[MAA 2.9] LOGARITHMS

SOLUTIONS

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O. Practice questions

1.

$\log_2 1 = 0$	$\log_2 2 = 1$	$\log_2 16 = 4$
$\log_5 1 = 0$	$\log_5 5 = 1$	$\log_5 25 = 2$
$\log_3 1 = 0$	$\log_3 3 = 1$	$\log_3 9 = 2$
$\log_3 27 = 3$	$\log_3 \frac{1}{3} = -1$	$\log_3 \sqrt{3} = \frac{1}{2}$

2.

$\log 100 = 2$	$\log 10 = 1$	$\log 1 = 0$
$\log \frac{1}{100} = -2$	$\log \frac{1}{10} = -1$	$\log 0.1 = -1$
$\log 10^{2020} = 2020$	$\log \sqrt{10} = \frac{1}{2}$	$\log \sqrt[3]{10} = \frac{1}{3}$

3.

$\ln 1 = 0$	$\ln \mathrm{e} = 1$	$\ln \mathrm{e}^2 = 2$
$\ln \frac{1}{\mathrm{e}} = -1$	$\ln \frac{1}{\mathrm{e}^2} = -2$	$\ln \sqrt{\mathrm{e}} = \frac{1}{2}$

4.

$\log_2 8 = x$	$x = 3$
$\log_2 x = 3$	$x = 8$
$\log_x 8 = 3$	$x = 2$

$\log 1000 = x$	$x = 3$
$\log x = 3$	$x = 1000$
$\ln x = 3$	$x = \mathrm{e}^3$

5.

$\log xy = \log x + \log y$	LHS = $\log 100000 = 5$
	RHS = $\log 1000 + \log 100 = 3 + 2 = 5$
$\log \frac{x}{y} = \log x - \log y$	LHS = $\log 10 = 1$
	RHS = $\log 1000 - \log 100 = 3 - 2 = 1$
$\log x^2 = 2 \log x$	LHS = $\log 1000000 = 6$
	RHS = $2 \log 1000 = 2 \times 3 = 6$

6. the answers for any set of 4 questions are 5, 5, 25 , 125

7. (a) C has equation $y = \log_2 x$

(b) Cuts x-axis $\Rightarrow \log_2 x = 0 \Rightarrow x = 2^{\circ} \Rightarrow x = 1$
Point is (1, 0)

8. (a) C has equation $y = \ln x$

(b) Cuts x-axis $\Rightarrow \ln x = 0 \Rightarrow x = \mathrm{e}^{\circ} \Rightarrow x = 1$
Point is (1, 0)

9.

$\log xy$	$= a + b$
$\log \frac{x}{y}$	$= a - b$
$\log x^3$	$= 3a$
$\log xyz$	$= a + b + c$
$\log x^2 y$	$= 2a + b$
$\log \sqrt{x}$	$= \frac{a}{2}$
$\log \frac{xy}{z}$	$= a + b - c$
$\log(10x)$	$= a + 1$
$\log(100x)$	$= a + 2$
$\log \frac{y}{10}$	$= b - 1$
$\log \frac{y}{100}$	$= b - 2$
$\log \frac{xy}{10z}$	$= a + b - c - 1$
$\log \frac{1}{z}$	$= -c$
$\log \frac{x^2 y^7}{\sqrt{z}}$	$= 2a + 7b - \frac{c}{2}$

10.

$\ln xy$	$= a + b$
$\ln \frac{x}{y}$	$= a - b$
$\ln x^3$	$= 3a$
$\ln xyz$	$= a + b + c$
$\ln x^2 y$	$= 2a + b$
$\ln \sqrt{x}$	$= \frac{a}{2}$
$\ln \frac{xy}{z}$	$= a + b - c$
$\ln(\mathrm{e}x)$	$= a + 1$
$\ln(\mathrm{e}^2 x)$	$= a + 2$
$\ln \frac{y}{\mathrm{e}}$	$= b - 1$
$\ln \frac{y}{\mathrm{e}^2}$	$= b - 2$
$\ln \frac{xy}{\mathrm{e}z}$	$= a + b - c - 1$
$\ln \frac{1}{z}$	$= -c$
$\ln \frac{x^2 y^7}{\sqrt{z}}$	$= 2a + 7b - \frac{c}{2}$

11.

