# Properties of Mathematical Sentences

"**Arithmetic is the Queen of Mathematics.**" Ones knowledge of arithmetic properties are mandatory to understanding Algebra, Geometry, and other fields of mathematics. The **Properties of Equality** are used to solve most equations. The following chart summarizes the Field Properties and Properties of Equality and Inequality. These properties apply to all real numbers: $a$, $b$, and $c$. They are used to solve arithmetic, algebraic, and most other math problems. {The underlined properties are central to doing Algebraic operations.}

## Field Properties

| **Property**<br>**Adjective Form**<br>**(Verb Form)** | **Addition** | **Multiplication** |
|---|---|---|
| **Associative**<br>**(associate)** | $(a + b) + c = a + (b + c)$ | $(ab)c = a(bc)$ |
| **Commutative**<br>**(commute)** | $a + b = b + a$ | $ab = ba$ |
| **Identity**<br>**(identify)** | $a + 0 = a = 0 + a$ | $a \cdot 1 = a = 1 \cdot a$ |
| **Inverse** | $a + (-a) = 0 = (-a) + a$ | $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a, if\ a \neq 0$ |
| **Inverse Applications** | $a - b = a + (-b)$<br>Directed Number Addition | $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$<br>Reciprocal Multiplication for Division |
| **Distributive**<br>**(distribute)** | $a(b + c) = ab + ac$ and $ab + ac = a(b + c)$<br>$a(b - c) = ab - ac$ and $ab - ac = a(b - c)$ | |

## Properties of Equality and Inequality

| **Property** | **Equality** | **Inequality** |
|---|---|---|
| **Multiplicative Property of Zero** | $a \cdot 0 = 0 = 0 \cdot a$ | |
| **Reflexive** | $a = a$ | |
| **Symmetric** | If $a = b$, then $b = a$. | |
| **Zero Product** | If $ab = 0$, then $a = 0$ or $b = 0$. | |
| **Transitive** | If $a = b$ and $b = c$, then $a = c$. | If a > b and b > c, then a > c.<br>If a < b and b < c, then a < c. |
| **Addition** | If $a = b$, then $a + c = b + c$. | If $a < b$, then $a + c < b + c$.<br>If $a > b$, then $a + c > b + c$. |
| **Subtraction** | If $a = b$, then $a - c = b - c$. | If $a < b$, then $a - c < b - c$.<br>If $a > b$, then $a - c > b - c$. |
| **Multiplication** | If $a = b$, then $ac = bc$. | If $a < b$ and $c > 0$, then $ac < bc$.<br>If $a < b$ and $c < 0$, then $ac > bc$.<br>If $a > b$ and $c > 0$, then $ac > bc$.<br>If $a > b$ and $c < 0$, then $ac < bc$. |
| **Division** | If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$. | If $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$.<br>If $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$.<br>If $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$.<br>If $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$. |
| **Substitution** | If $a = b$, then $b$ can be substituted for $a$ in any equation or inequality. | |

Many common errors are made by students with sign number arithmetic and inequality operations using signed numbers.

# Properties of Equality

**One-step Problems**

**Addition Properly of Equality**

The ***addition property of equality*** tells us that adding the same number to each side of an equation gives us an equivalent equation.
*If a = b, then a + c = b + c.*

$$x - 5 = 6$$
$$x - 5 + 5 = 6 + 5$$
$$x + 0 = 11$$
$$x = 11$$

**Subtraction Property of Equality**

The ***subtraction property of equality*** tells us that subtracting the same number to each side of an equation gives us an equivalent equation.
*If a = b, then a - c = b - c.*

$$x + 5 = 6$$
$$x + 5 - 5 = 6 - 5$$
$$x + 0 = 1$$
$$x = 1$$

**Multiplication Property of Equality**

The ***multiplication property of equality*** tells us that multiplying the same number to each side of an equation gives us an equivalent equation.
*If a = b, then ac = bc.*
*If* $\frac{a}{b} = c$, *and* $b \neq 0$, *then* $\frac{a}{b} \cdot b = c \cdot b$, *or* $a = cb$

$$\frac{x}{3} = 6$$
$$3 \times \frac{x}{3} = 3 \times 6$$
or $$\frac{3x}{3} = 3 \times 6$$
$$x = 18$$

