Lesson 1: A Different Kind of Change

Goals

- Describe (orally and in writing) a relationship that increases then decreases when represented by a graph.
- Given an interesting context, create drawings, tables, and graphs that represent a quadratic relationship.

Learning Targets

- I can create drawings, tables, and graphs that represent the area of a garden.
- I can recognize a situation represented by a graph that increases then decreases.

Lesson Narrative

In this lesson, students encounter a situation where a quantity increases then decreases. They don't yet have a name for this new pattern of change, but they recognize that it is neither linear nor exponential, and that the graph is unlike the graph of an exponential function.

Students make sense of this new kind of relationship in a geometric context and describe it in concrete and qualitative ways (MP2). Though some students may choose to represent the relationships with calculations or with expressions, these are not required or emphasized in the lesson. Students will have many opportunities to reason symbolically about quadratic patterns in upcoming lessons.

Technology isn't required for this lesson, but there are opportunities for students to choose to use appropriate tools to solve problems, so consider making technology available.

Alignments

Building On

- HSF-LE.A.1: Distinguish between situations that can be modeled with linear functions and with exponential functions.

Addressing

- HSF-BF.A.1.a: Determine an explicit expression, a recursive process, or steps for calculation from a context.

Building Towards

- HSF-LE.A: Construct and compare linear, quadratic, and exponential models and solve problems.

Instructional Routines

- MLR8: Discussion Supports

- Notice and Wonder

Required Materials

Graph paper

Required Preparation

Graph paper is optional for the activity Measuring a Garden.

Student Learning Goals

- Let’s find the rectangle with the greatest area.

1.1 Notice and Wonder: Three Tables

Warm Up: 5 minutes
This warm-up encourages students to notice a new pattern of change (quadratic) by contrasting it to two familiar patterns (linear and exponential). In the table showing a quadratic relationship, students are not expected to recognize how the input and output values are related. This prompt gives students opportunities to see and make use of structure (MP7). The specific structure they might notice is the output values don’t change by equal amounts or equal factors over equal intervals, and that the output values increase and then decrease.

Building On

- HSF-LE.A.1

Instructional Routines

- Notice and Wonder

Student Task Statement

Look at the patterns in the 3 tables. What do you notice? What do you wonder?

x	y
1	0
2	5
3	10
4	15
5	20

x	y
1	3
2	6
3	12
4	24
5	48

x	y
1	8
2	11
3	10
4	5
5	-4

Student Response

Sample responses:

Things students may notice:

- The x values are 1, 2, 3, 4, 5 in all three tables.
- In the first two tables the y values increase, while in the third table they increase and then decrease.
- The y values in the first table are all multiples of 5 and they grow linearly. In the second table, the y values grow by a factor of 2 each time x increases by 1. In the third table, there isn't an obvious pattern in how the y values change.

Things students may wonder:

- Is there a rule for the relationship in the third table?
- Will the y values in the third table continue to decrease, or will they increase again at some point?
- What would the third relationship look like if graphed?

Activity Synthesis

Invite students to share what they noticed and wondered. Record and display their responses for all to see. After all responses have been recorded without commentary or editing, ask students, "Is there anything on these lists that you are wondering about?" Encourage students to respectfully disagree, ask for clarification, point out contradicting information, etc.

If no one wondered about the rule for the relationship in the third table or how the outputs are changing, ask students to consider these questions. Tell students that in this unit they will investigate relationships such as shown in the third table.

1.2 Measuring a Garden

20 minutes

This activity gives students a concrete experience with a quadratic relationship in a familiar geometric context. Given a rectangle with a fixed perimeter, students experiment with how changes to one side of the rectangle affect its area. Along the way, they notice that as one length increases, the area does not continue to increase. Instead, at some point it begins to decrease.

Students are not expected to write an equation such as $\ell \cdot (25 - \ell)$ for the area of the rectangle. At this point, informal observations on how the values are changing are sufficient. Students will have ample opportunities throughout this unit to examine the formal structure of quadratic relationships in depth.

As students work, encourage them to consider side lengths that are not whole numbers. Look for students who organize their work in different systematic ways and select them to share their work later.

Considering the relationship between the dimensions and area of a rectangle given a fixed perimeter requires students to reason abstractly and quantitatively (MP2). Making spreadsheet technology available gives students an opportunity to choose appropriate tools strategically (MP5).

