A Junior Olympiad Geometry Problem

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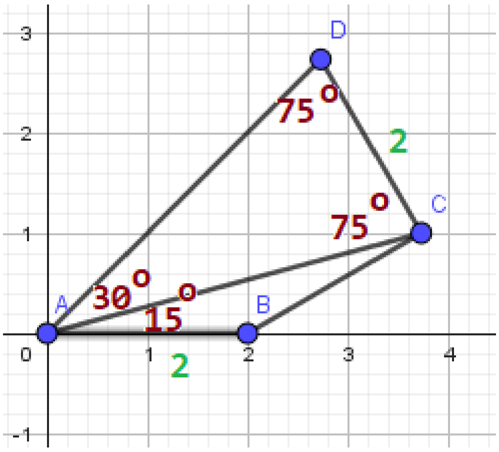
Problem

Let ABCD be a convex quadrilateral such that ∡BAD = 45°, ∡ADC = ∡ACD = 75° and AB = CD = 2. Find the length of BC.

Sketch

https://www.geogebra.org/m/qszdbxhx



If you make an accurate sketch and measure BC, it appears to be 2 units. If the answer must be a positive integer, then 2 is almost certainly correct.

Proof 1

Sine addition formula: $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$$\sin(75°) = \sin(45° + 30°)$$

$$= \sin(45°)\cos(30°) + \cos(45°)\sin(30°)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Similarly, $\sin(15°) = \sin(45° + (-30°))$

$$= \sin(45°)\cos(-30°) + \cos(45°)\sin(-30°)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{\sqrt{2}}\left(\frac{-1}{2}\right)$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Next, use the sine law in triangle ACD to get solve for AC:

$$\frac{AC}{\sin(75°)} = \frac{2}{\sin(30°)}$$

$$AC = \frac{2\sin(75°)}{\sin(30°)}$$

$= \frac{2(\frac{\sqrt{6}+\sqrt{2}}{4})}{\frac{1}{2}}$

$= \sqrt{6} + \sqrt{2}$

Finally, use the cosine law in triangle ABC to solve for BC:

$$\overline{BC}^2 = 2^2 + (\sqrt{6} + \sqrt{2})^2 - 2(2)(\sqrt{6} + \sqrt{2})\cos(15°)$$

We know $\cos(15°) = \sin(75°) = \frac{\sqrt{6}+\sqrt{2}}{4}$

Therefore, $\overline{BC}^2 = 4$

Hence, $\overline{BC} = 2$, as expected.

Proof 2

Find $AC = \sqrt{6} + \sqrt{2}$ as in Proof 1.

Convert the polar coordinates $C = (r, \theta)$ to cartesian coordinates (x, y)

$\cos(15°) = \frac{x}{\sqrt{6}+\sqrt{2}}$ $x = (\sqrt{6} + \sqrt{2})\cos(15°)$ $= (\sqrt{6} + \sqrt{2})\frac{\sqrt{6}+\sqrt{2}}{4}$ $= 2 + \sqrt{3}$	$\sin(15°) = \frac{y}{\sqrt{6}+\sqrt{2}}$ $y = (\sqrt{6} + \sqrt{2})\sin(15°)$ $= (\sqrt{6} + \sqrt{2})\frac{\sqrt{6}-\sqrt{2}}{4}$ $= 1$

C has the coordinates $(2 + \sqrt{3}, 1)$

Next, drop a perpendicular from C to the x-axis. Call the new point E.

In triangle BEC, BC is the hypotenuse

$\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2$

$= \sqrt{3}^2 + 1^2$

Hence, $\overline{BC} = 2$

Reflection

Is there a solution that starts by proving ∡ACD = 15°?

Then triangle ABC is isosceles and BC = 2 is trivial.