S4 – MATHEMATICS (M2) – CHAPTER 3 Limit and the number *e*
WORKSHEET 01

Name: ___________________ Class: ___________ [No.: __] Date: ___________________

Learning Objective

1 Idea of limits

- To understand the concept of the limit of a function.
- To learn finding the limit of a function by using the graph of the function.
- To recognize the definitions of intervals and absolute values.
- To understand the concept of continuous and discontinuous functions.

2 Limit of functions at a certain value

- To learn finding the limit at a certain value of a rational function and a function involving surds.
- To learn various theorems on limits of sum, difference, product, quotient, scalar multiple, composite functions and use them to find the limit.

1. Idea of limits

Consider the function $f(x)=\dfrac{x^3-8}{x-2}$. $f(x)$ is clearly undefined when $x=2$.
We want to observe the **behavior of $f(x)$ when *x* is close to 2** .

Task 1. Completing the following table. Give your answers corrected to 4 decimal places.

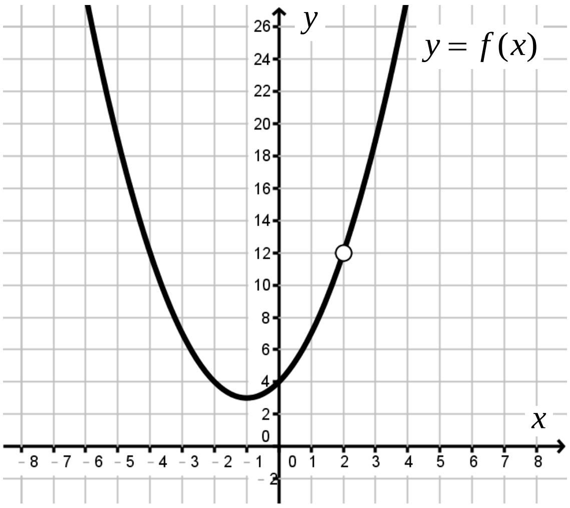
x	1	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1	3
$f(x)$						╳					

From the above result, when *x* moves closer and closer to __2__ , $f(x)$ gets closer and closer to __________.
We say "the limit of $f(x)$ is 12 when *x* tends to 2" and denote as $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}=12$.

> For a function $f(x)$, if the function value $f(x)$ gets closer and closer to a number *L* when *x* moves closer and closer to *a* (but $x\neq a$) , we say the function $f(x)$ has a limit *L* when *x* approaches to *a* .
> In symbol, we write $\lim\limits_{x\to a} f(x)=L$ or $f(x)\to L$ as $x\to a$.

Note: When we are talking about the limit of a function, $\lim\limits_{x\to a} f(x)$, we assume that $x\neq a$.

Task 2. Let $f(x)=\dfrac{x^3-8}{x-2}$. The figure shows the graph of $y=f(x)$. (The white dot ○ at (2 , 12) means that the point is excluded from the graph.)

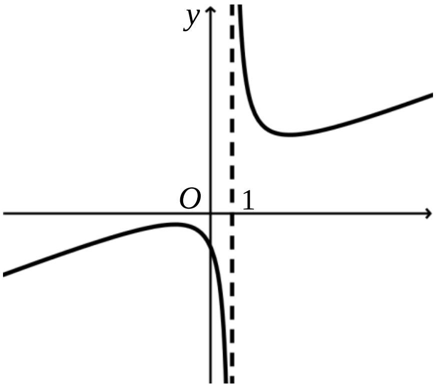


Determine whether $\lim\limits_{x\to 0} f(x)$ exists. Find the limit if it exists.

Conclusion: $\lim\limits_{x\to 0} f(x)$

Task 3. In each of the following, determine whether the limit exists. Find the limit if it exists.