Addressing

- HSF-BF.A.1.a

Building Towards

- HSF-LE.A

Instructional Routines

- MLR8: Discussion Supports

Launch

Display a rectangle for all to see and label the sides with some lengths. Ask students to find the perimeter of the rectangle and the area of the region enclosed by this rectangle. Then, ask for the definitions of perimeter and area. Before beginning the activity, make sure students are clear about the distinction of the two measures—that the perimeter is the distance all the way around a region and the area is the number of unit squares that cover a region without gaps or overlaps.

Give students access to graph paper and tell students that they can use graph paper for the first question if they wish. Also provide access to calculators. Some students may benefit from using them to get to the interesting part of the task.

For the second question, some students may choose to create a spreadsheet to keep track of (and perhaps sort) the lengths, widths, and areas of the rectangle. Make spreadsheet tool available, in case requested.

Support for Students with Disabilities

Representation: Internalize Comprehension. Differentiate the degree of difficulty or complexity by beginning with an example that illustrates how to use graph paper to determine the length and width of each side. Also, demonstrate how the spreadsheet tool can be used to illustrate the length and width of each rectangle.
Supports accessibility for: Conceptual processing

Anticipated Misconceptions

Some students may exclude a rectangle with side lengths 12.5 and 12.5 from their diagrams of Noah's garden, possibly because they think a square is not a rectangle, or possibly because they only generate whole numbers. Emphasize that a square is a type of rectangle that happens to have four sides of equal length. Prompt them to think of the definition of a rectangle and explain why a square meets all the criteria for a rectangle.

Student Task Statement

Noah has 50 meters of fencing to completely enclose a rectangular garden in the backyard.

1. Draw some possible diagrams of Noah's garden. Label the length and width of each rectangle.

2. Find the length and width of such a rectangle that would produce the largest possible area. Explain or show why you think that pair of length and width gives the largest possible area.

Student Response

1. Answers vary. (See examples in the Activity Synthesis.)
2. When the length and width are each 12.5 meters, the area is the largest (156.25 square meters). A table shows that as the rectangular shape gets closer and closer to being a square, the area gets larger.

Activity Synthesis

Display the work of a student who organized lengths, widths, and areas in a table. If no students did so, generate a table as a class. An example is shown here. (Pairs of length and area values will be needed in the next activity.)

length (meters)	width (meters)	area (square meters)
5	20	100
10	15	150
12	13	156
12.5	12.5	156.25
18	7	126
20	5	100
24	1	24

Discuss with students:

- “What do you notice about the relationship between the length and the width?” (They add up to 25. Each time the length increases by 1, the width decreases by 1. The relationship is linear.)
- “What do you notice about the relationship between the length and the area?” (As the length increases from 1 to 24, the area increases and then decreases. The relationship is not linear.)

Ask students to describe the rectangle they found to have the largest area and how they went about finding it. It is likely that many students will say that it has side lengths of 12 and 13, since these are the whole-number values that produce the greatest area. If no students tried 12.5 and 12.5, ask them to compute this area.

Solicit some ideas from students on how the area is related to the length. Ask questions such as:

- “How do we know if 12.5 and 12.5 would indeed produce the greatest area?”
- “Do you think going from 5 to 8 meters in length would produce a rectangle with a greater area? What about going from 15 to 18? Why or why not?”

Tell students that we’ll now try to get a better idea of what’s happening between the side lengths and the area of the rectangle by plotting some points.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion. For each observation that is shared, ask students to restate what they heard using precise mathematical language. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This provides more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

1.3 Plotting the Measurements of the Garden

10 minutes
In this activity, students plot the points that represent the relationship between a side length and the area of a rectangle with a perimeter of 50 meters. They encounter a graph where as one quantity increases, a second quantity increases and then decreases. Making sense of the graph in context helps them see why it is reasonable to expect the second quantity to decrease after a certain point.

As students work, look for those whose graphs include enough points to hint at a quadratic shape. Also monitor for those who can articulate why $(1, 25)$ cannot represent the length and area of the garden. Invite them to share their work later.

Making graphing technology available gives students an opportunity to choose appropriate tools strategically (MP5).

