AREA OF INTERSECTING CIRCLES

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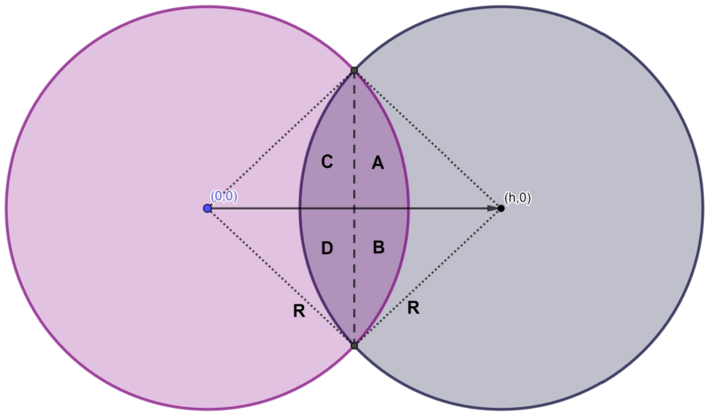

On the x-axis, two circles of equal radius R intersect. The leftmost circle is fixed, with its center at the origin. The second circle can move along the axis, with its center at (h,0). The position parameter h varies from zero to $2R$; if h is any larger than $2R$, there will be no intersection. Intuitively, it is clear that the intersection area (IA) can vary from zero (at $h = 2R$) to the full area of the circles (at $h = 0$).

First we need to find the x-coordinate of the intersection points. To do this, we have the equations for the two circles:

$$x^2 + y^2 = R^2 \qquad (x - h)^2 + y^2 = R^2$$

Then we equate the y-values, since they by definition are equal at the intersection points

$$y^2 = R^2 - x_I^{\,2} = R^2 - (x_I - h)^2$$

and this leads to

$$x_I = \frac{h}{2}$$

Next, we recognize that the IA consists of segments of the two circles, each segment defined by a line (a chord of the circle; dashed line, above) passing vertically through the intersection points, hitting the x-axis at x_I. If we can find the area of one segment, the total IA will be twice this, since the circles are the same size. In the diagram, regions A and B together comprise a segment of the left circle, and similarly for regions C and D for the moving circle. The sum of these is of course the IA.

To find the segment area, we integrate its top half, and will later double that result. The integral for the top half of the fixed circle's segment (region A) is

$$\int_{x_I}^{R} \sqrt{R^2 - x^2}\; dx$$

Using a trig substitution, let $x = R\cos(t)$; then $dx = -R\sin(t)\,dt$. We must also change the integration limits: when $x = R$, $t = 0$; when $x = x_I$, $t = \arccos(h/2R)$. With this our new integral is

$$\int_{\arccos\left(\frac{h}{2R}\right)}^{0} R\sin(t)\,\left[-R\sin(t)\,dt\right] \;=\; R^2\int_{0}^{\arccos\left(\frac{h}{2R}\right)} \sin^2(t)\,dt$$

This integral can be evaluated by parts, or we can look up the result in a table of integrals:

$$R^2\left[\frac{t}{2} \;-\; \frac{1}{4}\sin(2t)\right]_0^{\arccos\left(\frac{h}{2R}\right)} \;=\; R^2\left[\frac{1}{2}\arccos\left(\frac{h}{2R}\right) \;-\; \frac{1}{4}\sin\left(2\arccos\left(\frac{h}{2R}\right)\right)\right]$$

It will be convenient now to apply the multiplier of two, to account for the lower half of the segment (region B), and also to define the angle

$$\theta \equiv \arccos\left(\frac{h}{2R}\right)$$

Then the segment area is

$$A = R^2\arccos\left(\frac{h}{2R}\right) \;-\; \frac{R^2}{2}\sin(2\theta)$$

or

$$A = R^2\arccos\left(\frac{h}{2R}\right) - R^2\sin(\theta)\cos(\theta)$$

which is

$$A \;=\; R^2\arccos\left(\frac{h}{2R}\right) \;-\; \frac{R^2 h}{2R}\sin\left(\arccos\left(\frac{h}{2R}\right)\right)$$

and this will reduce to

$$A \;=\; R^2\arccos\left(\frac{h}{2R}\right) \;-\; \frac{hR}{2}\sqrt{1 \;-\; \left(\frac{h}{2R}\right)^2}$$

