# Activity – Linear Transformations

Goals:

- Understand the meaning of a linear transformation $T$ from one vector space $V$ to another $W$, $T: V \to W$.
- Linear transformations from a vector space $V$ to itself $T: V \to V$ moves points to different locations in the same vector space $V$. This kind of linear transformation is called a **linear operator**. (Note**,** change of basis keeps all points and vectors the same but changes how they are viewed, depending on the basis.)
- Understand the image of any *linear combination* in $V$ under a linear transformation is a *linear combination* in $W$.
- Understand the geometric meaning of the matrix representation of a linear transformation.
- Understand every linear transformation between finite dimensional vectors spaces can be written as a matrix representation.
- Transform any vector given in the Standard Cartesian system ($Basis$: $B_{e_1,e_2}$) into a vector in the uv-system ($Basis$: $B_{u,v}$) using linear transformations. Understand that the matrix constructed by columns of basis vectors is linear transformation.

***Definition:*** A **linear transformation** between two vector spaces $V$ and $W$ is a function $T: V \to W$ (i.e., $T$ assigns to each vector in $V$ a unique vector in $W$) such that the following holds:

1. $T(\vec{v} + \vec{u}) = T(\vec{v}) + T(\vec{u})$ for any vectors $\vec{v}$ and $\vec{u}$ in $V$; $T(\vec{v})$ and $T(\vec{u})$ are vectors in $W$, and

2. $T(c\vec{v}) = c\,T(\vec{v})$ for any scalar $c$. i.e., $T(\vec{v})$ in $W$, is scaled by the same $c$ as $\vec{v}$ in $V$

This can also be stated as follows:

$$T(c_1\vec{v} + c_2\vec{u}) = c_1T(\vec{v}) + c_2T(\vec{u}) \text{ for } \vec{v} \text{ and } \vec{u} \text{ in } V;\ c_1 \text{ and } c_2 \text{ scalars; } T(\vec{v}) \text{ and } T(\vec{u}) \text{ are vectors in } W$$

Note: Applying a **linear transformation** to a *linear combination* in $V$ results in a *linear combination* in $W$.

**Example:** Verify the following is a **linear transformation** from : $R^2 \to R^2$. $\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\boldsymbol{T}\left(\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{x}\begin{bmatrix} \mathbf{3} \\ \mathbf{1} \end{bmatrix} + \boldsymbol{y}\begin{bmatrix} \mathbf{-2} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{3}\boldsymbol{x} - \mathbf{2}\boldsymbol{y} \\ \mathbf{1}\boldsymbol{x} + \mathbf{2}\boldsymbol{y} \end{bmatrix}$$

We will use the succinct version of the definition of a linear combination.

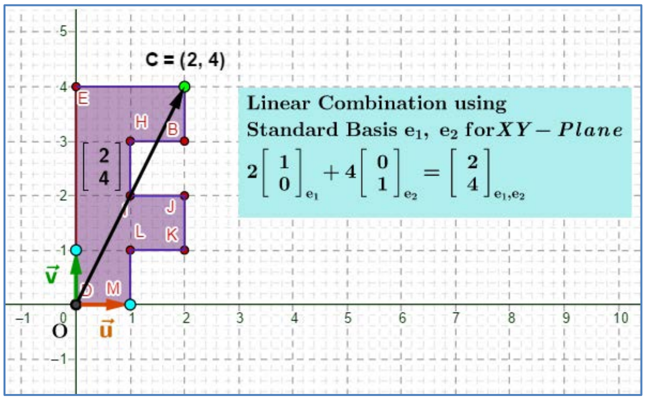
$$\boldsymbol{T}\left(\boldsymbol{c_1}\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{y_1} \end{bmatrix} + \boldsymbol{c_2}\begin{bmatrix} \boldsymbol{x_2} \\ \boldsymbol{y_2} \end{bmatrix}\right) = T\left(\begin{bmatrix} \boldsymbol{c_1}x_1 + \boldsymbol{c_2}x_2 \\ \boldsymbol{c_1}y_1 + \boldsymbol{c_2}y_2 \end{bmatrix}\right) = \begin{bmatrix} \mathbf{3}(\boldsymbol{c_1}x_1 + \boldsymbol{c_2}x_2) - \mathbf{2}(\boldsymbol{c_1}y_1 + \boldsymbol{c_2}y_2) \\ \mathbf{1}(\boldsymbol{c_1}x_1 + 1\boldsymbol{c_2}x_2) + \mathbf{2}(\boldsymbol{c_1}y_1 + \boldsymbol{c_2}y_2) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{3}\boldsymbol{c_1}x_1 + \mathbf{3}\boldsymbol{c_2}x_2 - \mathbf{2}\boldsymbol{c_1}y_1 - \mathbf{2}\boldsymbol{c_2}y_2 \\ \mathbf{1}\boldsymbol{c_1}x_1 + \mathbf{1}\boldsymbol{c_2}x_2 + \mathbf{2}\boldsymbol{c_1}y_1 + \mathbf{2}\boldsymbol{c_2}y_2 \end{bmatrix} = \begin{bmatrix} \mathbf{3}\boldsymbol{c_1}x_1 - \mathbf{2}\boldsymbol{c_1}y_1 \\ \mathbf{1}\boldsymbol{c_1}x_1 + \mathbf{2}\boldsymbol{c_1}y_1 \end{bmatrix} + \begin{bmatrix} \mathbf{3}\boldsymbol{c_2}x_2 - \mathbf{2}\boldsymbol{c_2}y_2 \\ \mathbf{1}\boldsymbol{c_2}x_2 + \mathbf{2}\boldsymbol{c_2}y_2 \end{bmatrix} = \boldsymbol{c_1}\,\boldsymbol{T}\left(\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{y_1} \end{bmatrix}\right) + \boldsymbol{c_2}\boldsymbol{T}\left(\begin{bmatrix} \boldsymbol{x_2} \\ \boldsymbol{y_2} \end{bmatrix}\right)$$

In the following use the **Linear Transformation** applet in the Linear Systems section of the GeoGebra Book: https://www.geogebra.org/m/XnfUWvvp