$\log_5 xy$	$= a + b$
$\log_5 \frac{x}{y}$	$= a - b$
$\log_5 x^3$	$= 3a$
$\log_5 \sqrt{x}$	$= \frac{a}{2}$
$\log_5 \frac{xy}{z}$	$= a + b - c$
$\log_5 \frac{xy}{5z}$	$= a + b - c - 1$
$\log_5 \frac{1}{z}$	$= -c$
$\log_5 \frac{x^2 y^7}{\sqrt{z}}$	$= 2a + 7b - \frac{c}{2}$
$\log_{25} x$	$= \frac{a}{2}$
$\log_x 5$	$= \frac{1}{a}$
$\log_x y$	$= \frac{b}{a}$
$\log_z xy$	$= \frac{a+b}{c}$
$\log_{25} xy$	$= \frac{a+b}{2}$

12.

$\log_3(x+1) = 2$	$x+1 = 9 \Leftrightarrow x = 8$
$\log(x+1) = 2$	$x+1 = 100 \Leftrightarrow x = 99$
$\ln(x+1) = 2$	$x+1 = \mathrm{e}^2 \Leftrightarrow x = \mathrm{e}^2 - 1$

13.

$\log_7(x+5) = 0$	$x+5 = 1 \Leftrightarrow x = -4$
$\log_7(x+5) = 1$	$x+5 = 7 \Leftrightarrow x = 2$
$\log(x+5) = 0$	$x+5 = 1 \Leftrightarrow x = -4$
$\log(x+5) = 1$	$x+5 = 10 \Leftrightarrow x = 5$
$\ln(x+5) = 0$	$x+5 = 1 \Leftrightarrow x = -4$
$\ln(x+5) = 1$	$x+5 = \mathrm{e} \Leftrightarrow x = \mathrm{e} - 5$

14.

$\log(2x) = 2$	$2x = 100 \Leftrightarrow x = 50$
$\ln(2x) = 2$	$2x = \mathrm{e}^2 \Leftrightarrow x = \dfrac{\mathrm{e}^2}{2}$
$\log(2x+4) = 1$	$2x + 4 = 10 \Leftrightarrow x = 3$
$\ln(2x+4) = 1$	$2x + 4 = \mathrm{e} \Leftrightarrow x = \dfrac{\mathrm{e}-4}{2}$
$\log(2x-5) = 0$	$2x - 5 = 1 \Leftrightarrow x = 3$
$\ln(2x-5) = 0$	$2x - 5 = 1 \Leftrightarrow x = 3$

15. (a) $\log_2 x(x+1) = \log_2 6 \Leftrightarrow x(x+1) = 6 \Leftrightarrow x^2 + x - 6 = 0 \Leftrightarrow x = 2$ (–3 is rejected)

(b) $\log_2 x(x+1) = 1 \Leftrightarrow x(x+1) = 2 \Leftrightarrow x^2 + x - 2 = 0 \Leftrightarrow x = 1$ (–2 is rejected)

(c) $\log_2 \dfrac{x+5}{x} = 1 \Leftrightarrow \dfrac{x+5}{x} = 2 \Leftrightarrow x + 5 = 2x \Leftrightarrow x = 5$

16. (a) $\log x(x+1) = \log 6 \Leftrightarrow x(x+1) = 6 \Leftrightarrow x^2 + x - 6 = 0 \Leftrightarrow x = 2$ (–3 is rejected)

(b) $\log x(x+3) = 1 \Leftrightarrow x(x+3) = 10 \Leftrightarrow x^2 + 3x - 10 = 0 \Leftrightarrow x = 2$ (–5 is rejected)

(c) $\log \dfrac{x+18}{x} = 1 \Leftrightarrow \dfrac{x+18}{x} = 10 \Leftrightarrow x + 18 = 10x \Leftrightarrow 18 = 9x \Leftrightarrow x = 2$

17. (a) $\log_2(x+14) - 2\log_2 x = 2 \Leftrightarrow \dfrac{x+14}{x^2} = 4 \Leftrightarrow x + 14 = 4x^2 \Leftrightarrow 4x^2 - x - 14 = 0$

$x = 2$ ($x = -\dfrac{14}{8}$ is rejected)

(b) $x = 2$ (it is in fact the same equation as in (a))

(c) $x = 2$ (it is in fact the same equation as in (a))

