Transforming Linear Algebra Education with GeoGebra Applets
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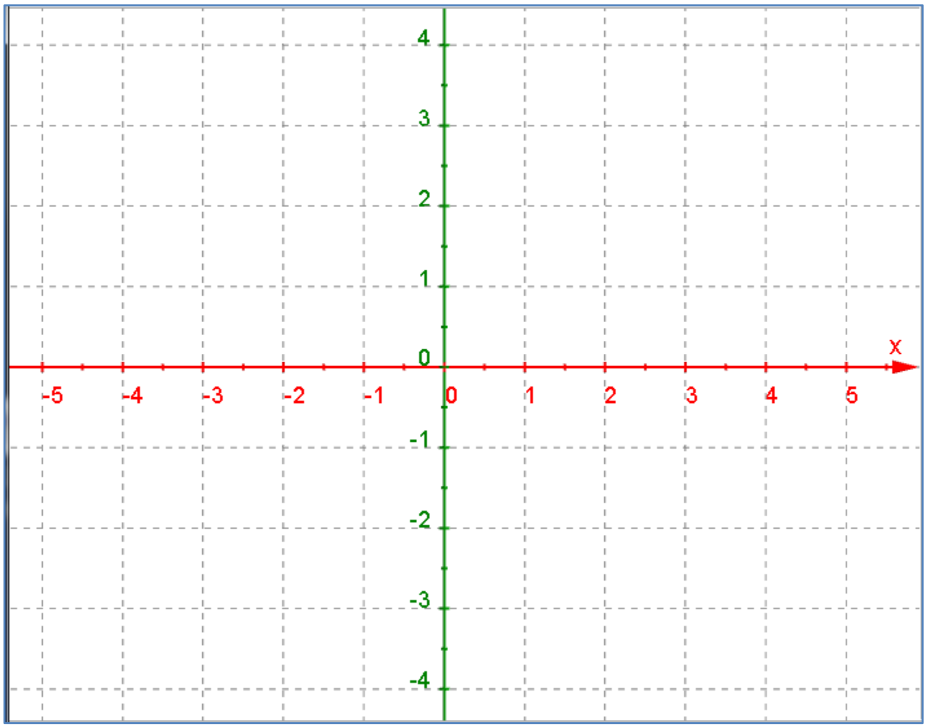
Gram-Schmidt Process - Activity

Orthonormal Vectors (Unit Vectors that are orthogonal - There is a 90° angle between the unit vectors)

The *orthogonal* vectors $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, form a basis for $\mathbb{R}^2$ Draw and label these vectors.



Prove that $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are *orthogonal* vectors, i.e. a 90° *angle is between* $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Use $\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos \boldsymbol{\theta}$

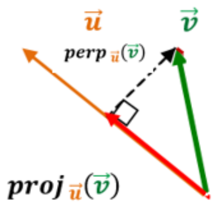
The *orthonormal basis* for $\mathbb{R}^2$ is obtained by finding the unit vector of each vector in the orthogonal set v
$\vec{\mathbf{u}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

Calculate the *orthonormal basis*. Graph and label it on the above graph.

Orthogonal Projection

Recall: In $\mathbb{R}^2$, the projection of a vector $\vec{v}$ onto a nonzero vector $\vec{u}$ is given by

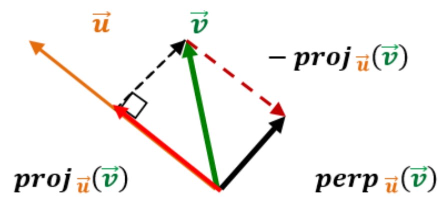
$$\boldsymbol{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$



Note:

$\boldsymbol{perp}_{\vec{u}}(\vec{v}) = \vec{v} - \boldsymbol{proj}_{\vec{u}}(\vec{v})$ is orthogonal to $\boldsymbol{proj}_{\vec{u}}(\vec{v})$ and therefore