Building Towards

- HSF-LE.A

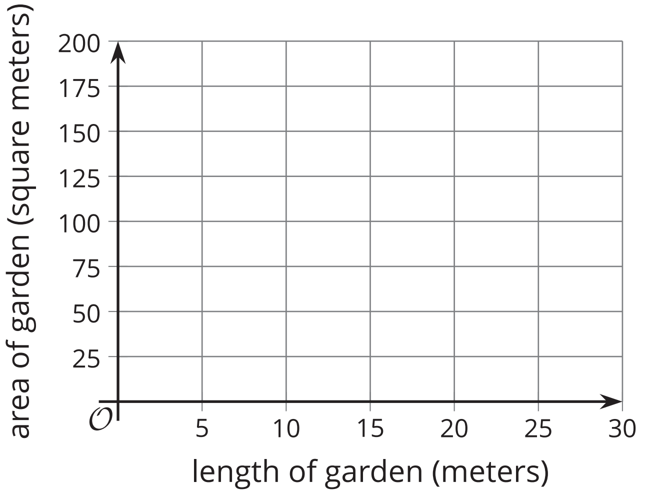
Launch

Keep a table from the previous activity displayed. If students created their own table, encourage them to use the values in their table.

If students used a spreadsheet tool to organize the lengths and area for the earlier activity, they may choose to use graphing technology to plot the data.

Student Task Statement

1. Plot some values for the length and area of the garden on the coordinate plane.



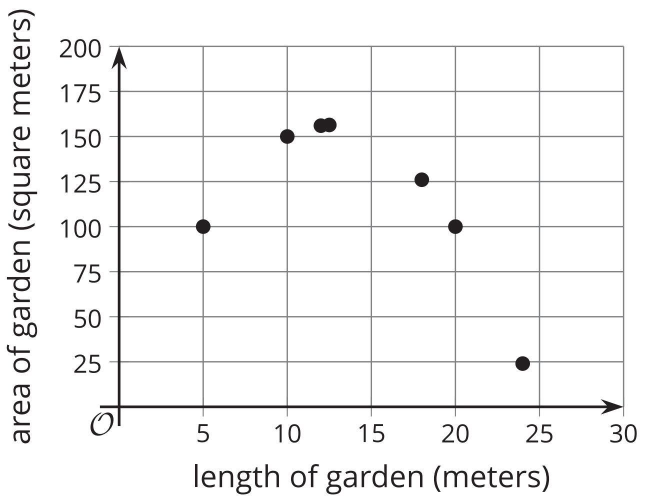
2. What do you notice about the plotted points?

3. The points $(3, 66)$ and $(22, 66)$ each represent the length and area of the garden. Plot these 2 points on coordinate plane, if you haven't already done so. What do these points mean in this situation?

4. Could the point $(1, 25)$ represent the length and area of the garden? Explain how you know.

Student Response

1. Sample response:



2. Sample response: Notice that at first the area grows as the length increases but then it decreases. Also the area increases more and more slowly as the length increases.

3. The point $(3, 66)$ means that if the length of the garden is 3 meters then the area is 66 square meters. This is right because the width is 22 meters and $3 \cdot 22 = 66$. The point $(22, 66)$ means that the length of the garden is 22 meters so its width is 3 meters and the area is again 66 square meters.

4. The point $(1, 25)$ cannot represent the length and area of the garden. If the length of the garden is 1 meter, the width would be 24 meters and so the area is 24 square meters, not 25 square meters.

Are You Ready for More?

1. What happens to the area when you interchange the length and width? For example, compare the areas of a rectangle of length 11 meters and width 14 meters with a rectangle of length 14 meters and width 11 meters.

2. What patterns would you notice if you were to plot more length and area pairs on the graph?

Student Response

1. The area stays the same if the length and width are interchanged because the area is the product of the length and the width.

2. Many of the areas would come with two different pairs: for example, 100 comes from the pair $(5, 100)$ but also from the pair $(20, 100)$ because a rectangle with length 5 meters and width 20 meters is the same as a rectangle with length 20 meters and width 5 meters. Using a different length and corresponding width, the 156 comes from the pair $(12, 156)$ and also from the pair $(13, 156)$. The points on the graph would have reflectional symmetry about the line $x = 12.5$.