**Part 1** – Consider different kinds of linear transformations from $R^2$ to $R^2$

Identity




The basis: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ defines the **linear transformation**

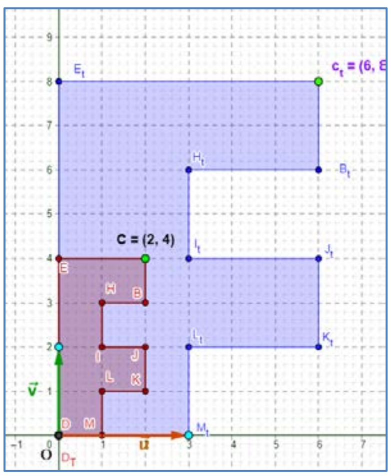
$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

For $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$, $\quad T\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = 2\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$

Here no points are moved or vectors changed

$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ is the Identity Linear Transformation

Scaled




The basis $\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}$ defines the **linear transformation**

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}3\\0\end{bmatrix} + y\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}3x\\2y\end{bmatrix}$$

For $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$, $\quad T\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = 2\begin{bmatrix}3\\0\end{bmatrix} + 4\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix}$

Here points are moved and vectors changed.

The figure is scaled horizontally by 3 and vertically by 2. The vector $\begin{bmatrix}2\\4\end{bmatrix}$ is changed to $\begin{bmatrix}6\\8\end{bmatrix}$ and the point **C = (2, 4)** to **$C_t$ = (6,8).** The entire red polygon **F** is transformed into the blue polygon **F**.

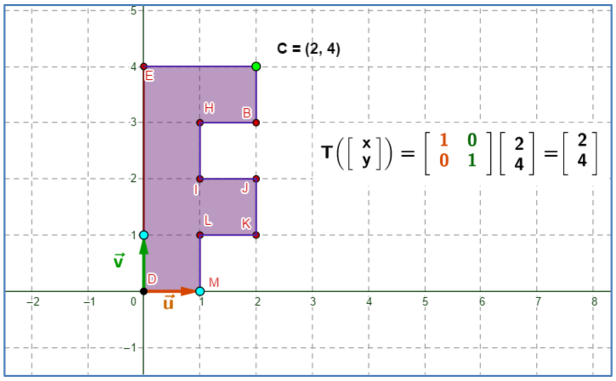
**Verify:** $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ is a linear transformation $T: R^2 \to R^2$.

Use this form of the definition: $T(c_1\vec{v} + c_2\vec{u}) = c_1T(\vec{v}) + c_2T(\vec{u})$, where $\vec{v}$, $\vec{u}$ in $R^2$

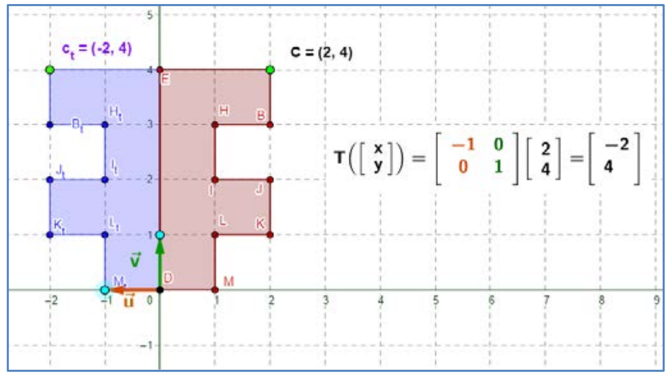
**<u>Excercises:</u>** Consider the **linear transformation** on each graph below. Review this sequence of reflections.

## <u>Reflections</u>

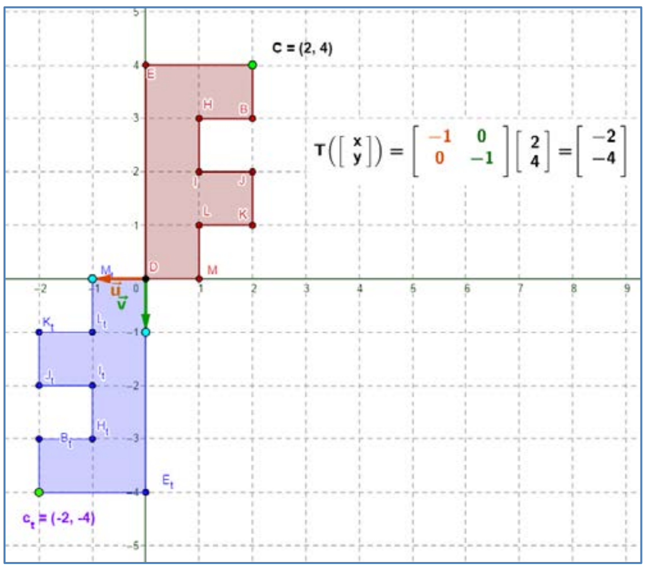
Identity → R**I**



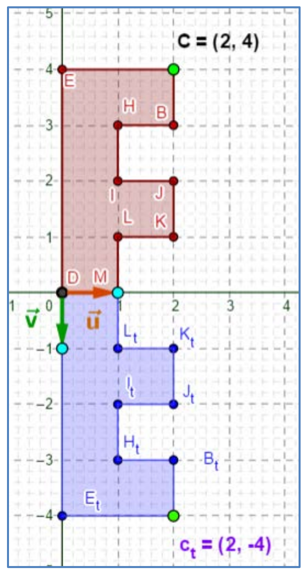

Reflection in **(Y+)** → R(**Y+**)




Reflection in (**X**−) → R( **X**−)




Reflection in (**Y**−) → R( **Y**−)




Define the matrix form of this Linear Transformation

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$

Find $T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) =$

a) Perform matrix multiplication to get a simplified form of the composition of linear transformations.