18. In all three cases we obtain the quadratic $y^2 - 2y + 1 = 0 \Leftrightarrow y = 1$

(a) $x = e$ (b) $x = e$ (c) $x = 10$

A. Exam style questions (SHORT)

19. (a) 5

(b) **METHOD 1**

$$\log_2\left(\frac{32^x}{8^y}\right) = \log_2 32^x - \log_2 8^y = x\log_2 32 - y\log_2 8 = 5x - 3y$$

$p = 5, q = -3$

METHOD 2

$$\frac{32^x}{8^y} = \frac{(2^5)^x}{(2^3)^y} = \frac{2^{5x}}{2^{3y}} = 2^{5x-3y}$$

$\log_2 (2^{5x-3y}) = 5x - 3y$

$p = 5, q = -3$

20. $\log_{10}\left(\frac{P}{QR^3}\right)^2 = 2\log_{10}\left(\frac{P}{QR^3}\right) = 2(1\log_{10}P - \log_{10}Q - 3\log_{10}R) = 2(x - y - 3z) = 2x - 2y - 6z$

21. $\log\left(\frac{x^2\sqrt{y}}{z^3}\right) = 2\log x + \frac{1}{2}\log y - 3\log z = 2a + \frac{1}{2}b - 3c$

22. $\log_{10}\left(\frac{x}{y^2\sqrt{z}}\right) = \log_{10} x - 2\log y - \frac{1}{2}\log z = p - 2q - \frac{1}{2}r$

23. (a) $\ln a^3b = 3\ln a + \ln b = 3p + q$

(b) $\ln \frac{\sqrt{a}}{b} = \frac{1}{2}\ln a - \ln b = 1/2\ p - q$

24. (a) $\log_a 10 = \log_a (5 \times 2) = \log_a 5 + \log_a 2 = p + q$

(b) $\log_a 8 = \log_a 2^3 = 3\log_a 2 = 3q$

(c) $\log_a 2.5 = \log_a \frac{5}{2} = \log_a 5 - \log_a 2 = p - q$

25. (a) (i) $\log_c 15 = \log_c 3 + \log_c 5 = p + q$

(ii) $\log_c 25 = 2\log_c 5 = 2q$

(b) $d^{\frac{1}{2}} = 6 \Leftrightarrow d = 36$

26. (a) $\log_5 x^2 = 2\log_5 x = 2y$

(b) $\log_5 \frac{1}{x} = -\log_5 x = -y$

(c) $\log_{25} x = \frac{\log_5 x}{\log_5 25} = \frac{1}{2}y$

27. (a) $\log_2 5 = \frac{\log_a 5}{\log_a 2} = \frac{y}{x}$

(b) $\log_a 20 = \log_a 4 + \log_a 5 = 2\log_a 2 + \log_a 5 = 2x + y$

28. (a) $5^{x+1} = 5^4 \Leftrightarrow x + 1 = 4 \Leftrightarrow x = 3$

(b) $3x + 5 = a^2 \Leftrightarrow x = \frac{a^2 - 5}{3}$

29. $\log_2(x(x-2)) = 3 \Leftrightarrow x(x-2) = 2^3 \Leftrightarrow x^2 - 2x - 8 = 0 \Leftrightarrow (x-4)(x+2)$

$x = 4$

30. (a) $\log_3 x - \log_3 (x-5) = \log_3\left(\frac{x}{x-5}\right) \qquad A = \frac{x}{x-5}$

(b) $\log_3\left(\frac{x}{x-5}\right) = 1 \Leftrightarrow \frac{x}{x-5} = 3^1 (= 3) \Leftrightarrow x = 3x - 15 \Leftrightarrow x = \frac{15}{2}$

31. $\log_3(x+2) = 1 + \frac{\log_3 x}{2} \Leftrightarrow 2\log_3(x+2) = 2 + \log_3 x \Leftrightarrow \log_3 \frac{(x+2)^2}{x} = 2$

$\Leftrightarrow \frac{(x+2)^2}{x} = 9 \Leftrightarrow (x+2)^2 = 9x \Leftrightarrow x^2 - 5x + 4 = 0 \Leftrightarrow x = 1$ or $x = 4$

32. **METHOD 1**

$\log 81 + \log_9\left(\frac{1}{9}\right) + \log_9 3 = \log_9 x \Rightarrow \log_9\left[81\left(\frac{1}{9}\right)3\right] = \log_9 x$

