Grade: 9-12 (AS)

Duration: 30 min

Tools: one Logifaces Set / 2-3 pairs or 4-6 students

Individual / Pair work

Keywords: Cosine, Formula, Proof, Heron's formula

538 - Ratio of Areas

MATHS / TRIGONOMETRY

LOGIFACES
METHODOLOGY
Erasmus+

TEACHER

Logifaces

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DESCRIPTION

In the 9 pcs or 16 pcs Set students choose those blocks that have two vertical edges with the same height and one with different height. These are blocks 112, 113, 122, 133, 223 and 233.
They denote the area of the base triangle by A_b and the area of the top triangle by A_t. The following connection holds between the angle between the planes of the top and base triangles (α) and the areas A_b and A_t: $cos(\alpha) = \frac{A_b}{A_t}$.

LEVEL 1 Students use this formula to complete the table below. Enter the data in the table by measurement or calculation, and then use the formula above to calculate the angle in the last column (with grey background).

To calculate the area of the triangles use Heron's formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the edges of the triangle and $s = \frac{a+b+c}{2}$.

Block	**Base triangle** ($a = b = c = 4$)	**Top triangle**					**Angle of the planes** α
	A_b	a	b	c	s	A_t	
112							
113							
122							
133							
223							
233							

LEVEL 2 Students prove the formula $cos(\alpha) = \frac{A_b}{A_t}$.

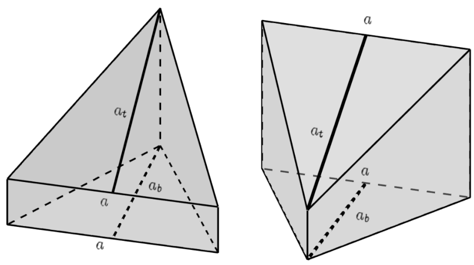
Hint: Use the results of [537 - Ratio of Heights] and the fact that both triangles have an edge that is parallel to the common line of the two planes and the heights a_b and a_t are perpendicular to that edge. In fact, the proof works for any triangle with this property.

SOLUTIONS / EXAMPLES

LEVEL 1 The solutions are given in standard units, but can also be calculated with real lengths (see exercise 404 - Top Edges for the calculated edge lengths and exercise 413 - Area with Heron's Formula for the areas calculated using Heron's formula). The results in the last column of the table are the same in both cases.

Block	**Base triangle** $(a = b = c = 4)$	**Top triangle**					**Angle of the planes** α
	A_b	a	b	c	s	A_t	
112	$4\sqrt{3}$	4	$\sqrt{17}$	$\sqrt{17}$	$2+\sqrt{17}$	$2\sqrt{13}$	$\alpha \approx 16°$
113	$4\sqrt{3}$	4	$\sqrt{20}$	$\sqrt{20}$	$2+\sqrt{20}$	8	$\alpha = 30°$
122	$4\sqrt{3}$	4	$\sqrt{17}$	$\sqrt{17}$	$2+\sqrt{17}$	$2\sqrt{13}$	$\alpha \approx 16°$
133	$4\sqrt{3}$	4	$\sqrt{20}$	$\sqrt{20}$	$2+\sqrt{20}$	8	$\alpha = 30°$
223	$4\sqrt{3}$	4	$\sqrt{17}$	$\sqrt{17}$	$2+\sqrt{17}$	$2\sqrt{13}$	$\alpha \approx 16°$
233	$4\sqrt{3}$	4	$\sqrt{17}$	$\sqrt{17}$	$2+\sqrt{17}$	$2\sqrt{13}$	$\alpha \approx 16°$

LEVEL 2 We use the results of exercise 537 - Ratio of Heights where in Level 2 we proved the following. Denote the altitude of the base triangle by a_b and the altitude of the top triangle starting from the vertex of the different height by a_t. Then the following connection holds between the angle between the planes of the top and base triangles (α) and the altitudes a_b and a_t: $cos(\alpha) = \frac{a_b}{a_t}$.



The edges of the base and top triangles perpendicular to the altitudes a_b and a_t have equal length. Denote this length by a. It follows that the proportion of the triangle areas depends only on the proportion of their altitudes. This gives the formula $\frac{A_b}{A_t} = \frac{\frac{a \times a_b}{2}}{\frac{a \times a_t}{2}} = \frac{a_b}{a_t} = cos(\alpha)$, as required.

PRIOR KNOWLEDGE

Angle between two planes, Area of triangle, Trigonometric ratios

RECOMMENDATIONS / COMMENTS

The exercise is suitable for differentiation, as proving the formula is more difficult than applying it.

The following exercises are recommended before this exercise: exercises 404 - Top Edges for calculating the edge lengths, exercise 413 - Area with Heron's Formula for calculating the areas of the triangles using Heron's formula, 537 - Ratio of Heights for another way to calculate the angle between the planes and for the proof of the formula.

While discussing the proof, it is worth drawing students' attention to the fact that there is more than one way to determine the area of a triangle.

Exercise 536 - Different Slopes is recommended before this exercise in order to clarify the concept of the angle between two planes and the difficulty of measuring this angle.
The calculations can be verified using GeoGebra, see exercise 528 - Read the Results in GeoGebra.