps (because of the infinite number of addends into the brackets), we would have concluded the poor Alice will never meet again her lovely Bob… Although she will always be closer to him (what a terrible fate….)

But we are modern mathematicians, and we conclude that Alice can reach her Bob…

Today we know that, despite the infinite number of addends, sometimes the sum into the brackets can be a finite number. And this is one of those cases: the geometrical series….

$$\left(1-\frac{1}{2^{n+1}}\right)=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}\right)$$

it is a famous identity you learned at Algebra's course from which it is easy to get

$$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}\right)=2\left(1-\frac{1}{2^{n+1}}\right)$$

The quantity $\left(1-\frac{1}{2^{n+1}}\right)$ gets closer and closer to 1 as n gets bigger and bigger…

In that sense we state

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}\ \rightarrow\ 2$$

as n goes to infinity

Don't worry little Alice, you will reach once again your Bob.

Alice's total length to reach Bob is $2\pi r_A\,2=4\pi r_A$ (two rounds long) and total time duration is $2T_A$

In general, if Alice is running along her circle with a frequency $f_A > f_B$ then the path length and time duration needed to reach Bob are:

$$\textbf{total length} = 2\pi r_A \frac{1}{1-\frac{f_B}{f_A}} = 2\pi r_A \frac{f_A}{f_A - f_B}$$

$$\textbf{total duration} = T_A \frac{f_A}{f_A - f_B} = \frac{1}{f_A - f_B} = \frac{T_A T_B}{T_B - T_A}$$

If we assume that Alice's frequency is bigger than Bob's one, the event:

"**Alice and Bob are aligned on the same side** with respect to the center"

is a periodic event with Period

$$T = \frac{1}{f_A - f_B} = \frac{T_A T_B}{T_B - T_A}$$

And to reach Bob, Alice rotates by an angle:

$$\alpha = 2\pi \frac{f_A}{f_A - f_B} > 2\pi$$

I told you the story of two.

When you watch to Mars in the night sky you can see Mars do the same kind of race.

Alice is none other than the Earth along her orbit around the Sun and Bob is none other than Mars along his orbit around the Sun.

In this case $f_{Earth} = 1\frac{1}{year}$ and $f_{Mars} = 0.5316\frac{1}{year}$ is not exactly the half of Earth frequency as in the case of Bob and Alice…

Thus, the Earth and Mars will be aligned on the same side with respect to the Sun every 2.13 years that is about every 2 years and 47 days.

Furthermore, the paths of Earth and Mars around the Sun have an elliptical shape, they are not circular shape. In first approximation, we can assume that paths to be circles because the eccentricities are very close to zero and assume the velocity value to be constant to its mean value.

It was well known To The ancient astronomers The empirical fact that every two years the Earth “meets” her lovely Mars… and Mars, for the happiness, makes in the sky a beautiful dance. The astronomers called this kind of event “conjunction” and the beautiful dance “retrograde motion”.

Now is the time you watch the following Geogebra app I wrote for this topic….