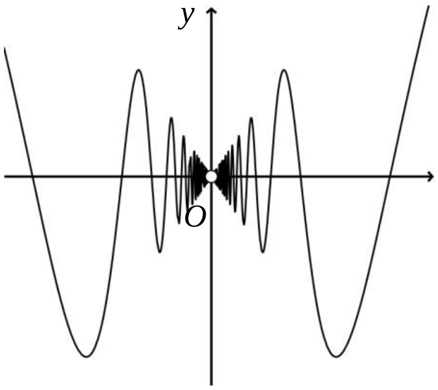
(a)



$y = \dfrac{x^2 + 2x + 4}{x - 1}$

$\lim_{x \to 1} \dfrac{x^2 + 2x + 4}{x - 1}$

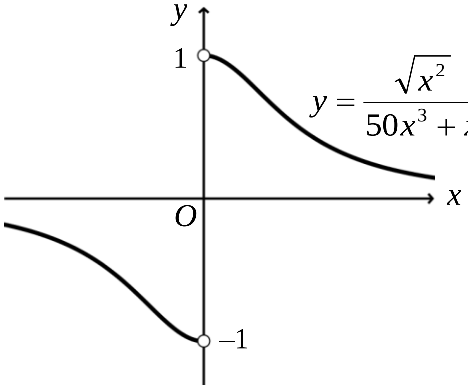
(b)



$y = x \sin \dfrac{1}{x}$

$\lim_{x \to 0} x \sin \dfrac{1}{x}$

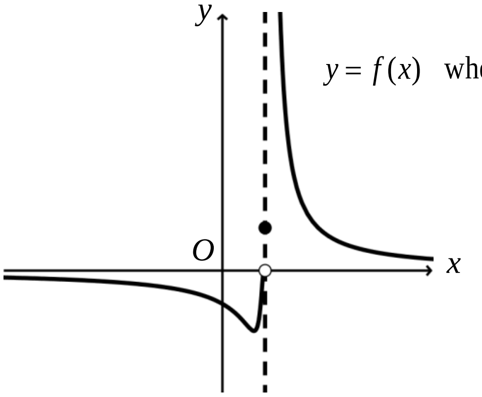
(c)



$y = \dfrac{\sqrt{x^2}}{50x^3 + x}$

$\lim_{x \to 0} \dfrac{\sqrt{x^2}}{50x^3 + x}$

(d)



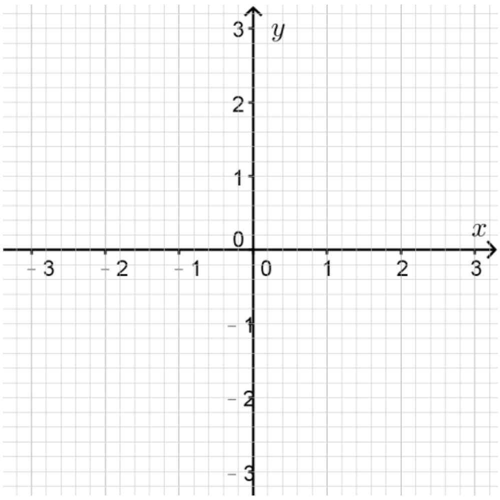
$y = f(x)$ where $f(x) = \begin{cases} \dfrac{1.3^{\frac{1}{x-1}}}{x-1}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$

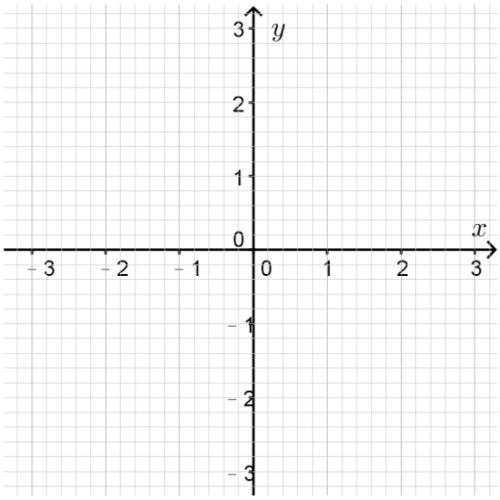
$\lim_{x \to 1} f(x)$

Note: Students can visit the following link for the demonstration
https://goo.gl/zbolX5

Example 1. Given that $f(x)=\begin{cases} x\,, & \text{if } x>1 \\ 1\,, & \text{if } x\le 1 \end{cases}$ and $g(x)=\begin{cases} 1-x\,, & \text{if } x\ge 0 \\ x+2\,, & \text{if } x<0 \end{cases}$, let $h(x)=f(x)+g(x)$.