Doubling this to get the entire IA, we have the final result

$$A_{Total} \;=\; 2R^2\arccos\left(\frac{h}{2R}\right) \;-\; hR\sqrt{1-\left(\frac{h}{2R}\right)^2} \qquad (1)$$

A series expansion of this, to low order, will give the approximation

$$A_{Total} \;\approx\; \pi R^2 - 2hR \;+\; \frac{h^3}{12R} \qquad (2)$$

The next step is to make a new independent variable, where we will re-use x to now be $x = h/2R$. The purpose is to make the displacement range from 0 to 1, and we will also scale the IA to its maximum value, the area of the circle(s). This will give

$$A_x = 2R^2\left[\;\arccos(x) \;-\; x\sqrt{1-x^2}\;\right] \qquad (3)$$

and its series approximation

$$A_x \approx \pi R^2 \; - \; 4\,x\,R^2 \; + \frac{2}{3} R^2\,x^3 \; = \; R^2 \left[\pi \; - 4x \; + \frac{2}{3} x^3 \right] \tag{4}$$

and then the scaled IA will be

$$AreaRatio = \frac{2R^2 \left[\arccos(x) \; - \; x\sqrt{1 - x^2} \; \right]}{\pi R^2} \; = \frac{2}{\pi} \left[\arccos(x) \; - \; x\sqrt{1 - x^2} \; \right] \tag{5}$$

or approximately

$$AreaRatio \; \approx \; \frac{R^2 \left[\pi \; - 4x \; + \frac{2}{3} x^3 \right]}{\pi R^2} = 1 \; - \; \frac{4}{\pi} x \; + \; \frac{2}{3\pi} x^3 \tag{6}$$

It will be of interest to plot these last two functions, to see how the scaled area varies with the scaled displacement of the second circle. For Eq(6) we have another option, by dropping the cubic term; this gives a simple linear approximation for the scaled area. Figures 1 and 2 show respectively the functions and fractional error for the approximations.