$\Rightarrow \log_9 27 = \log_9 x \Rightarrow x = 27$

METHOD 2

$\log_9 81 + \log_9\left(\frac{1}{9}\right) + \log_9 3 = 2 - 1 + \frac{1}{2}$

$\Rightarrow \frac{3}{2} = \log_9 x \Rightarrow x = 9^{\frac{3}{2}} \Rightarrow x = 27$

33. $\log \frac{10x+20}{x^2} = 1 \Leftrightarrow \frac{10x+20}{x^2} = 10 \Leftrightarrow 10x + 20 = 10x^2 \Leftrightarrow x + 2 = x^2 \Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow x = 2$

34. $\log_2(4x \cdot x^2) = 5 \Leftrightarrow 4x^3 = 32 \Leftrightarrow x^3 = 8 \Leftrightarrow x = 2$

35. Given $\log_2(5x^2 - x - 2) = 2 + 2\log_2 x$

$\Rightarrow \log_2(5x^2 - x - 2) = \log_2 4 + 2\log_2 x$

$\Rightarrow \log_2(5x^2 - x - 2) = \log_2 4x^2$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow x = 2$

(the negative solution is rejected)

36. $16^{\frac{1}{2}} = \sqrt[3]{100 - x^2}$

$4 = \sqrt[3]{100 - x^2}$

$64 = 100 - x^2$

$x^2 = 36$, so $x = \pm 6$

37. $\log_{27}(x(x - 0.4)) = 1 \Leftrightarrow x^2 - 0.4x = 27 \Leftrightarrow x = 5.4$ or $x = -5$

So $x = 5.4$

OR directly by GDC $x = 5.4$

38.

$2\log_3(x-3) - \log_3(x+1) = 2$

$\log_3 \frac{(x-3)^2}{x+1} = 2$

$3^2 = \frac{(x-3)^2}{x+1}$

$9x + 9 = x^2 - 6x + 9$

$0 = x^2 - 15x$

$x = 15$

3.

39. $\log_2 x = \frac{\log_2(x+6)}{\log_2 4} \Leftrightarrow 2\log_2 x = \log_2(x+6) \Leftrightarrow \log_2 x^2 = \log_2(x+6) \Leftrightarrow x^2 = x + 6$

$x^2 - x - 6 = 0 \Leftrightarrow x = 3$ ($x = -2$ is rejected)

40.

$\ln(x+3)=1 \Rightarrow x=\mathrm{e}-3$

$\ln(x+3)=-1 \Rightarrow x=\frac{1}{\mathrm{e}}-3 \quad (=\mathrm{e}^{-1}-3)$

41. $2(\ln x)^2-3\ln x+1=0$

$\ln x=\frac{1}{2},\ \ln x=1$

$x=\sqrt{\mathrm{e}},\ x=\mathrm{e}$

42. $9\log_x 5=\log_5 x \Leftrightarrow 9\frac{1}{\log_5 x}=\log_5 x \Leftrightarrow (\log_5 x)^2=9 \Leftrightarrow \log_5 x=\pm 3 \Leftrightarrow x=5^{\pm 3}$

$x=125$ or $x=\frac{1}{125}$

43. $9\log_5 x=25\log_x 5 \Leftrightarrow 9\log_5 x=\frac{25}{\log_5 x} \Leftrightarrow (\log_5 x)^2=\frac{25}{9} \Leftrightarrow \log_5 x=\pm\frac{5}{3} \Leftrightarrow x=5^{\pm\frac{5}{3}}$

44. **METHOD A**

$9\log_8 x=6+8\log_x 8 \Leftrightarrow 9\frac{\log_2 x}{\log_2 8}=6+8\frac{\log_2 8}{\log_2 x} \Leftrightarrow 9\frac{\log_2 x}{3}=6+8\frac{3}{\log_2 x}$

$\Leftrightarrow 3\log_2 x=6+8\frac{3}{\log_2 x} \Leftrightarrow \log_2 x=2+\frac{8}{\log_2 x}$

Let $y=\log_2 x$

$y=2+\frac{8}{y} \Leftrightarrow y^2=2y+8 \Leftrightarrow y^2-2y-8=9$

$y=4,\ y=-2$

$\log_2 x=4 \Leftrightarrow x=16$

$\log_2 x=-2 \Leftrightarrow x=\frac{1}{4}$

METHOD B

Let $y=\log_8 x$. We similarly obtain the quadratic

$9y=6+\frac{8}{y} \Leftrightarrow 9y^2-6y-8=0$ …

$x=16,\ x=\frac{1}{4}$

45. $x=4,\ y=8$

46. $x=\frac{3}{22},\ y=\frac{24}{11}$

47. $x=1,\ y=3$ or $x=\frac{1}{4},\ y=\frac{3}{2}$

48. $x=64,\ y=16$

49. (a) $55\ln 2$ (b) $\frac{\ln 2[(1-(\ln 2)^{10}]}{1-\ln 2}$ (c) $\frac{\ln 2}{1-\ln 2}$