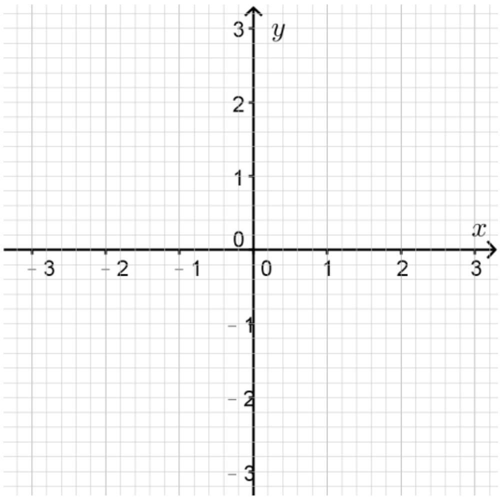
(a) Draw the graphs of $y=f(x)$ and $y=g(x)$ in the following.





(b) Find $h(x)$.

(c) In the following graph paper, draw the graph of $y=h(x)$.



(d) Using the graph, determine whether the following limits exist or not. If it exists, find the value.

(i) $\lim_{x\to 1} h(x)$

(ii) $\lim_{x\to 0} h(x)$

(iii) $\lim_{x\to 2} h(x)$

Solution

(b)

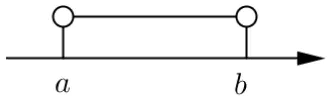
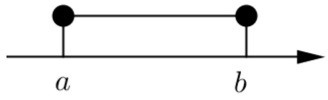
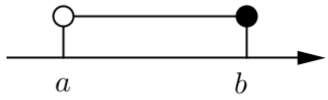
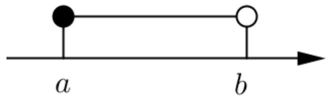
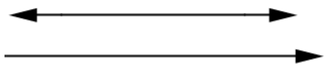
(d)

Intervals

An interval is a set of real numbers that lies between two numbers. The two numbers are called the endpoints of the interval.

There are different kinds of intervals defined as follows.

Let a and b are real numbers such that $a < b$.

Notation	Definition	Representation on the number line
(a,b)	all real numbers x , $a < x < b$	
$[a,b]$	all real numbers x , $a \le x \le b$	
$(a,b]$	all real numbers x , $a < x \le b$	
$[a,b)$	all real numbers x , $a \le x < b$	
$(-\infty,\infty)$	all real numbers	

For example,
$(0\ ,1]$ represents all real numbers that lies between 0 and 1 , excluding 0 and including 1 ;
$[-5\ ,\infty)$ represents all real numbers greater than or equal to 5 .

Example 2. Express the following in notation of intervals.

(a) All real numbers greater than 3 Ans: ____________

(b) All real numbers between 2 and 4 **<u>inclusively</u>** Ans: ____________

(c) The domain of $\log x$ Ans: ____________

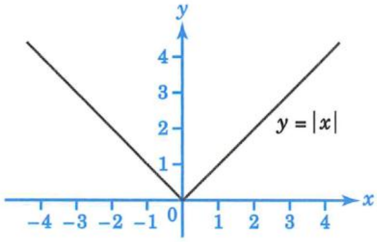
(d) The domain of $\sqrt{x-1}$ Ans: ____________

(e) All real number x such that $x^2 < 1$ Ans: ____________

Absolute value

The absolute value of a real number is defined as

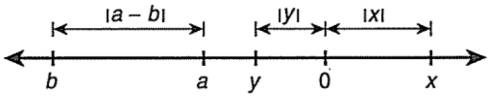
$$|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}.$$



For examples, $|5| = 5$, $|-1| = 1$, $|0| = 0$, $|\sqrt{2}| = \sqrt{2}$.

Note: (a) $|x| \ge 0$

(b) On the number line, $|x|$ means the distance of x from the origin and $|a-b|$ means the distance between a and b .



Example 3. Evaluate the following.