**Division Property of Equality** says
The ***division property of equality*** tells us that dividing the same number to each side of an equation gives us an equivalent equation.
*If* $a = b$ *and* $c \neq 0$, *then* $\frac{a}{c} = \frac{b}{c}$.

$$4x = 16$$
$$\frac{4x}{4} = \frac{16}{4}$$
$$x = 4$$

The above example are done with the constant terms of the equation; however, the One-Step Problems can be working with variables being the terms added or subtracted. Examples when multiplied or divided are rare at the HSE level.

$$2x = x + 5$$
$$2x – x = x + 5 – x$$
$$x = 5$$

$$-2x = 9 – 3x$$
$$-2x + 3x = 9 – 3x + 3x$$
$$x = 9$$

There could be a **General Property of Equality**

Each ***property of equality*** tells us that whatever operation you perform on one side of an equation you must perform on the other side of the equation. The result is an equivalent equation. This extends to all mathematical operations performed on both sides of the equation. {Whatever you do to the left-side of the equation symbol, you must do to the right-side of the equation symbol and vice versa.}

**Steps for solving equations/inequalities**:

1) Always! **Simplify when possible** each side! (Simplifying is needed before and after each algebraic step.)
2) ***Add/Subtract*** *the same value from both sides* if needed and *simplify* again. Repeat until no addition/subtraction exist.
   a. If there are variables on both side, Add/Subtract so result has a positive coefficient.
   b. Add/Subtract constant terms to isolate the variable on one side.
3) ***Multiply/Divide*** *both sides* by the same value and simplify. Repeat until no multiplication/division exist.
4) ***Exponential/Roots***
5) ***Grouping Symbols***

The **<u>Substitution Property</u>**

The ***substitution property*** states that if two quantities are equal to each other, then one can be substituted for the other in an equation or inequality without changing the solution.

*If a = b, then b can be substituted for a in any equation or inequality.*

This example is a demonstration of how we can check a solution to an algebra problem we just solve.

The substitution can be with equations. This will be shown later when we work with solving linear systems of equations.

Given $\begin{cases} y = 2x + 11 \\ x = 5 \end{cases}$

**Substitute**:

$$y = 2x + 11$$
$$y = 2(\mathbf{5}) + 11$$
$$y = 10 + 11$$
$$y = 21$$

## Two-Step Equalities

Two-Step Equalities just mean you will be doing two operations from the four methods used above to solve equations. Simplifying either side of an equation is an arithmetic operation, two-step equation mean you will be using two or more operations on both sides of the equation (inequality) one-step at a time.

Identify each of the steps for following examples:

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______________

$$2x - 3 = x - 5$$
$$2x - 3 - x = x - 5 - x$$
$$x - 3 = -5$$
$$x - 3 + 3 = -5 + 3$$
$$x = -2$$

______________

______________

$$2x + 3 = x - 5$$
$$2x + 3 - x = x - 5 - x$$
$$x + 3 = -5$$
$$x + 3 - 3 = -5 - 3$$
$$x = -8$$

______________

______________

$$-3x + 3 = 24$$
$$-3x + 3 - 3 = 24 - 3$$
$$-3x = 21$$
$$\frac{-3}{-3}x = \frac{21}{-3}$$
$$x = -7$$

______________

______________

$$\frac{x}{-7} - 6 = -3$$
$$\frac{x}{-7} - 6 + 6 = -3 + 6$$
$$-7 \times \frac{x}{-7} = -7 \times 3$$
$$x = -21$$

# Properties of Inequality

The Properties of Inequalities

The ***property of inequality*** tells us that whatever operation you perform on one side of an inequality you must perform on the other side of the equality results in an equivalent inequality. Many times we show inequality solutions with a number line graph for better understanding.