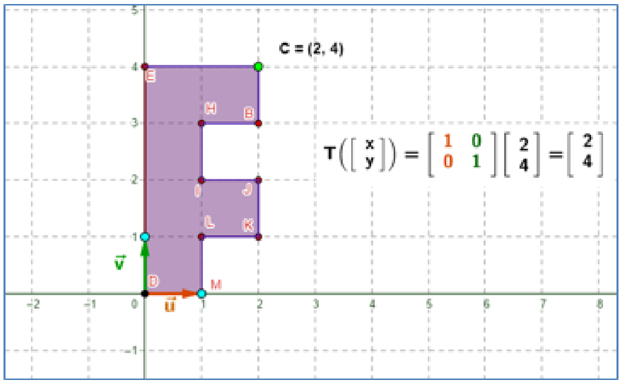
$$\mathrm{R}(\mathbf{X}-)\left[\mathrm{R}(\mathbf{Y}+)\begin{bmatrix} x \\ y \end{bmatrix}\right] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} =$$

Let $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ Find $\mathrm{R}(\mathbf{X}-)\left[\mathrm{R}(\mathbf{Y}+)\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right] =$

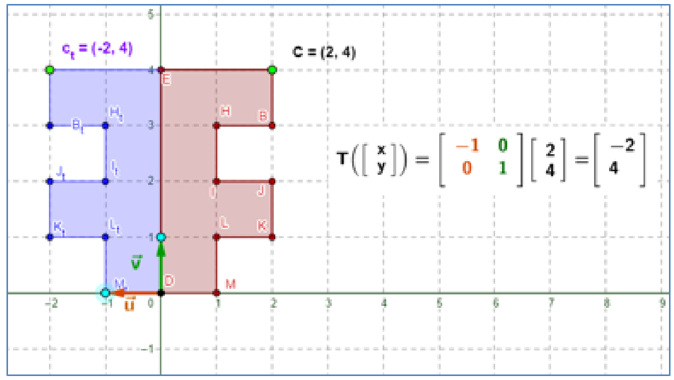
How does this compare to your answer above? ______

## Reflections

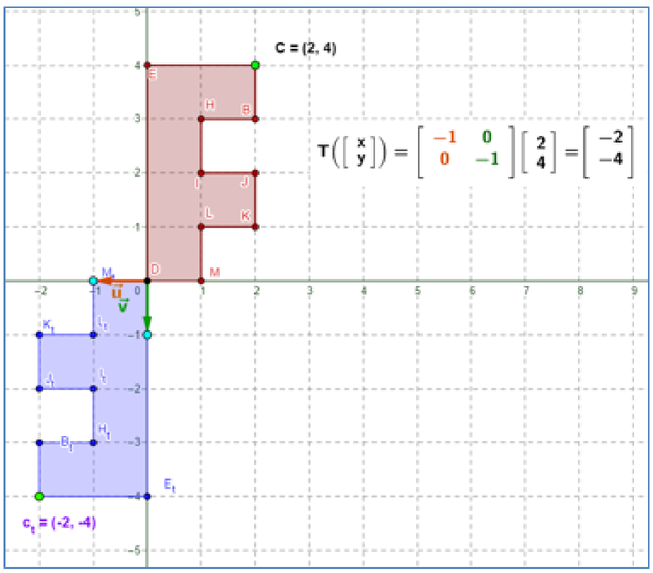
Identity → RI



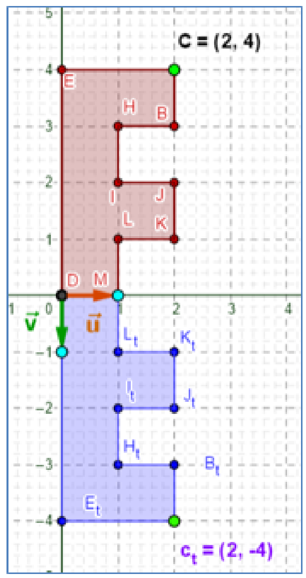

Reflection in (**Y+**) → R(**Y+**)




Reflection in (**X−**) → R( **X−**)




Reflection in (**Y−**) → R( **Y−**)




Define the matrix form of this Linear Transformation

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$

Find $T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) =$

b) What does this composition mean in terms of the above squence of reflections? What image does it start with and which image does it end with? State what each transformation does.

$$\mathrm{R}(\mathbf{Y}-)\left[R(\mathbf{X}-)\left[\mathrm{R}(\mathbf{Y}+)\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}\right]\right] =$$

_______________________________________________________________________

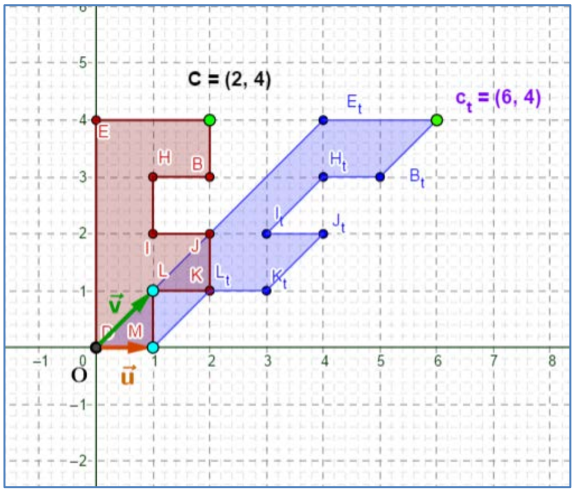
State the resulting simplified matrix form of the linear transformatiuon for this composition?

_____________________________________________________________________________

Let $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Find $\mathrm{R}(\mathbf{Y}-)\left[R(\mathbf{X}-)\left[\mathrm{R}(\mathbf{Y}+)\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right]\right] =$

Consider the **linear transformation** on each graph below.

Shear




Define the matrix form of the linear transformation represented.

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$

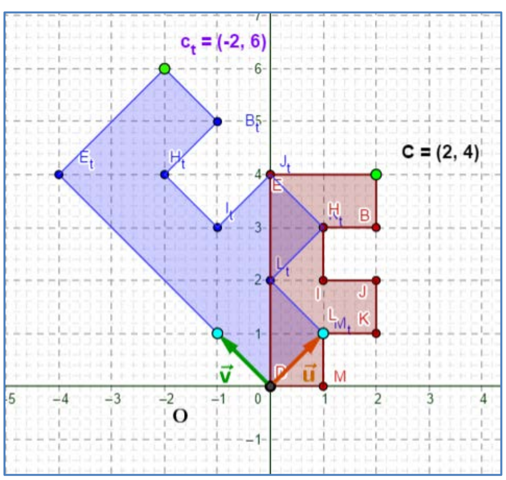
Find $T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) =$

Find the inverse of this Linear Transformation.