(a)	$\lvert 2\cdot(-5)\rvert$	Ans: ______		(b)	$\lvert 2\rvert\cdot\lvert(-5)\rvert$	Ans: ______
(c)	$\left\lvert \frac{8}{-2} \right\rvert$	Ans: ______		(d)	$\frac{\lvert 8\rvert}{\lvert -2\rvert}$	Ans: ______
(e)	$\lvert 7-3\rvert$	Ans: ______		(f)	$\lvert 3-7\rvert$	Ans: ______
(g)	$\lvert 2+(-5)\rvert$	Ans: ______		(h)	$\lvert 2\rvert+\lvert(-5)\rvert$	Ans: ______

From the above examples, the conclusions below are obvious.

Properties of absolute value

For all real numbers x and y .

(a) $|-x| = |x|$

(b) $|x|^2 = x^2$ and $|x| = \sqrt{x^2}$

(c) $|x \cdot y| = |x| \cdot |y|$

(d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ where $y \ne 0$

(e) $|x+y|$ is NOT necessary equal to $|x|+|y|$

Example 4. Simplify $\left(|1-\sqrt{2}|+1\right)^2$.

Solution

ANS: 2

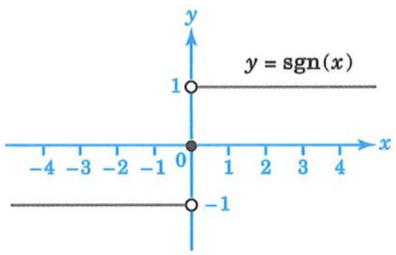
Signum function

The ***signum function*** sgn(x) is defined as:

$$\text{sgn}(x) = \begin{cases} -1 & \text{when } x < 0 \\ 0 & \text{when } x = 0 \\ 1 & \text{when } x > 0 \end{cases}$$

e.g. $\text{sgn}(3.6) = 1$, $\text{sgn}(-7) = -1$, $\text{sgn}(0) = 0$.

sgn(x) is discontinuous at $x = 0$.



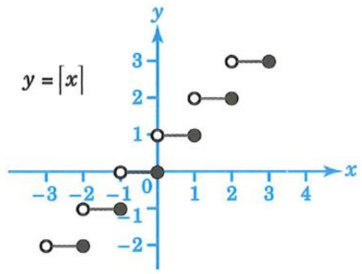
Ceiling function and Floor function

The ***ceiling function*** $\lceil x \rceil$ gives the smallest integer greater than or equal to x. Those non-integral values are rounded up to the nearest integer.

e.g. $\lceil 0.9 \rceil = 1$, $\lceil 1.6 \rceil = 2$, $\lceil 2 \rceil = 2$,

$\lceil -0.5 \rceil = 0$, $\lceil -1 \rceil = -1$, $\lceil 0 \rceil = 0$.

$\lceil x \rceil$ is discontinuous at every integral value of x.

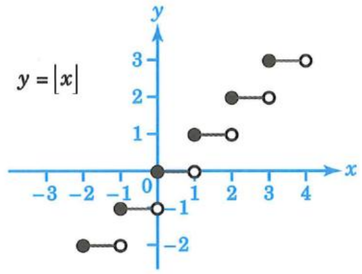


The ***floor function*** $\lfloor x \rfloor$ gives the largest integer less than or equal to x. Those non-integral values are rounded down to the nearest integer.

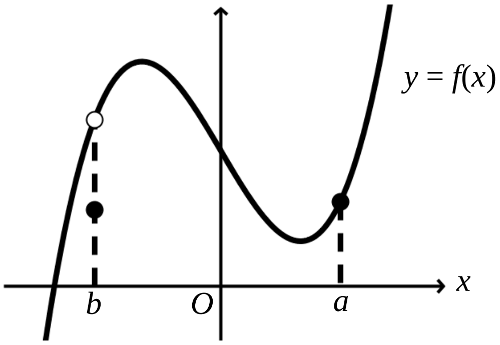
e.g. $\lfloor 0.9 \rfloor = 0$, $\lfloor 1.6 \rfloor = 1$, $\lfloor 2 \rfloor = 2$,

$\lfloor -0.5 \rfloor = -1$, $\lfloor -1 \rfloor = -1$, $\lfloor 0 \rfloor = 0$.

$\lfloor x \rfloor$ is discontinuous at every integral value of x.