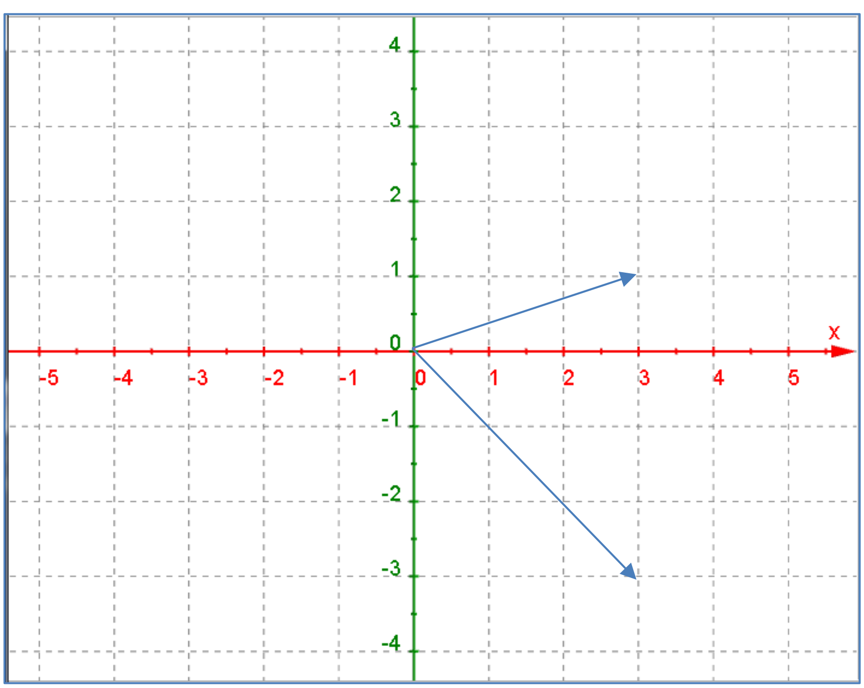
$$\vec{v} = \boldsymbol{proj}_{\vec{u}}(\vec{v}) + \boldsymbol{perp}_{\vec{u}}(\vec{v})$$



Therefore $\vec{u}$ and $\boldsymbol{perp}_{\vec{u}}(\vec{v})$ are orthogonal and $\vec{v}$ can be written as a linear combination of them.

Exercise: For the subspace spanned by the basis vectors $\vec{u} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, obtain an *orthogonal* basis that spans that same subspace.

Graph and label each vector.

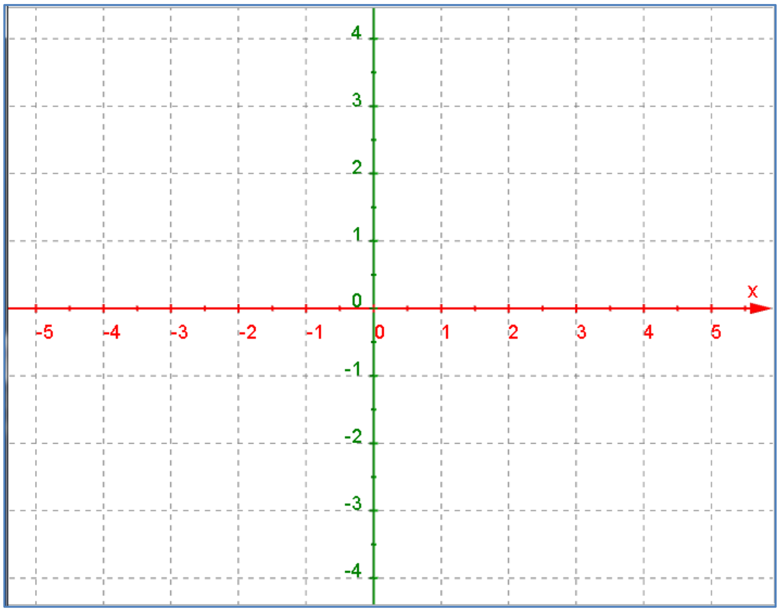


Steps: Obtaining an *orthogonal* basis.