**Major caveat**, on inequalities you must take extra care when working with **negative values**. Wherever, you do the multiplication/division operations, the direction of the inequality must change to its opposite direction on that line.

Addition Properly of Inequality

The ***addition property of inequality*** tells us that adding the same number to each side of an inequality gives us an equivalent inequality.

If $a < b$, then $a + c < b + c$.

If $a > b$, then $a + c > b + c$.

$$x - 5 < 6$$
$$x - 5 + 5 < 6 + 5$$
$$x + 0 < 11$$
$$x < 11$$

Subtraction Property of Inequality

The ***subtraction property of inequality*** tells us that subtracting the same number to each side of an inequality gives us an equivalent inequality.

*If $a < b$, then $a - c < b - c$.*

*If $a > b$, then $a - c > b - c$.*

$$x + 5 > 6$$
$$x + 5 - 5 > 6 - 5$$
$$x + 0 > 1$$
$$x > 1$$

Multiplication Property of Inequality

The ***multiplication property of inequality*** tells us that multiplying the same positive number to each side of an inequality gives us an equivalent inequality.

*If $a < b$ and $c > 0$, then $ac < bc$.*

*If $a < b$ and $c < 0$, then $ac > bc$.* **Required inequality change!**

*If $a > b$ and $c > 0$, then $ac > bc$.*

*If $a > b$ and $c < 0$, then $ac < bc$.* **Required inequality change!**

$$\frac{x}{3} < 6$$
$$3 \times \frac{x}{3} < 3 \times 6$$
or $$\frac{\cancel{3}x}{\cancel{3}} < 3 \times 6$$
$$x < 18$$

$$\frac{-x}{3} < 6$$
$$-3 \times \frac{-x}{3} > -3 \times 6$$
or $$\frac{-1\times\cancel{3}(-x)}{\cancel{3}} > -3 \times 6$$
$$x > -18$$

Notice the required inequality sign change.

Division Property of Inequality says

The ***division property of inequality*** tells us that dividing by the same positive number to each side of an inequality gives us an equivalent inequality.

If $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$.

If $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$. **Required inequality change!**

If $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$.

If $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$. **Required inequality change!**

$$4x > 16$$
$$\frac{\cancel{4}x}{4} > \frac{16}{4}$$
$$x > 4$$

$$-4x > 16$$
$$\frac{\cancel{-4}x}{\cancel{-4}} < \frac{16}{-4}$$
$$x < -4$$

Notice the required inequality sign change.

The **Properties of Inequality** Two Step Problems

The ***property of inequality*** tells us that whatever operation you perform on one side of an inequality you must perform on the other side of the inequality results in an equivalent inequality.

**Positive** Multiplication/Division Examples (Fractional Coefficient):

| | Multiply then Divide (2-steps) | Using a **reciprocal** (1-step) |
|---|---|---|
| If you have a fraction as a coefficient of the variable, there are two methods to solve the problem. It is your choice as to which method you choose. | $\frac{3}{7}x < 21$<br>$7 \times \frac{3}{7}x < 7 \times 21$<br>$\frac{3}{3}x < \frac{147}{3}$<br>$x < 49$ | $\frac{3}{7}x < 21$<br>$\frac{7}{3} \times \frac{3}{7}x < \frac{7}{3} \times 21$<br>$x < 49$ |

**Negative** Multiplication/Division Examples (Fractional Coefficient):

| | Multiply then Divide (2-steps) | Using a **reciprocal** (1-step) |
|---|---|---|
| If you have a fraction as a coefficient of the variable, there are two methods to solve the problem. It is your choice as to which method you choose.<br>Notice change in relation symbol when operating on both sides with a negative value. | $-\frac{3}{8}x > 24$<br>$8 \times \frac{-3}{8}x > 7 \times 24$<br>$\frac{-3}{-3}x < \frac{168}{-3}$<br>$x < -56$ | $-\frac{3}{8}x > 24$<br>$\left(-\frac{8}{3}\right) \times \left(-\frac{3}{8}x\right) < \left(-\frac{8}{3}\right) \times 24$<br>$x < -56$ |