$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$

Find $T^{-1}\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) =$

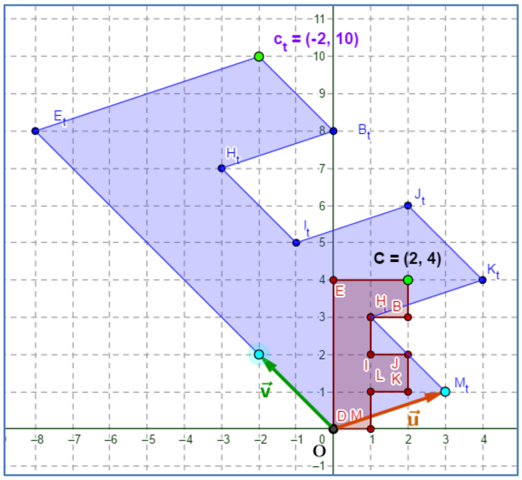
What does $T^{-1}$ do?




Define the matrix form of the linear transformation represented.

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$

Find $T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) =$




Define the matrix form of the linear transformation represented on the left

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$$

Find $T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) =$

Find the inverse of this Linear Transformation.

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$$

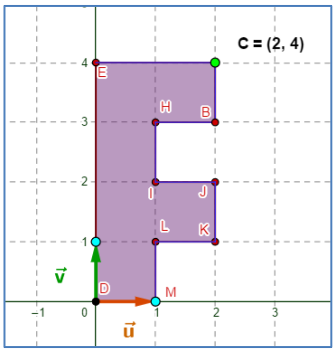
Find $T^{-1}\left(\begin{bmatrix} -8 \\ 8 \end{bmatrix}\right) =$

**<u>Rotation</u>**

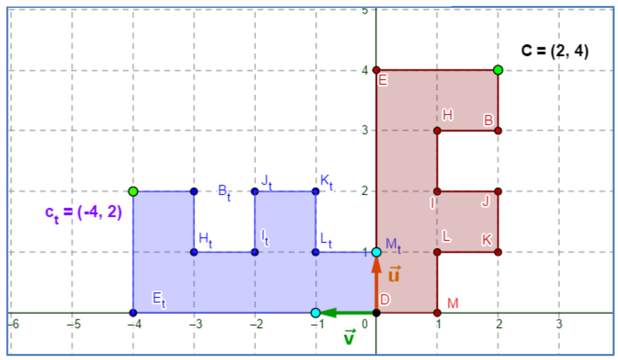
**Note:** $\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$ rotates $\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}$ by $\theta$ $\qquad$ $\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$ rotates $\vec{v} = \begin{bmatrix}0\\1\end{bmatrix}$ by $\theta$

$\mathbf{T}\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ is a linear transformation that rotates the vector $\begin{bmatrix}x\\y\end{bmatrix}$ by $\theta$

**For example**: Let $\theta = \frac{\pi R}{2}$ $\qquad$ $\mathbf{T}\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}\cos\left(\frac{\pi R}{2}\right) & -\sin\left(\frac{\pi R}{2}\right)\\ \sin\left(\frac{\pi R}{2}\right) & \cos\left(\frac{\pi R}{2}\right)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$




$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$




In the above linear transformation notice how the vecto**r** $\begin{bmatrix}2\\4\end{bmatrix}$ is transformed

$\mathbf{T}\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}-4\\2\end{bmatrix}$ $\qquad$ Therefore, **C = (2, 4)** transforms to $C_t$ = **(-4, 2)**

Find the linear transformation for rotating the polygon $\theta = \frac{5\pi R}{4}$

State the corresponding modified vectors $\vec{u}$ and $\vec{v}$

State the matrix form of the linear transformation created from these vectors.

Apply it to the vector $\begin{bmatrix}2\\4\end{bmatrix}$. $\qquad$ What vector is $\begin{bmatrix}2\\4\end{bmatrix}$ transformed to? $\qquad$ What point is **(2, 4)** transformed to?

To verify your results use the **Linear Transformation** applet.
Paste a copy of your results here.

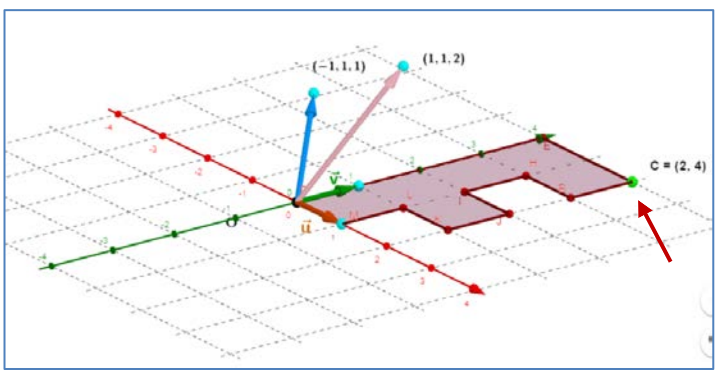
**Part 2** -- Consider Linear Transformations from $R^2$ to $R^3$

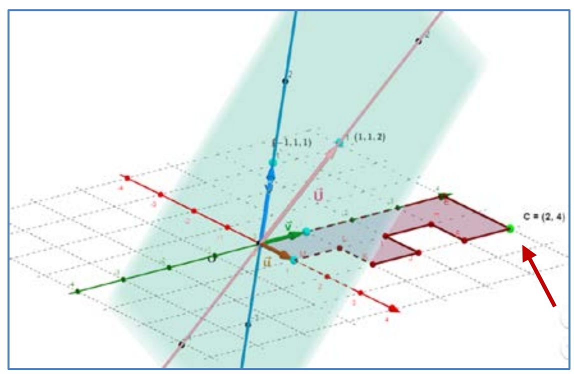
Verify the following matrix multiplication is a **linear** transformation $T: R^2 \rightarrow R^3$.
$$\boldsymbol{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