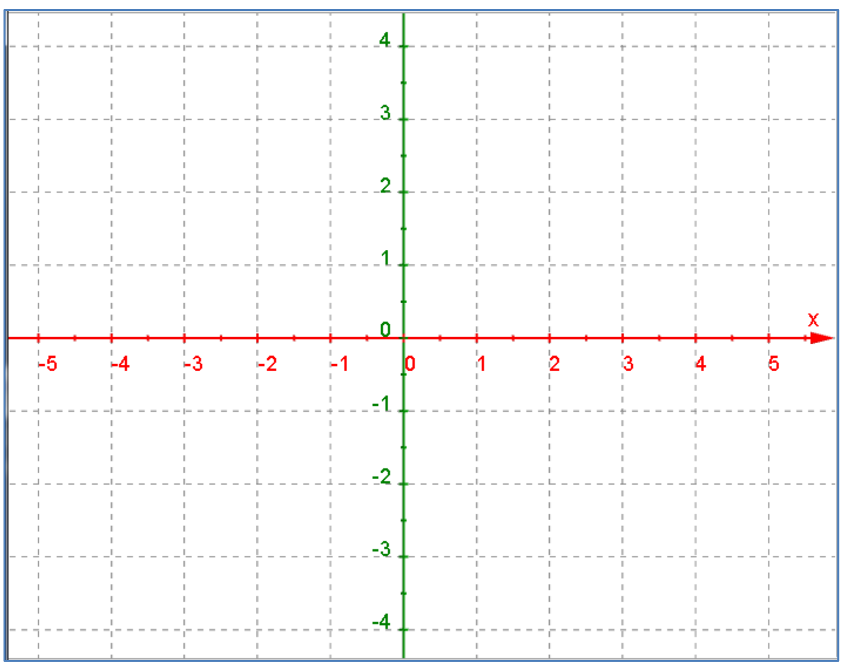
1. Algebraically, find the projection of $\vec{v}$ onto $\vec{u}$, i.e. $\boldsymbol{proj}_{\vec{u}}(\vec{v})$**.** Draw and label it on the graph.

2. Algebraically, find $\boldsymbol{perp}_{\vec{u}}(\vec{v})$**.** Draw and label it on the graph.

Draw and label the new *orthogonal* basis. Draw their grid lines



Draw and label the *orthonormal basis*. Draw their grid lines

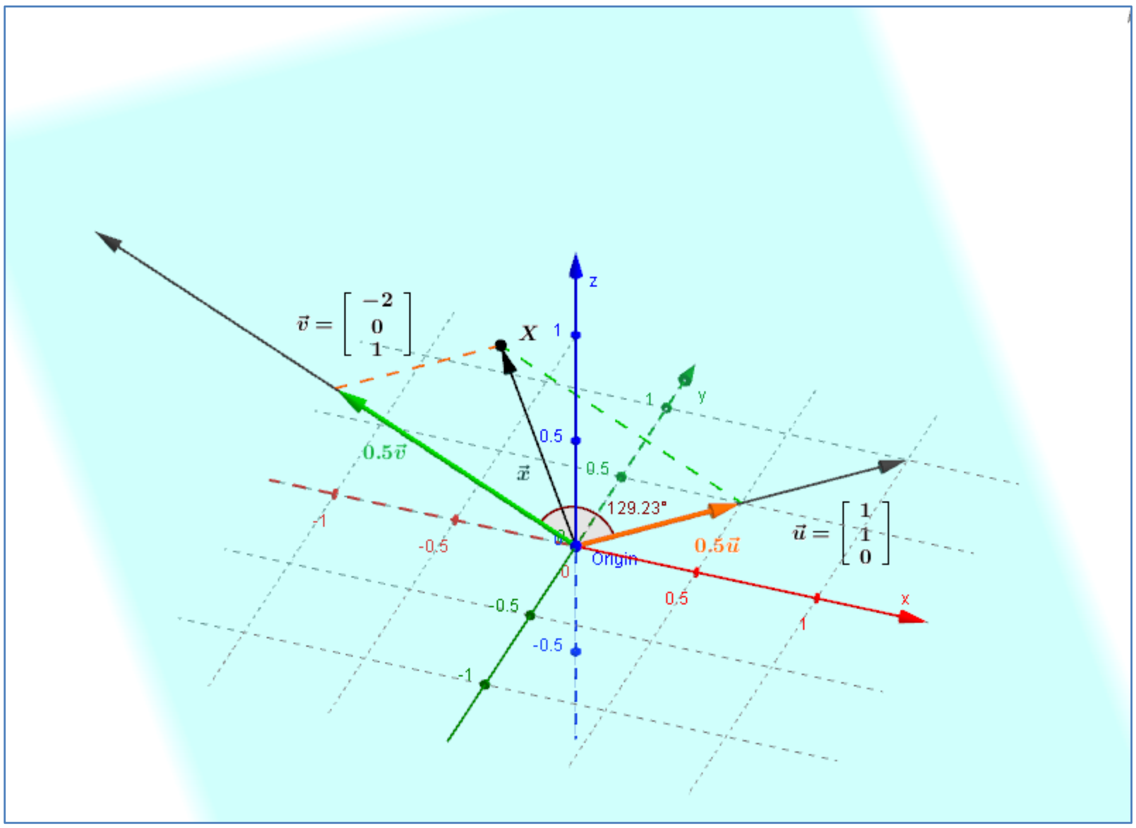


Obtain an *orthonormal* basis from the orthogonal basis. Draw and label *orthonormal* basis

The Gram-Schmidt Process

The Gram-Schmidt Process is a simple method for constructing an orthogonal (or orthonormal) basis for any subspace $\boldsymbol{W}$ of $\mathbb{R}^n$. The idea is to begin with an arbitrary basis $\{\vec{x}_1, \cdots, \vec{x}_k\}$ for $\boldsymbol{W}$ and to "orthogonalize" it one vector at a time.