Continuous functions



If the graph of a function f is "**connected**" when $x = a$, we say f is **continuous** at $x = a$; however, if the graph of a function f "**breaks**" when $x = b$, we say f is **discontinuous** at $x = b$.

y

Here is an important result from continuous function.

If f is continuous at $x = a$, then $\lim_{x \to a} f(x) = f(a)$

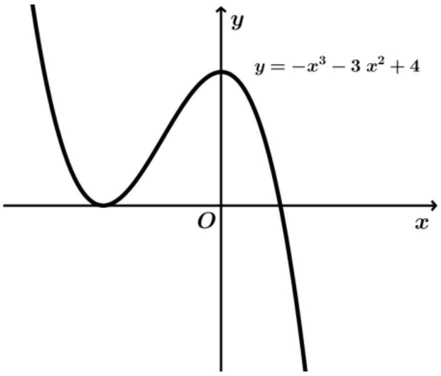
If a function is "connected" at any point on an interval, then the function is called a continuous function on the interval.

All elementary functions (four operations and composition functions of power function, exponential function, logarithmic function and trigonometric functions) are **continuous** on their **domains**.

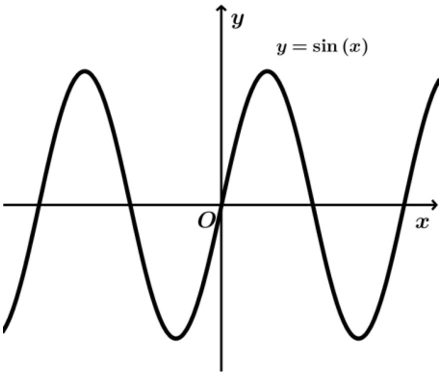
In the previous examples, $|x|$ is a continues function. (Why?)
$\text{sgn}(x)$, ceiling function and floor function are discontinuous.

The followings are examples of continuous functions and their graphs.