Activity Synthesis

Select a student to present their graph, or display a graph with some points already plotted and amend it with additional points students provide. Discuss with students:

- "Is the graph linear? Is it exponential?" (It is neither.)
- "If we plot a bunch more points—say for every whole-number length between 0 and 25—what do you think the graph would look like?" (Possible predictions: A stretched out upside-down U, or an arch. A line with a positive slope that then turns into one with a negative slope. A curve. Some students might even recall the term parabola.)
- "How can we tell whether $(1, 25)$ does or does not represent the length and area of the garden?" (By multiplying 1 and 24 (if the length is 1, the width is 24) to see if it gives 25. Because 1 times 24 is 24, $(1, 25)$ does not represent the length and area.)
- "Can we tell if a point represents the length and area of the garden simply by plotting it on the graph? For example, does $(1.5, 200)$ represent the length and area? What about $(23, 60)$?" (It depends. If a point is far away from other points (for example, $(1.5, 200)$, we can probably tell it does not. If it seems to belong with the general trend of the other points, we may need to check if the input times the width equals the output value.)

Tell students that this unit will focus on functions that are like that relating the length and area of the garden. The output of the function may both increase and decrease, so we know they are neither linear nor exponential, but they also don't change in a random way.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Provide a graphic organizer that focuses on increasing the length of on-task orientation in the face of distractions. For example, during the whole-class discussion create a graphic organizer that includes the questions that will be asked in one column and a blank space to record the answers in the other column.
Supports accessibility for: Attention; Social-emotional skills

Lesson Synthesis

Invite students to reflect on how the relationship between the side lengths and the area of a rectangle differs from other relationships they've seen. Consider asking students to comment on:

- the values in the table relating the length and the area of the rectangle
- the graph representing the length-area relationship
- the rule that relates the input and output of the function

It is not essential that students frame their observations in precise ways at this point. Their capacity to do so will be developed in the coming lessons.

1.4 100 Meters of Fencing

Cool Down: 5 minutes

Building Towards

- HSF-LE.A

Student Task Statement

A rectangular yard is enclosed by 100 meters of fencing. The table shows some possible values for the length and width of the yard.

length (meters)	width (meters)	area (square meters)
10	40	400
20	30	
25	25	625
35	15	525
40		

1. Complete the table with the missing values.
2. If the values for length and area are plotted, what would the graph look like?
3. How is the relationship between the length and the area of the rectangle different from other kinds of relationships we've seen before?

Student Response

1. The missing value in the second row is 600. The missing values in the last row are 10 and 400.
2. Sample response: The points would form a stretched out upside-down U shape, or an arch.
3. Sample response: As one quantity increases, the other quantity first increases then decreases, instead of always increasing or always decreasing.

Student Lesson Summary

In this lesson, we looked at the relationship between the side length and the area of a rectangle when the perimeter is unchanged.

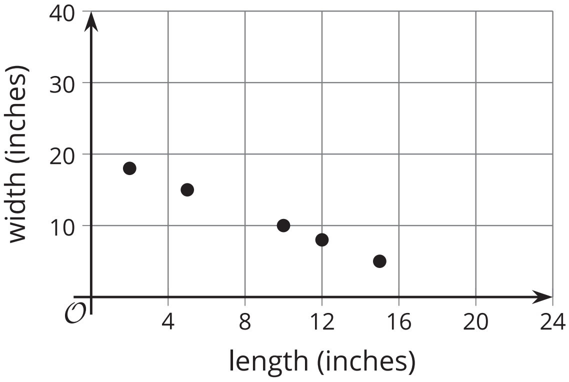
If a rectangle has a perimeter of 40 inches, we can represent the possible lengths and widths as shown in the table.

length (inches)	width (inches)
2	18
5	15
10	10
12	8
15	5

We know that twice the length and twice the width must equal 40, which means that the length plus width must equal 20, or $\ell + w = 20$.

To find the width given a length ℓ, we can write: $w = 20 - \ell$.

The relationship between the length and the width is linear. If we plot the points from the table representing the length and the width, they form a line.

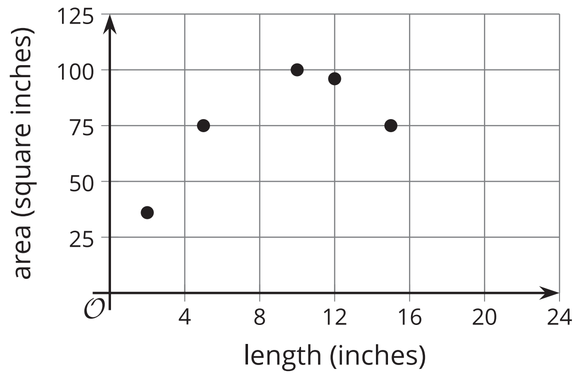


What about the relationship between the side lengths and the area of rectangles with perimeter of 40 inches?