50. $\sum_{r=1}^{50} \ln(2^r) = \sum_{r=1}^{50} r(\ln 2)$

Arithmetic Sequence with $u_1 = \ln 2$ and $d = \ln 2$,

so $S_n = \frac{50}{2}(2\ln 2 + 49\ln 2) = 1275\ln 2$

OR $\sum_{r=1}^{50} r(\ln 2) = (\ln 2)\sum_{r=1}^{50} r = (\ln 2)\left(\left(\frac{50}{2}\right)51\right) = 1275\ln 2$

51. $\ln x^2 + \ln\frac{x^2}{y} + \ln\frac{x^2}{y^2} + \ln\frac{x^2}{y^3} + \ldots$

$= \ln x^2 + (\ln x^2 - \ln y) + (\ln x^2 - 2\ln y) + (\ln x^2 - 3\ln y) + \ldots$

Arithmetic Sequence with $u_1 = \ln x^2$ and $d = -\ln y$,

$S_{35} = \frac{n}{2}(2u_1 + (n-1)\mathrm{d}) = \frac{35}{2}(2\ln x^2 - 34\ln y) = 35\ln x^2 - 595\ln y^2$

$= \ln x^{70} - \ln y^{595}$

$= \ln\frac{x^{70}}{y^{595}}$ (Accept $m = 70$, $n = 595$)

52. (a) $f(x) = \ln\frac{x(x-2)}{x^2-4}$

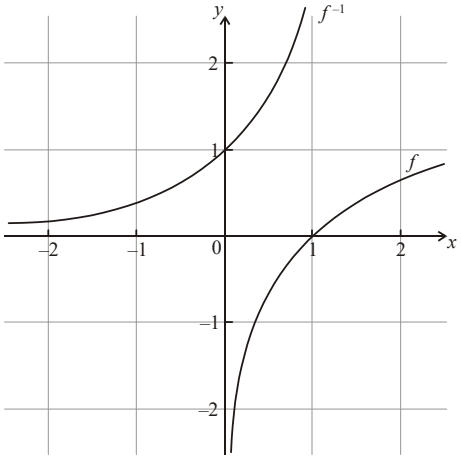
$= \ln\frac{x}{x+2}$ (Accept $a = 2$)

(b) $\ln\frac{x}{x+2} = y \Leftrightarrow \frac{x}{x+2} = \mathrm{e}^y \Leftrightarrow x = x\mathrm{e}^y + 2\mathrm{e}^y \Leftrightarrow x(1-\mathrm{e}^y) = 2\mathrm{e}^y \Leftrightarrow x = \frac{2\mathrm{e}^y}{1-\mathrm{e}^y}$

$f^{-1}(x) = \frac{2\mathrm{e}^x}{1-\mathrm{e}^x}$

53. (a) (i) $f(a) = 1$ (ii) $f(1) = 0$ (iii) $f(a^4) = 4$

(b)



54. (a) **METHOD 1**

$f(8) = 1 \Leftrightarrow 1 = k \log_2 8 \Leftrightarrow 1 = 3k \Leftrightarrow k = \frac{1}{3}$

METHOD 2

find the inverse of $f(x) = k \log_2 x$

$y = 2^{\frac{x}{k}}$

substituting 1 and 8

$2^{\frac{1}{k}} = 8 \Leftrightarrow k = \frac{1}{3}$

(b) **METHOD 1**

$f(x) = \frac{2}{3} \Leftrightarrow \frac{2}{3} = \frac{1}{3}\log_2 x \Leftrightarrow \log_2 x = 2 \Leftrightarrow x = 4 \quad f^{-1}\left(\frac{2}{3}\right) = 4$

METHOD 2

inverse of $f(x) = \frac{1}{3}\log_2 x$ is $f^{-1}(x) = 2^{3x} \quad f^{-1}\left(\frac{2}{3}\right) = 4$