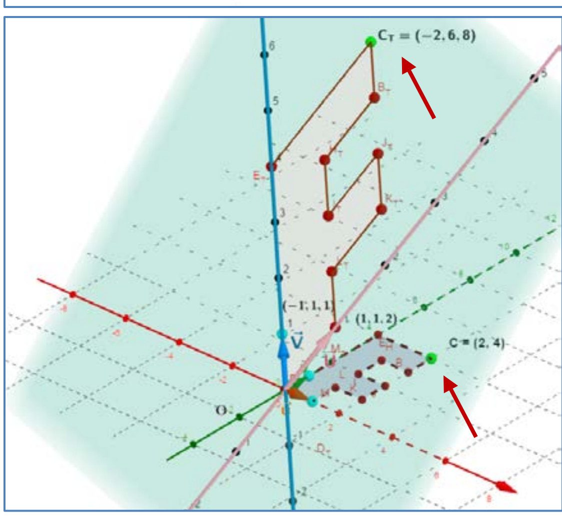
$$\boldsymbol{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1x - 1y \\ 1x + 1y \\ 2x + 1y \end{bmatrix}$$

$$\boldsymbol{T}\left(c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{bmatrix}\right) =$$

## Geometric meaning:




**Consider:** the linear transformation of the vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$\boldsymbol{T}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 8 \end{bmatrix}$$

Therefore,

**C = (2, 4)** is transformed to **$C_T$ = (-2, 6, 8)**

**Transforming Linear Algebra Education with GeoGebra Applets**
NSF TUES Grant Award ID: 1141045

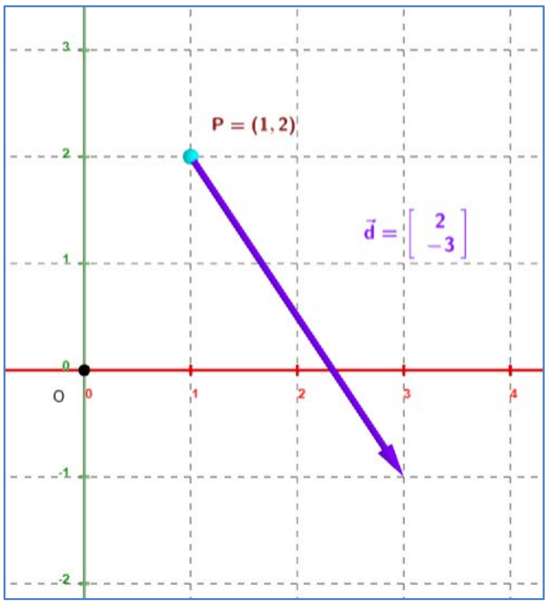
**<u>Example</u>: The Linear Transformation $T: R^{2} \rightarrow R^{3}$ of the Vector Form of a Line.**
(For examples use the applet in the GeoGebra Book: **Lines & Planes: Vector Form** )

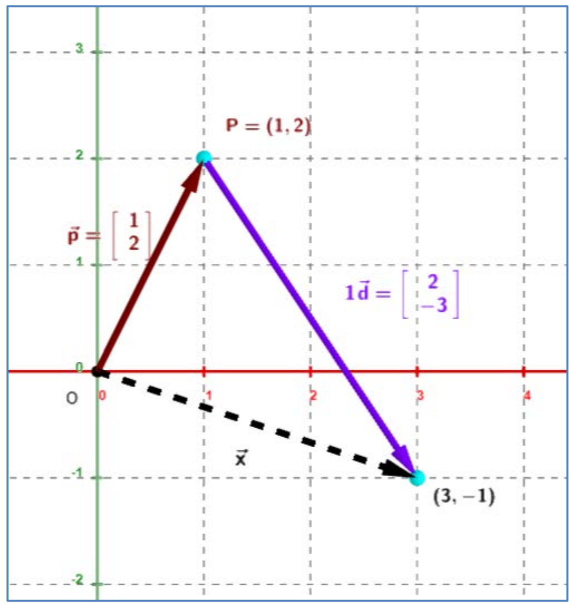
Given a point **P**, with vector representation $\vec{\mathbf{p}}$**,** and a direction vector $\vec{\mathbf{d}}$**,** the vector form of a line is defined by this linear combination:

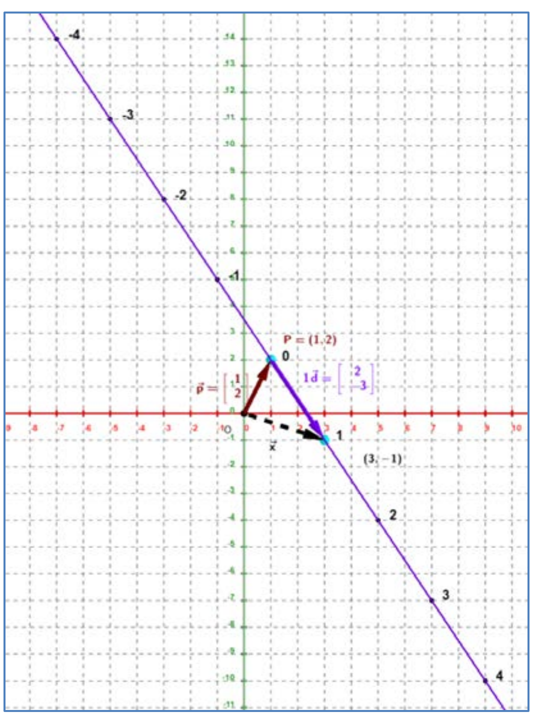
$$\vec{\mathbf{x}} = \vec{\mathbf{p}} + c \cdot \vec{\mathbf{d}} \quad -\infty < c < \infty.$$

Let the vectors $\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{\mathbf{p}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\vec{\mathbf{d}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Then this is the vector form of a line