Here are some possible areas of different rectangles whose perimeter are all 40 inches.

length (inches)	width (inches)	area (square inches)
2	18	36
5	15	75
10	10	100
12	8	96
15	5	75

Here is a graph of the lengths and areas from the table:



Notice that, initially, as the length of the rectangle increases (for example, from 5 to 10 inches), the area also increases (from 75 to 100 square inches). Later, however, as the length increases (for example, from 12 to 15), the area decreases (from 96 to 75).

We have not studied relationships like this yet and will investigate them further in this unit.

Lesson 1 Practice Problems

Problem 1

Statement

Here are a few pairs of positive numbers whose sum is 50.

a. Find the product of each pair of numbers.

b. Find a pair of numbers that have a sum of 50 and will produce the largest possible product.

c. Explain how you determined which pair of numbers have the largest product.

first number	second number	product
1	49	
2	48	
10	40	

Solution

a. The products are 49, 96, and 400.

b. 25 and 25. The product is 625.

c. Sample response: I listed different pairs and found their products, and noticed that when the two numbers are the same, the product is the largest.

Problem 2

Statement

Here are some lengths and widths of a rectangle whose perimeter is 20 meters.

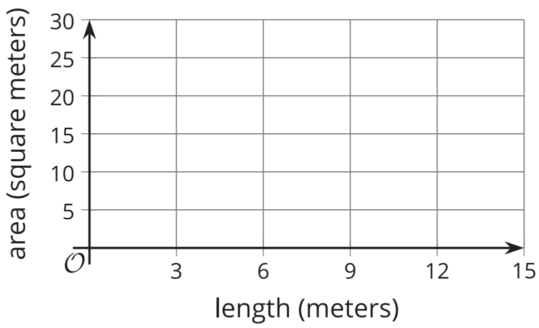
a. Complete the table. What do you notice about the areas?

length (meters)	width (meters)	area (square meters)
1	9	
3	7	
5		
7		
9		

b. Without calculating, predict whether the area of the rectangle will be greater or less than 25 square meters if the length is 5.25 meters.

c. On the coordinate plane, plot the points for length and area from your table.

Do the values change in a linear way? Do they change in an exponential way?



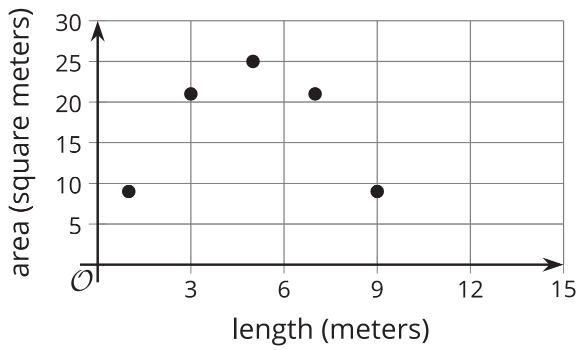
Solution

a. Sample response: The values of the area repeat. The order of the factors (the length and width) changes but the product is the same.

length (meters)	width (meters)	area (square meters)
1	9	9
3	7	21
5	5	25
7	3	21
9	1	9

b. Predictions vary. Sample response: When the length is 5.25, the width will be 4.75, and I estimate that the product of those numbers will be less than 25.

c. The values change in neither a linear nor an exponential way. The area increases and then decreases.