Two-Step Example: Subtraction/Division

| | Traditional | Using signed numbers Subtraction |
|---|---|---|
| Adding -6 is the same as subtracting 6.<br><br>Note: Change in inequality! | $-7x + 6 \le 27$<br>$-7x + 6 - 6 \le 27 - 6$<br>$-7x \le 21$<br>$\frac{-7}{-7}x \ge \frac{21}{-7}$<br>$x \ge -3$ | $-7x + 6 \le 27$<br>$-7x + 6 + (-6) \le 27 + (-6)$<br>$-7x \le 21$<br>$\frac{-7}{-7}x \ge \frac{21}{-7}$<br>$x \ge -3$ |

a - b = a + (-b), Rule of Subtraction for signed numbers, change the sign of value subtracted and add values.

Examples… more

# Compound Inequalities

The <u>Properties of Inequalities</u>

The ***property of inequality*** instructs user perform the same operation on both sides of an inequality results in an equivalent inequality. With compound inequalities, this applies to all parts of the compound statement.

**Major caveat:** <u>With inequalities use extra care when working with **negative values**</u>. <u>If you use either *multiplication* or *division* operations, the direction of the inequality must change to its opposite direction on that line of work.</u>

**$-6 < 4x + 2 \leq 14$** is actually two inequalities connected by an **"and"**;
**$-6 < 4x + 2$ and $4x + 2 \leq 14$**, which can be solved independently or combined.
**$-6 < 4x + 2$ or $4x + 2 > 14$**, each part must be solved independently.

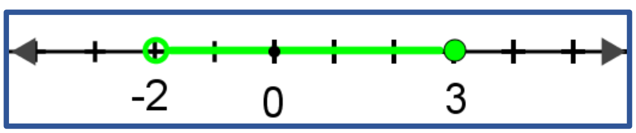
In solving in equations/inequalities, the first step is to *add/subtract the same value from both sides* if needed and simplify. Repeat until no addition/subtraction exist.

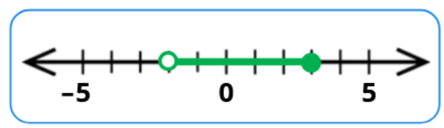
The second step is to *multiply/divide both sides* by the same value and simplify. Repeat until no more multiply/divide exist. <u>The caveat, on inequalities you must take extra care when working with **negative values**. Where you do that operation, the direction of the inequality must change to its opposite direction.</u>

Solving:

| | **$-6 < 4x + 2 \leq 14$** | |
|---|---|---|
| Rewrite | $-6 < 4x + 2$ and $4x + 2 \leq 14$ | Do NOT try until after a lot of examples. |
| Subtract 2 from both sides. | $-6 - 2 < 4x + 2 - 2$ and $4x + 2 - 2 \leq 14 - 2$ | $-6 - 2 < 4x + 2 - 2 \leq 14 - 2$ |
| | $-8 < 4x$ and $4x \leq 12$ | $-8 < 4x \leq 12$ |
| Divide both sides by 4. | $-\frac{8}{4} < \frac{4}{4}x$ and $\frac{4}{4}x \leq \frac{12}{4}$ | $-\frac{8}{4} < \frac{4}{4}x \leq \frac{12}{4}$ |
| | $-2 < x$ and $x \leq 3$ | |
| Recombine inequalities. | $-2 < x \leq 3$ | $-2 < x \leq 3$ |

Create the number line, where x is between -2 and 3, including 3 (in words).