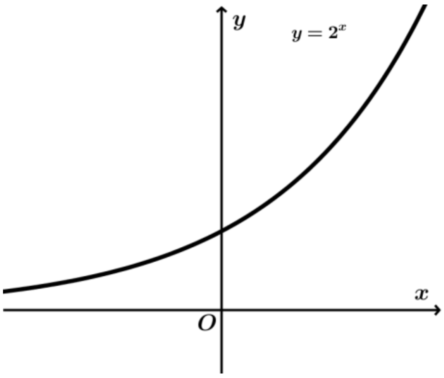
(a) $y = -x^3 - 3x^2 + 4$



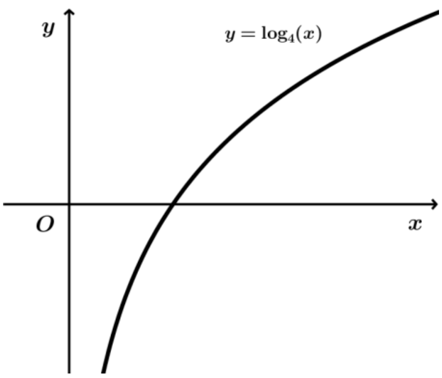
(b) $y = \sin x$



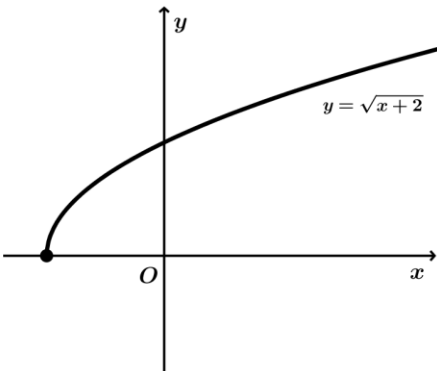
(c) $y = 2^x$



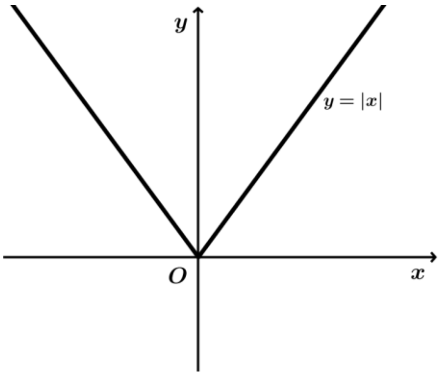
(d) $y = \log_4 x$ for $x > 0$



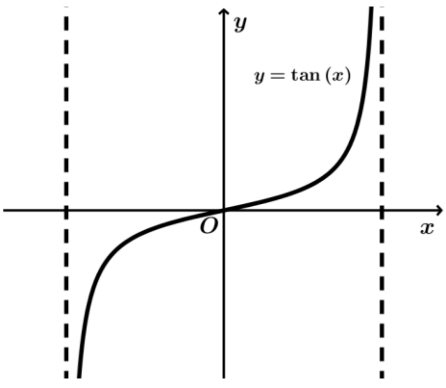
(e) $y = \sqrt{x+2}$ for $x \geq -2$



(f) $y = |x|$



(g) $y = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$



Note: the tangent function, $\tan x$, is **continuous** on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but it is **not continuous** on $(-\infty, \infty)$.

2. Limit of functions at a certain value

Example 5. Evaluate each of the following.

(a) $\lim_{x \to 1}(2x^2 + x - 3)$

(b) $\lim_{x \to \frac{-1}{2}} \frac{x^3 - 4x + 1}{x + 1}$

Explanation

Since the above functions are **continuous** at the given points, the limits can be evaluated by **substitution**.

Solution

(a)

(b)

ANS: (a) 0
(b) 5.75

Example 6. Evaluate $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$.

Explanation

At the point $x = 1$, the above function is undefined and thus **discontinuous**. However,
$\frac{x^3 - 1}{x - 1} \equiv x^2 + x + 1$ for $x \neq 1$. $\therefore$ $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1}(x^2 + x + 1)$.

Solution

ANS: 3

Example 7. Find $\lim_{x \to 8} \frac{x-8}{\sqrt{x+1}-3}$.

Explanation

At the point $x = 8$, the above function is undefined and thus **discontinuous** . However, $\frac{x-8}{\sqrt{x+1}-3} \equiv \sqrt{x+1}+3$ for $x \neq 8$. $\therefore$ $\lim_{x \to 8} \frac{x-8}{\sqrt{x+1}-3} = \lim_{x \to 8}(\sqrt{x+1}+3)$.

Solution

ANS: 6

Example 8. Evaluate $\lim_{x \to 2} \frac{x^3-2}{x-2}$.

Explanation

When x is close to 2 , x^3-2 approaches to 6 thus is **non-zero** and $x-2$ **is close to 0** .

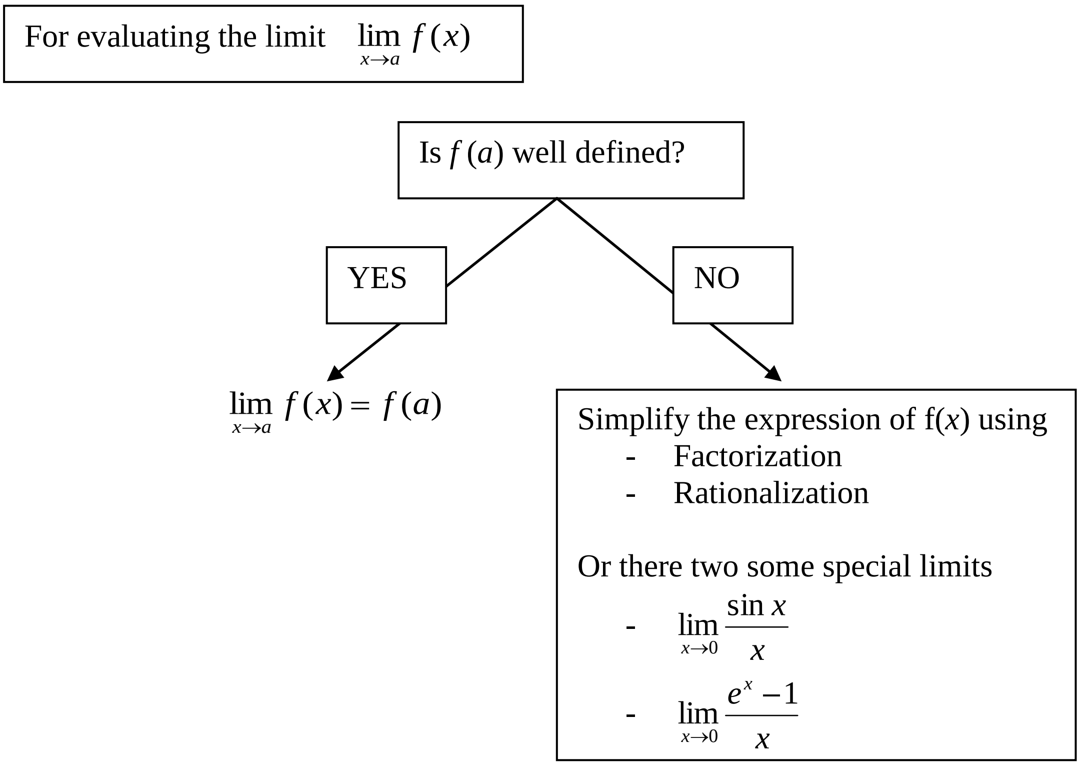
As x approaches 2 from values greater than 2 , $\frac{x^3-2}{x-2}$ increases indefinitely $\left(\lim_{x \to 2^+} \frac{x^3-2}{x-2} = +\infty\right)$;