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad -\infty < c < \infty$$



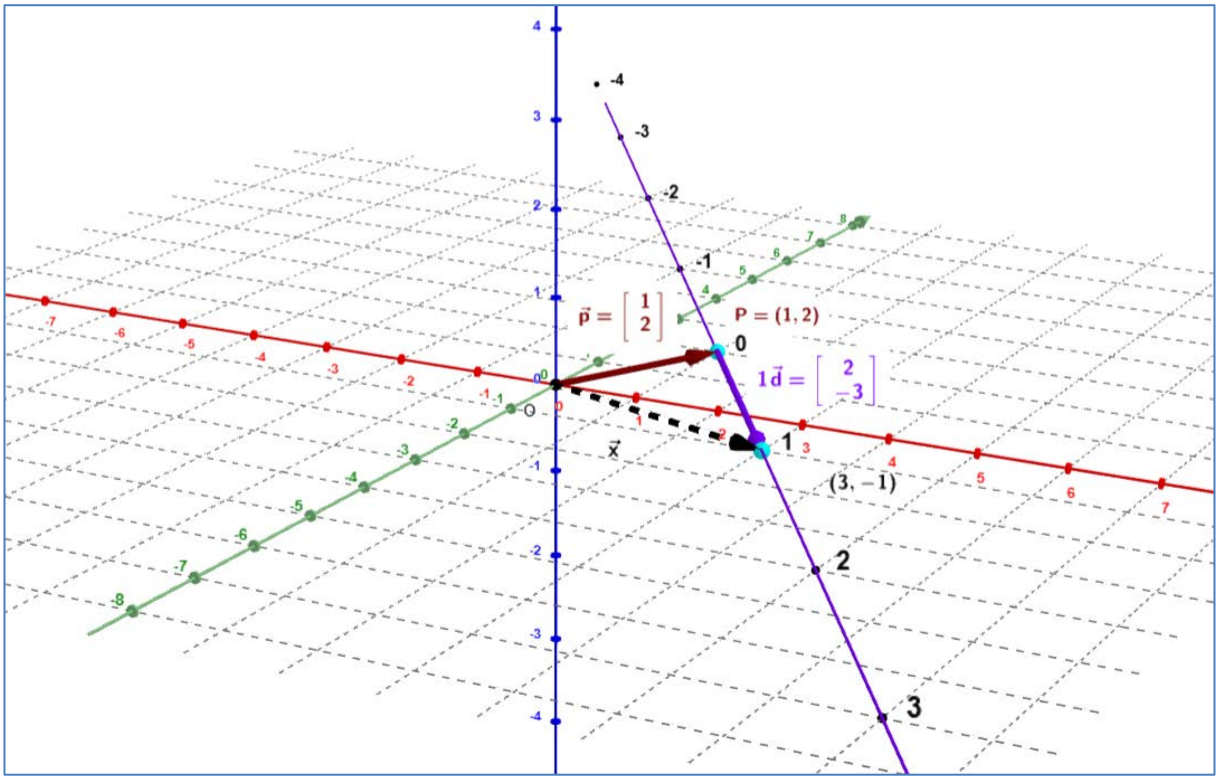



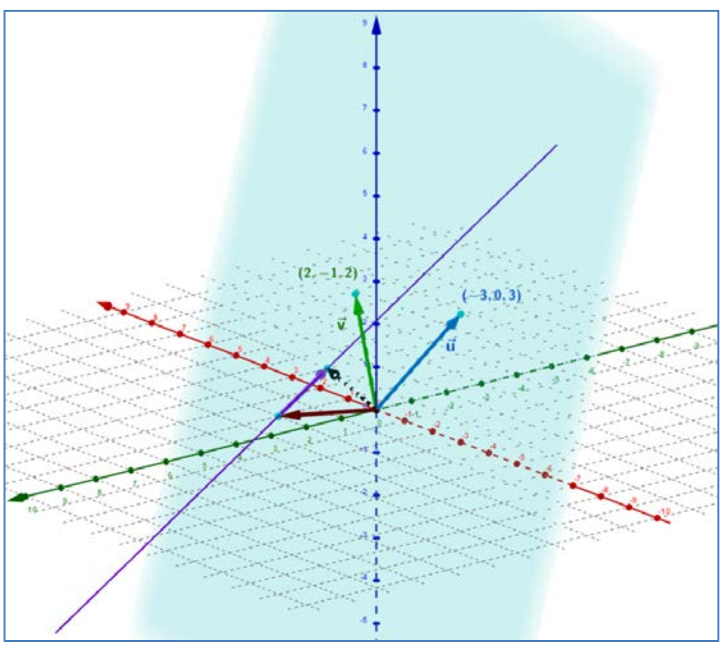

Consider:

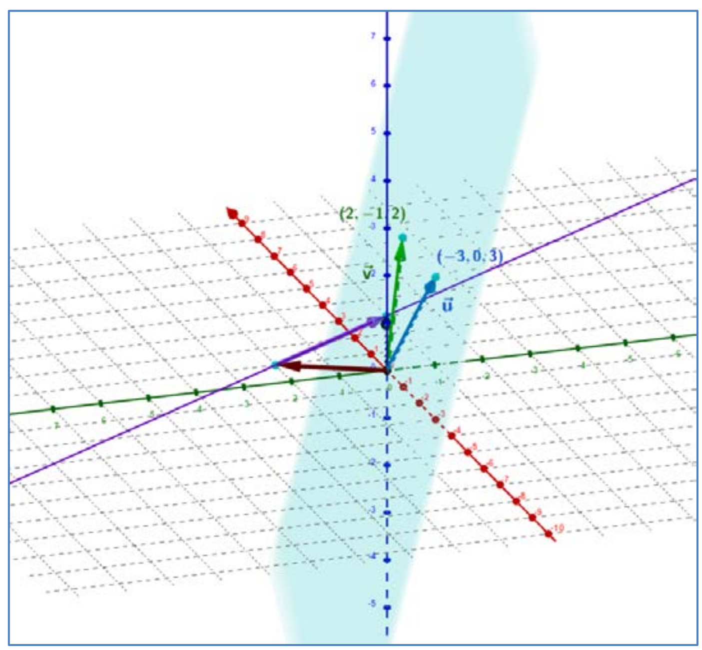
| $c$ | $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad -\infty < c < \infty$ | |
|---|---|---|
| -2 | $\begin{bmatrix} -3 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ | **point (-3, 8)** |
| -1 | $\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -1 \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ | **point (-1, 5)** |
| 0 | $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ | **point (1, 2)** |
| 1 | $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ | **point (3, -1)** |
| 2 | $\begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ | **point (5, -4)** |

View this line in the perspective of 3D space.




