NOTES: VECTOR ANALYSIS FOR PHYSICS

VECTOR RESOLUTION

A vector has magnitude and "direction" but "direction" must be defined w.r.t. some coordinate system. We will most often use the usual Cartesian (rectangular) coordinates, with *x* horizontal and *y* vertical. However, in 3D work we would take *z* to be the vertical direction, perpendicular to the *x*-*y* plane.

The approach we will use for combining (or "resolving") multiple vectors into a single resultant is the *components* method. The components of a vector along the *x* and *y* axes are found using:

$$x_i = r_i \cos(\theta_i) \qquad y_i = r_i \sin(\theta_i) \tag{1}$$

where *r* is the magnitude (length) of the i-th vector, and *θ* is the angle (positive counter-clockwise from the positive *x* axis) of that vector. *Be careful to carry along the correct signs for these components*. For vectors with specific names, we use bold letters for the name (e.g., **A**), and then write the components as A_x and A_y. Notice that the components are not bold letters.

To resolve multiple vectors into a resultant vector, all we need to do is add up the components; thus

$$R = \sqrt{\left(\sum_{i=1}^{n} x_i\right)^2 + \left(\sum_{i=1}^{n} y_i\right)^2} \tag{2}$$

and

$$\theta_R = \tan^{-1}\left(\frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}\right) \tag{3}$$

are the magnitude and direction of the resultant vector **R**. This is simple enough. The only complication is that we must be careful about the signs of the sums in the argument of the arctan. These signs will determine the quadrant in which the resultant will be. **WARNING**: *The arctan function on your calculators will not directly produce the correct angle for all combinations of the signs.*

UNIT VECTORS

A unit vector, logically enough, has a length of one. Its only interesting property is its direction. A vector **A** pointing in any direction in 3D, can be made a unit vector as follows:

$$\mathbf{u}_A = \frac{\mathbf{A}}{\|\mathbf{A}\|} = \frac{\mathbf{A}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{4}$$

However, the unit vectors we will use most of the time are those aligned with the *x,y,z* axes. These are designated **i**, **j**, **k**, respectively. These three unit vectors are mutually orthogonal, which means that they are all perpendicular to each other. These vectors point in the positive *x,y,z* directions, along the axes.

When we decompose a vector into its *x,y,z* components it is then very simple and convenient to describe the vector using

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k} \tag{5}$$

Finding the components in 3D is difficult; we will not need to do this. For 3D problems you will be given the components directly, or the problem will be in 2D so that you can use Eq(1) to find them.

VECTOR PRODUCTS: DOT PRODUCT

There are two defined ways to multiply vectors. One way produces a scalar result, the other produces a vector result. The scalar product is the **dot product.** This is found using the *definition*

$$\mathbf{A} \bullet \mathbf{B} \equiv \|\mathbf{A}\| \, \|\mathbf{B}\| \cos(\theta) \tag{6}$$

where the double vertical bars mean the magnitude of that vector, and *θ* is the angle between the vectors. Note the use of bold letters to denote vectors. A simpler way to write this is

$$\mathbf{A} \bullet \mathbf{B} = A \; B \cos(\theta) \tag{7}$$

where it is understood that the non-bold letters denote the vector magnitudes. The dot product is widely used in mechanics, e.g., for finding the work done by a force.

VECTOR PRODUCTS: CROSS PRODUCT

Multiplication of two vectors can produce a third vector. This is the vector or **cross product**, which is written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{8}$$

and its magnitude is

$$C = A \; B \sin(\theta) = \|\mathbf{A} \times \mathbf{B}\| \tag{9}$$

The direction of the new vector **C** is found using the so-called *right-hand rule*. The vector **C** is perpendicular to the plane defined by **A** and **B**. That is easy enough, but **C** could point "up" or "down" in that direction. As your book says: "The four fingers of the right hand are pointed along **A** and then 'wrapped' into **B** through the angle *θ*." Then the direction of the thumb will point in the direction of **C**.

A very convenient way to evaluate a cross product, given the *x,y,z* components of the vectors, is

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \tag{10}$$

If the vectors **A**, **B** are in 2D in the *x*-*y* plane (zero *z* component), then this determinant simplifies to

$$\mathbf{A} \times \mathbf{B} = \left(A_x \, B_y - B_x \, A_y \right) \mathbf{k} \tag{11}$$

which is perpendicular to the *x*-*y* plane, as expected. The cross product appears in mechanics when we deal with rotational dynamics, namely, torque. (It is also used a lot in E&M.)

SOME VECTOR OPERATIONS AND RESULTS

$$\mathbf{A} \pm \mathbf{B} = \left(A_x \pm B_x \right) \mathbf{i} + \left(A_y \pm B_y \right) \mathbf{j} + \left(A_z \pm B_z \right) \mathbf{k} \tag{12}$$

$$k\,\mathbf{A} = k\,A_x\,\mathbf{i} + k\,A_y\,\mathbf{j} + k\,A_z\,\mathbf{k} \tag{13}$$

$$\mathbf{A} \bullet \mathbf{B} = \left(A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k} \right) \bullet \left(B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k} \right) = A_x B_x + A_y B_y + A_z B_z \tag{14}$$

$$\cos(\theta) = \frac{A_x\,B_x + A_y\,B_y + A_z\,B_z}{A\,B} = \frac{\mathbf{A} \bullet \mathbf{B}}{A\,B} \qquad \sin(\theta) = \frac{\|\mathbf{A} \times \mathbf{B}\|}{A\,B} \tag{15}$$