Problem 3

Statement

The table shows the relationship between x and y, the side lengths of a rectangle, and the area of the rectangle.

a. Explain why the relationship between the side lengths is linear.

b. Explain why the relationship between x and the area is neither linear nor exponential.

x (cm)	y (cm)	area (sq cm)
2	4	8
4	8	32
6	12	72
8	16	128

Solution

Sample response:

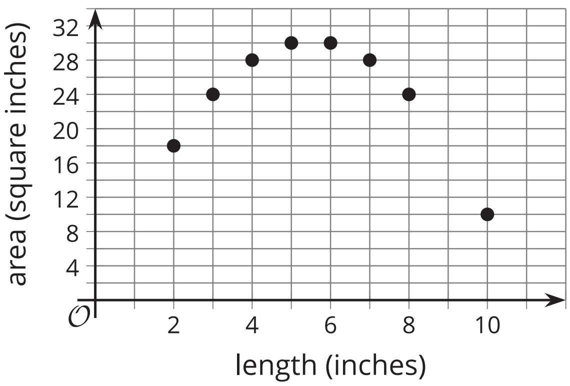
a. When x increases by 2 cm, y increases by an equal amount, 4 cm.

b. When x increases by 2 cm, the area does not increase by an equal amount or by an equal factor. It changes by 24 sq cm, 40 sq cm, 56 sq cm, and so on. It also changes by different factors at each step.

Problem 4

Statement

Which statement best describes the relationship between a rectangle's side length and area as represented by the graph.



A. As the side length increases by 1, the area increases and then decreases by an equal amount.

B. As the side length increases by 1, the area increases and then decreases by an equal factor.

C. As the side length increases by 1, the area does not increase or decrease by an equal amount.

D. As the side length increases by 1, the area does not change.

Solution

C

Problem 5

Statement

Copies of a book are arranged in a stack. Each copy of a book is 2.1 cm thick.

a. Complete the table.

b. What do you notice about the differences in the height of the stack of books when a new copy of the book is added?

c. What do you notice about the factor by which the height of the stack of books changes when a new copy is added?

d. How high is a stack of b books?

copies of book	stack height in cm
0	
1	
2	
3	
4	

Solution

a. See table.

b. Each time a new copy of the book is added, the height of the stack grows by 2.1 cm.

c. Each time a new copy of the book is added, the factor by which the height of the stack grows changes. There is no growth factor from 0 books to 1 book, since no multiple of 0 will give 2.1.

d. $(2.1) \cdot b$ centimeters

copies of book	stack height in cm
0	0
1	2.1
2	4.2
3	6.3
4	8.4

(From Unit 5, Lesson 2.)

Problem 6

Statement

The value of a phone when it was purchased was \$500. It loses $\frac{1}{5}$ of its value a year.

a. What is the value of the phone after 1 year? What about after 2 years? 3 years?

b. Tyler says that the value of the phone decreases by \$100 each year since $\frac{1}{5}$ of 500 is 100. Do you agree with Tyler? Explain your reasoning.

Solution

a. After 1 year it is worth \$400, after 2 years it is worth \$320, and after 3 years it is worth \$256.

b. Tyler is not correct. The phone loses \$100 in value the first year, but after that, the value it loses each year is less because it is $\frac{1}{5}$ of its value at the beginning of the new year.

(From Unit 5, Lesson 4.)

Problem 7

Statement

Technology required. The data in the table represents the price of one gallon of milk in different years.

Use graphing technology to create a scatter plot of the data.

a. Does a linear model seem appropriate for this data? Why or why not?

b. If the data seems appropriate, create the line of best fit. Round to two decimal places.

c. What is the slope of the line of best fit, and what does it mean in this context? Is it realistic?

d. What is the y-intercept of the line of best fit, and what does it mean in this context? Is it realistic?

x, time (years)	price per gallon of milk (dollars)
1930	0.26
1935	0.47
1940	0.52
1940	0.50
1945	0.63
1950	0.83
1955	0.93
1960	1.00
1965	1.05
1970	1.32
1970	1.25
1975	1.57
1985	2.20
1995	2.50
2005	3.20
2018	2.90
2018	3.25

Solution

a. Sample response: When I look at the scatter plot, the data appears to be in a somewhat linear relationship; as the years increase, so does the price of milk.

b. $y = 0.03x - 67.12$

c. 0.03; The slope means that for every year that passes, we expect the price of milk to increase by $0.03. This seems realistic, because the price of milk is increasing steadily over time.

d. $(0, \text{-}67.12)$ The y-intercept means that in year 0, we would expect milk to cost -$67.12. This is not realistic since the price of milk should not be negative.

(From Unit 3, Lesson 5.)

Problem 8

Statement

Give a value for r that indicates that a line of best fit has a negative slope and models the data well.

Solution

Sample solution: -0.92. Correct answers should be between -0.8 and -1.

(From Unit 3, Lesson 7.)