55. (a) $\ln 5x^3 = \ln 5 + \ln x^3 = \ln 5 + 3\ln x$

$g(x) = f(x) + \ln 5$

(b) translation by $\begin{pmatrix} 0 \\ \ln 5 \end{pmatrix}$ **OR** shift up by ln 5 **OR** vertical translation of ln 5

56. (a) $\log_3 \sqrt{x} = y \Leftrightarrow \sqrt{x} = 3^y \Leftrightarrow x = 3^{2y}$

$f^{-1}(x) = 3^{2x}$

(b) $y > 0$

(c) **METHOD 1**

$g(2) = \log_3 2$

$(f^{-1} \circ g)(2) = f(x) = 3^{2\log_3 2} = 3^{\log_3 2^2} = 4$

METHOD 2

$(f^{-1} \circ g)(x) = 3^{2\log_3 x} = 3^{\log_3 x^2} = x^2$

$(f^{-1} \circ g)(2) = 4$

57. (a) $x = -1$

(b) (i) $f(-1.999) = \ln(0.001) = -6.91$ (ii) $g(4) = 1$

(c) (4.64, 1.89)

B. Exam style questions (LONG)

58. (a) $f^{-1}(x) = \ln x$

(b) $g^{-1}(x) = \dfrac{e^x - 1}{2}$

(c) (i) $(g \circ f)(x) = \ln(1 + 2e^x)$

(ii) $(f \circ g)(x) = e^{\ln(1+2x)} = 1 + 2x$

(d) $y = 1 + 2x \Leftrightarrow x = (y - 1)/2$

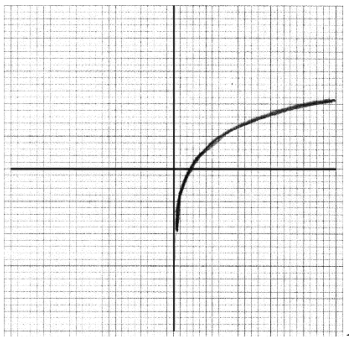
$(f \circ g)^{-1}(x) = \dfrac{x-1}{2}$

59. (a) $\log_3 \dfrac{1}{2}x + \log_3 4 = \log_3 \dfrac{4x}{2} = \log_3 2x$

(b) $f(0.5) = 0, f(4.5) = 2$

(c) (i) $a = 2, b = 3$

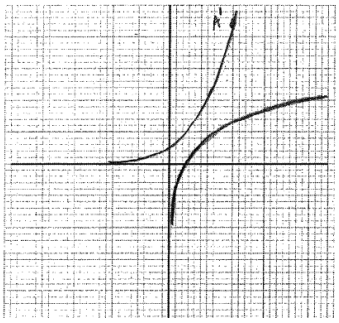
(ii)



(iii) $x = 0$ (must be an equation)

(d) $f^{-1}(0) = 0.5$

(e)



60. (a) $a = 135$

(b) 1.61 2.71 3.81 4.91

(c) all differences are 1.1 they are in arithmetic sequence

(d) 0.699 1.18 1.65 2.13

The differences are 0.481 0.47 0.48

The values are almost equal; he difference is due to rounding. In fact we still have an arithmetic sequence.

(e) a $3a$ $9a$

(f) $\ln 3a - \ln a = \ln \frac{3a}{a} = \ln 3$ $\ln 9a - \ln 3r = \ln \frac{9a}{3a} = \ln 3$ common difference = $\ln 3$

(g) If the sequence u_n is geometric then the sequence $v_n = \ln u_n$ is arithmetic

$u_n = u_1 r^{n-1}$ and $u_{n+1} = u_1 r^n$

$v_{n+1} - v_n = \ln v_{n+1} - \ln v_n = \ln u_1 r^n - \ln u_1 r^{n-1} = \ln \frac{u_1 r^n}{u_1 r^{n-1}} = \ln r$

So the common difference is $d = \ln r$

61. (a) $x^2 = 49$ $x = \pm 7$ $x = 7$

(b) $2^x = 8$ $x = 3$

(c) $x = 25^{-\frac{1}{2}}$ $x = \frac{1}{\sqrt{25}}$ $x = \frac{1}{5}$

(d) $\log_2 (x(x-7)) = 3$
$\log_2 (x^2 - 7x) = 3$
$2^3 = x^2 - 7x$
$x^2 - 7x - 8 = 0$
$(x-8)(x+1) = 0$ $(x = 8, x = -1)$
$x = 8$