Consider two linearly independent vectors $\vec{\mathbf{u}} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ which define a plane.

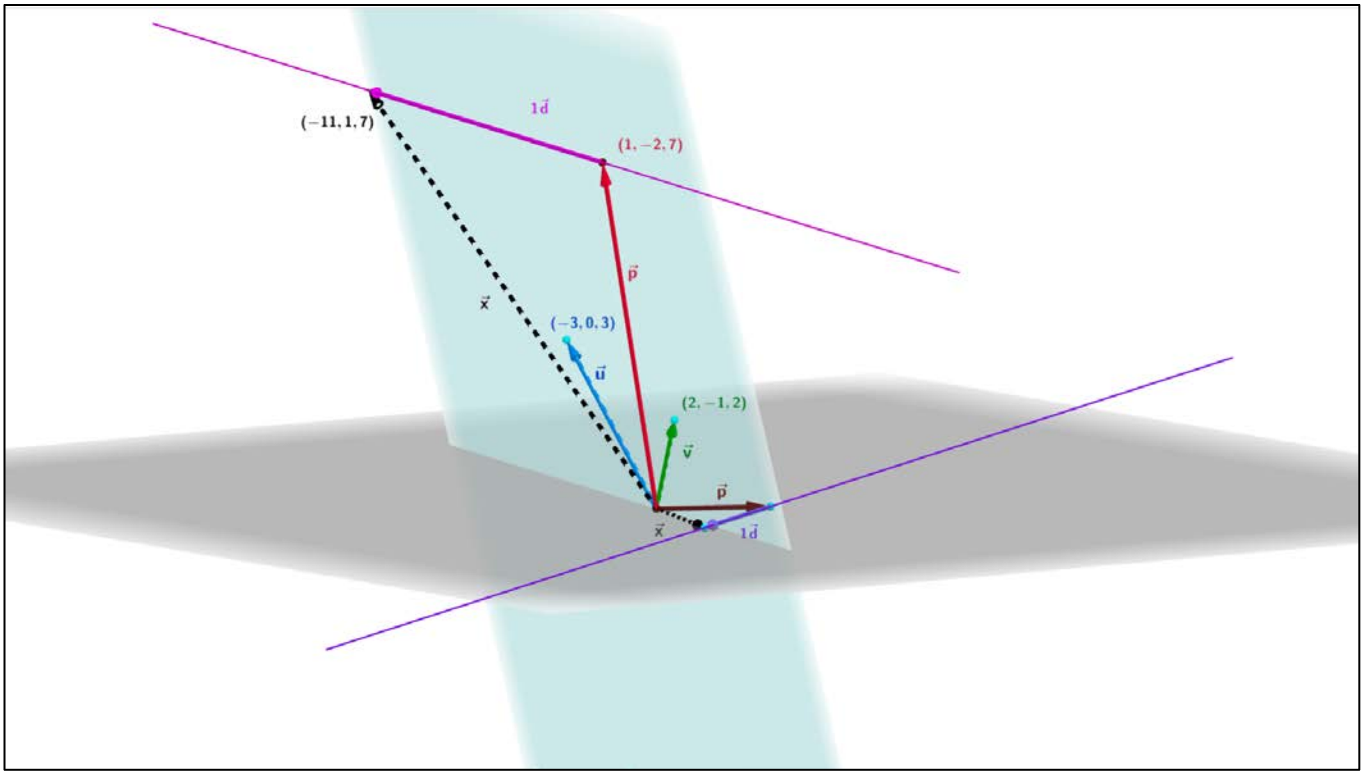


**Recall:** The vector form of a line is a *linear combination* and linear transformations preserve *linear combinations*.

The picture below is a view of the purple line, in vector form, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad -\infty < c < \infty$ in the xy-plane and its image is the magenta line, in vector form, in the $\vec{u}\,\vec{v}$-plane, whose basis vectors are $\vec{u} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Using these vectors as columns of a matrix gives us the linear transformation: $\mathbf{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3 & 2 \\ 0 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$




Use this linear transformation to determine the vector form of the line in the $\vec{u}\,\vec{v}$-plane.

$\mathbf{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{T}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) =$

**<u>Example</u>: The Linear Transformation $T: R^2 \rightarrow R^3$ of the Unit Circle.**

The unit circle can be defined as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(c) \\ \sin(c) \end{bmatrix}$, where $-\infty < c < \infty$ , where $c$ is the angle in radians. It is defined by the linear combination $\begin{bmatrix} x \\ y \end{bmatrix} = \cos(c)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(c)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