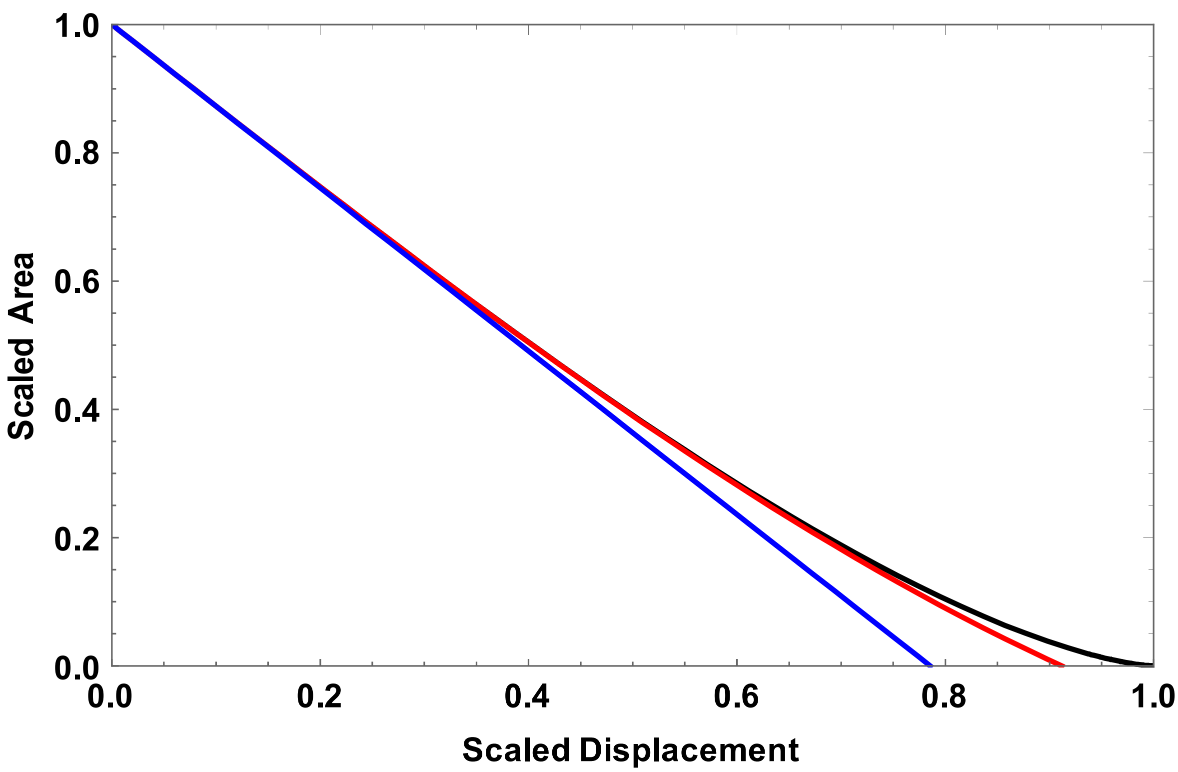

Figure 1. Plot of the exact area ratio Eq(5) (black) and the linear (Eq(6) without the cubic term, blue) and cubic series approximation Eq(6) (red).

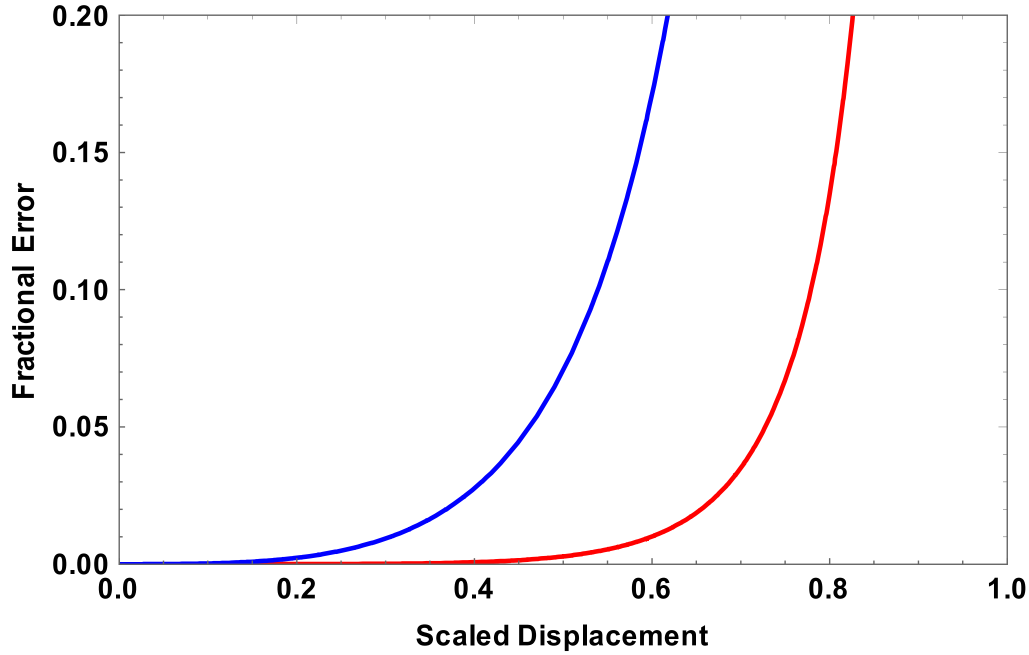

Figure 2. Plot of the fractional error in the approximate areas. Blue, linear; red, cubic. The linear is no worse than 10% for displacements up to about 0.5, while the cubic is the same out to about 0.75. The linear is very good (1% or better) out to about 0.3, and about 0.6 for the cubic.

To verify that the above calculations are correct, a Monte Carlo simulation was done to estimate the intersection area. The procedure was: (1) uniformly randomly sample a known rectangular region that encloses the subset region of interest (ROI); (2) find and keep count of the number of x,y pairs that fall within the ROI; (3) find the ratio of the number of hits to the number of trials; (4) multiply that fraction by the known area of the rectangular region. This was coded in MATLAB, and the figure below shows the results of a run of 10000 trials. In repeated runs, the Monte Carlo area consistently agreed (within expected variability) with the area as calculated by Eq(1) above.