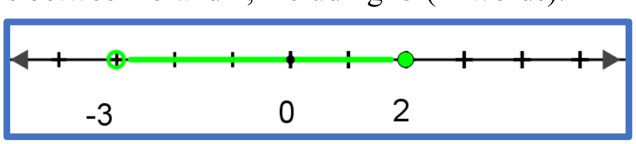



However, if coefficient of x is negative, we do the following: Do **NOT** try the shortcut.

| | **$-6 < -4x + 2 \leq 14$** | |
|---|---|---|
| Rewrite | $-6 < -4x + 2$ and $-4x + 2 \leq 14$ | Note: Subtracting two in each part |
| Subtract 2 from both sides. | $-6 - 2 < -4x + 2 - 2$ and $-4x + 2 - 2 \leq 14 - 2$ | $-6 - 2 < -4x + 2 - 2 \leq 14 - 2$ |
| | $-8 < -4x$ and $-4x \leq 12$ | $-8 < -4x \leq 12$ |
| Divide both sides by -4. | $\frac{-8}{-4} > \frac{-4}{-4}x$ and $\frac{-4}{-4}x \geq \frac{12}{-4}$ | $-\frac{8}{-4} > \frac{-4}{-4}x \geq \frac{12}{-4}$ Note: Change in inequality. |
| | $2 > x$ and $x \geq -3$ | $2 > x \geq -3$ |
| Rewrite inequalities from low-high. | $-3 < x \leq 2$ | $-3 \leq x < 2$ |

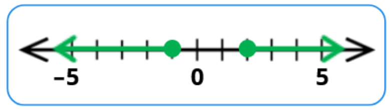
Create the number line, where x is between -3 and 2, including -3 (in words).



Compound Example with **“or”**

| | | | |
|---|---|---|---|
| | $-x \geq 1$ or $3x - 4 \geq 2$ | | |
| | There are two ways to do left side. | | |
| Solve each part of the “or” statement. | $-x \geq 1$ | $-x \geq 1$ | $3x - 4 \geq 2$ |
| Add a quantity to both sides. | | $-x + x \geq 1 + x$ | $3x - 4 + 4 \geq 2 + 4$ |
| Note: Inequality change in left column | $\frac{-x}{-1} \leq \frac{1}{-1}$ | $0 \geq 1 + x$ | $3x \geq 6$ |
| Subtract a quantity on both sides. | | $0 - 1 \geq 1 + x - 1$ | $\frac{3}{3}x \geq \frac{6}{3}$ |
| Divide by a quantity on both sides. | | $-1 \geq x$ | |
| | $x \leq -1$ | $x \leq -1$ | $x \geq 2$ |
| Rewrite the “or” statement. | $x \leq -1$ or $x \geq 2$ | | |

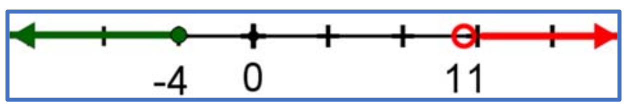
Graph the solution:




Recall: **a - b = a + (-b)**, the Rule of Subtraction for signed numbers.

| | | |
|---|---|---|
| | $-7x + 2 < -75$ or $-8x - 2 \geq 30$ | |
| | $-7x + 2 < -75$ | $-8x - 2 \geq 30$ |
| Add on both sides | $-7x + 2 + (-2) < -75 + (-2)$ | $-8x + (-2) + 2 \geq 30 + 2$ |
| | $-7x + 0 < -77$ | $-8x + 0 \geq 32$ |
| Simplify | $-7x < -77$ | $-8x \geq 32$ |
| Divide both sides by negatives, invert inequality | $\frac{-7x}{-7} > \frac{-77}{-7}$ | $\frac{-8x}{-8} \leq \frac{32}{-8}$ |
| Simplify | $x > 11$ | $x \leq -4$ |
| Rewrite “or” statement | $x \leq -4$ or $x > 11$ | |

Graph the inequality:




**2009 *Mathematics Standards of Learning***

Grade 3: Commutative Properties
Grade 4: Associative Properties
Grade 5: Distributive Property
Grade 6: Identity Properties, multiplicative property of 0, and inverse properties of multiplication
Grade 7: Apply all properties
Grade 8: Identify properties of operations used to solve an equation
Algebra: Solve multistep linear and quadratic equations in two variables…