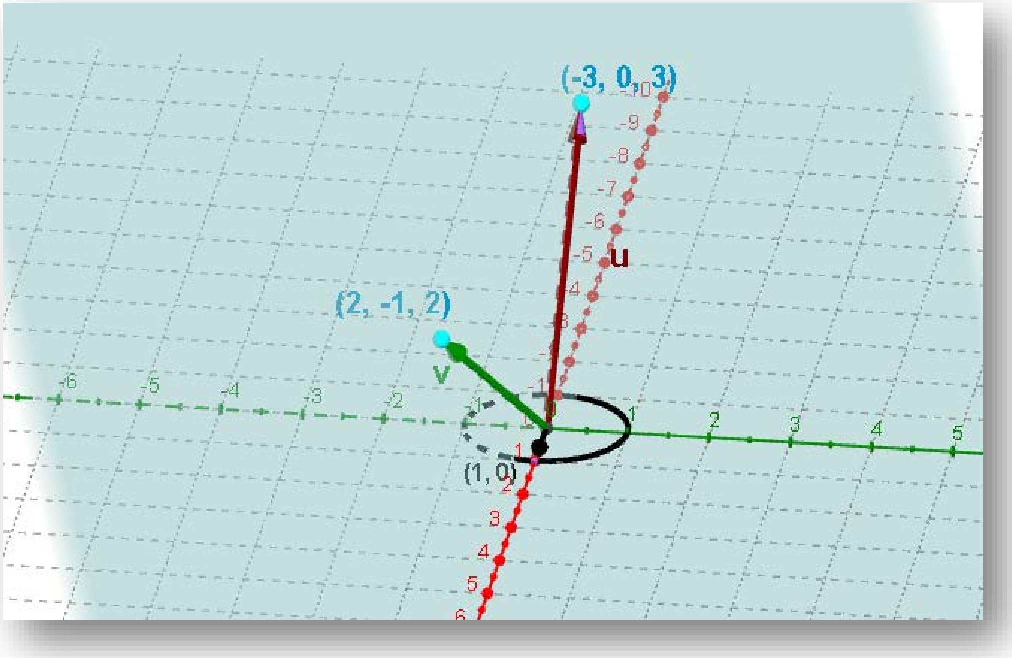
The unit circle in the xy-plane will be transformed into the $\vec{u}\,\vec{v}$-plane by the linear transformation

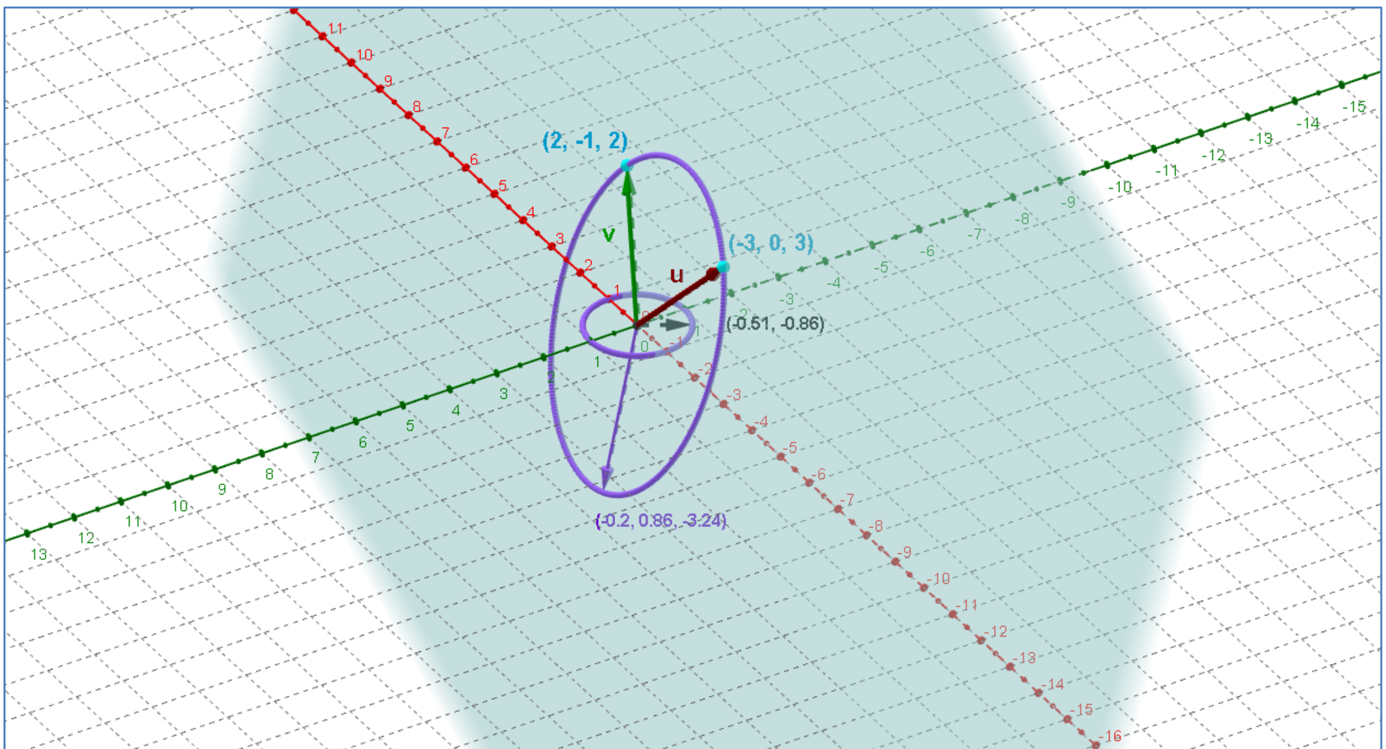
$\mathbf{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3 & 2 \\ 0 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ defined by the linearly independent vectors $\vec{u} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ .

$$\mathbf{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{T}\left(\begin{bmatrix} \cos(c) \\ \sin(c) \end{bmatrix}\right) = \begin{bmatrix} -3 & 2 \\ 0 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} \cos(c) \\ \sin(c) \end{bmatrix} = \cos(c)\begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} + \sin(c)\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Use the applet to see the figure that the unit circle is transformed into. Paste it here. Note it is an ellipse. Why?

Change the basis $\vec{u} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ so that the unit circle image is a unit circle in the $\vec{u}\,\vec{v}$-plane.
(Hint: use Gram-Schmidt Process)




Unit Circle is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(c) \\ \sin(c) \end{bmatrix}$, The unit circle is mapped by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ into the plane defined by $u = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

where $-\infty < c < \infty$,

$c = 4.18$

$$T\left(\begin{bmatrix} -0.51 \\ -0.86 \end{bmatrix}\right) = T\left(\begin{bmatrix} \cos(4.18) \\ \sin(4.18) \end{bmatrix}\right) = \begin{bmatrix} -3 & 2 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -0.51 \\ -0.86 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.86 \\ -3.24 \end{bmatrix}$$