on the other hand, when x approaches 2 from values smaller than 2 , $\frac{x^3-2}{x-2}$ decreases indefinitely $\left(\lim_{x \to 2^-} \frac{x^3-2}{x-2} = -\infty\right)$. We say $\lim_{x \to 2} \frac{x^3-2}{x-2}$ does not exist $\left(\text{or} \lim_{x \to 2} \frac{x^3-2}{x-2} = \pm\infty\right)$.

Solution

ANS: Limit does not exist

Summary for evaluating limit at a point



Example 9. Evaluate the following limits.

(a) $\lim_{x\to -4} \frac{x+4}{x+1}$

(b) $\lim_{x\to -1} \frac{x+4}{x+1}$

(c) $\lim_{x\to 0} \left(\frac{x+4}{x+1}\right)^{\frac{1}{2}}$

Solution

ANS: (a) 0
(b) not exist
(c) 2

Example 10. Evaluate the following limits.

(a) $\lim_{x\to\sqrt{5}} \frac{\sqrt{x^2+4}-2}{x}$

(b) $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x}$

Solution

ANS: (a) $\frac{\sqrt{5}}{5}$

(b) 0

More Examples of limits at a point

Example 11. Evaluate $\lim_{x\to 1} \frac{\sqrt[3]{x+7}-2}{x-1}$.

Solution

ANS: $\frac{1}{12}$

Example 12. Find the limit $\lim_{x\to 0}\frac{(x+1)^n-1}{x}$ where n is an integer greater than 2 .

Solution

ANS: n

Example 13. It is known that $\lim_{x\to -1}\frac{x^3+ax^2-2x+3}{x+1}$ exists. Find the value of a and the limit.

Solution

ANS: $a=-4$, 9

Example 14. Let $f(x)=x^2+x$. Evaluate $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Solution

ANS: $2x+1$

∗ **Example 15.** Let $f(x,y)=\frac{y}{x+y}$.

(a) Evaluate $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$.

(b) Evaluate $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$.

(c) Is $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,y)$?

Solution

ANS: (a) 1
(b) 0
(c) NO

Theorems on limits

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

(a) $\lim_{x\to a} k = k$

(b) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(c) $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$ where k is a constant

(d) $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

(e) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ where $\lim_{x\to a} g(x) \neq 0$

(f) If $f(x)$ is a continuous function and $\lim_{x\to a} g(x)$ exists, then

$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$

Example 16. It is known that $\lim_{x\to 2} \frac{f(x)}{x+7} = 5$. Does $\lim_{x\to 2} f(x)$ exist? If yes, find the limit.

Solution

Example 17. Let $f(x) = 2x^2 + \frac{1}{x}$ and $g(x) = 3 - \frac{1}{x}$. Does $\lim_{x\to 0}[f(x)+g(x)] = \lim_{x\to 0} f(x) + \lim_{x\to 0} g(x)$ hold? Explain.

Solution

Note: The above example shows that the above theorems fail if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist.