Ratio of Hyperbolic Cosine Area to Arc Length

William C. Evans ***April 2019***

A cable or chain or rope, etc. that hangs freely from two suspension points will, under the influence of gravity on its own weight, form a special type of curve called a "catenary." This looks very much like a parabola, and under some circumstances the difference between the two shapes is small, but they *are* different. It is shown in many engineering mechanics textbooks that this catenary curve can be written as

$$y(x) = a\cosh\left(\frac{x}{a}\right)$$

so that $y(0) = a$; however, in the mechanics texts the catenary form is often stated as

$$y(x) = a\left\{\cosh\left(\frac{x}{a}\right) - 1\right\}$$

which will make $y(0) = 0$. Note that x is measured from the minimum point on the curve, where the tangent has zero slope. The parameter "a" here stands for some physical properties of the cable or chain, etc. that is hanging from two support anchors at some elevation above grade. For now, all we are interested in is a certain property of this mathematical function.

The area A under the catenary graph, between any two x-values, is given by

$$A = \int_{x_1}^{x_2} a\cosh\left(\frac{x}{a}\right) dx$$

which we find without much difficulty to be

$$A = a^2\left\{\sinh\left(\frac{x_2}{a}\right) - \sinh\left(\frac{x_1}{a}\right)\right\}$$

Next, we wish to find the corresponding arc length L, so that we need to evaluate this integral (found in any calculus text):

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2}\, dx \quad \Rightarrow \quad \int_{x_1}^{x_2} \sqrt{1 + \left\{\sinh\left(\frac{x}{a}\right)\right\}^2}\, dx$$

which uses the derivative

$$\frac{d}{dx}\, a\cosh\left(\frac{x}{a}\right) = \sinh\frac{x}{a}$$

Then, using *Mathematica* for this integral, we will find that

$$L = a\cosh\left(\frac{x_2}{a}\right)\tanh\left(\frac{x_2}{a}\right) - a\cosh\left(\frac{x_1}{a}\right)\tanh\left(\frac{x_1}{a}\right)$$

which simplifies to just

$$L = a\left\{\sinh\left(\frac{x_2}{a}\right) - \sinh\left(\frac{x_1}{a}\right)\right\}$$

and so we see the remarkable fact that **the ratio of the area *A* to the arc length *L* is simply *a***, which is a constant. Interestingly, this property is shared with the far simpler function $y(x) = k$, a horizontal line. If we go a distance L in the x-direction, the rectangular area under $y(x)$ is of course kL, and the "arc length" is L, so the ratio is the constant k.

NOTE: A useful, and affordable, reference on the engineering aspects of this material is J. P. Den Hartog, ***Mechanics****, Dover, pp. 63-67.*