<u>Example</u>: Let $\boldsymbol{W} = span(\vec{u}, \vec{v})$ where $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Construct an orthogonal basis for $\boldsymbol{W}$. Draw diagram.



Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ **and** $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

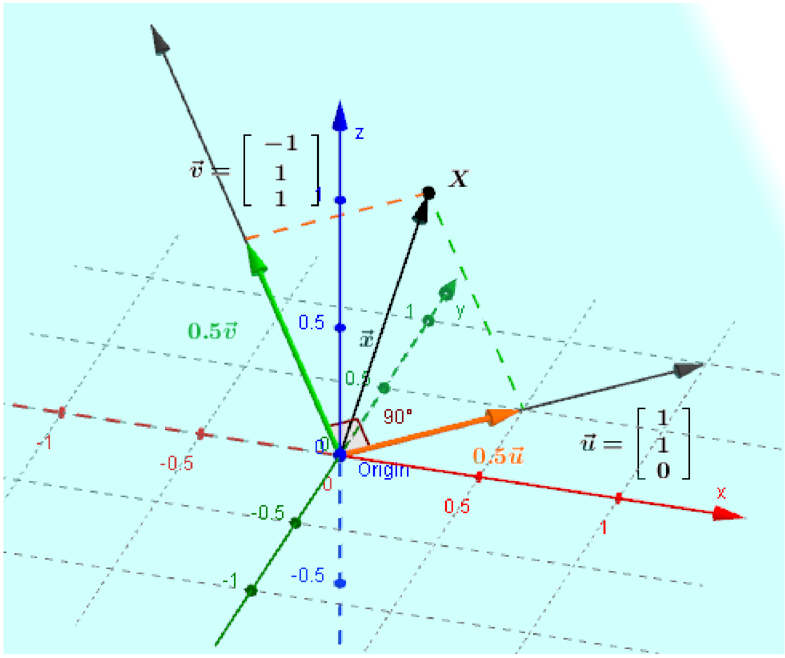
$$\vec{v}\,' = perp_{\vec{u}}(\vec{v}) = \vec{v} - proj_{\vec{u}}(\vec{v})$$

$$= \vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{-2}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\{\vec{u}, \vec{v}\,'\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal basis for $\boldsymbol{W}$



$\{\vec{u}, \vec{v}\,'\}$ **is an orthogonal set of vectors in** $\boldsymbol{W}$. Hence, $\{\vec{u}, \vec{v}\,'\}$ is a linearly independent set and therefore a basis for $\boldsymbol{W}$, since $\mathbf{dim}(\boldsymbol{W}) = \mathbf{2}$.

Note: This method depends on the order of the original basis vectors. If we still have had $\vec{u} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ we would have obtained a different orthogonal basis for $\boldsymbol{W}$. Verify this:

For $\vec{u} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{v}\,' = perp_{\vec{u}}(\vec{v}) = \vec{v} - proj_{\vec{u}}(\vec{v})$$

$$= \vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{-2}{5}\right)\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \left(\frac{1}{5}\right)\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} - \left(\frac{1}{5}\right)\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} = \left(\frac{1}{5}\right)\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1 \\ 2/5 \end{bmatrix}$$

$$\{\vec{u}, \vec{v}\,'\} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 1 \\ 2/5 \end{bmatrix} \right\}$$

Which is an orthogonal basis for $\boldsymbol{W}$.

They form a linearly independent set and therefore a basis for $\boldsymbol{W}$, since $\mathbf{dim}(\boldsymbol{W}) = \mathbf{2}$.

The Gram-Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$$\mathbf{v}_1 = \mathbf{x}_1, \qquad W_1 = \text{span}(\mathbf{x}_1)$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1, \qquad W_2 = \text{span}(\mathbf{x}_1, \mathbf{x}_2)$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2, \qquad W_3 = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

$$\vdots$$

$$\mathbf{v}_k = \mathbf{x}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_k}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_k}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{x}_k}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}, \qquad W_k = \text{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

Then for each $i = 1, \ldots, k$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W.