SIMPLE HARMONIC MOTION NOTES

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INTRODUCTION

Simple Harmonic Motion (SHM) results when a force acts on an object in such a manner that (1) the direction of the force is always towards the “rest” or equilibrium position of the object, and (2) the magnitude of the force is linearly proportional to the displacement from this rest position. Condition (1) means that the direction of the force changes during the object’s motion. Condition (2) means that the magnitude of the force is not constant.

Another way to look at this is from a potential-energy point of view. When the PE has a “well” shape, as in Fig. 1, then small departures from the stable equilibrium point at the bottom of the well will result in SHM. For many problems, the PE is a parabola, but for the pendulum it is a cosine function. The latter approximates a parabola for small displacements.

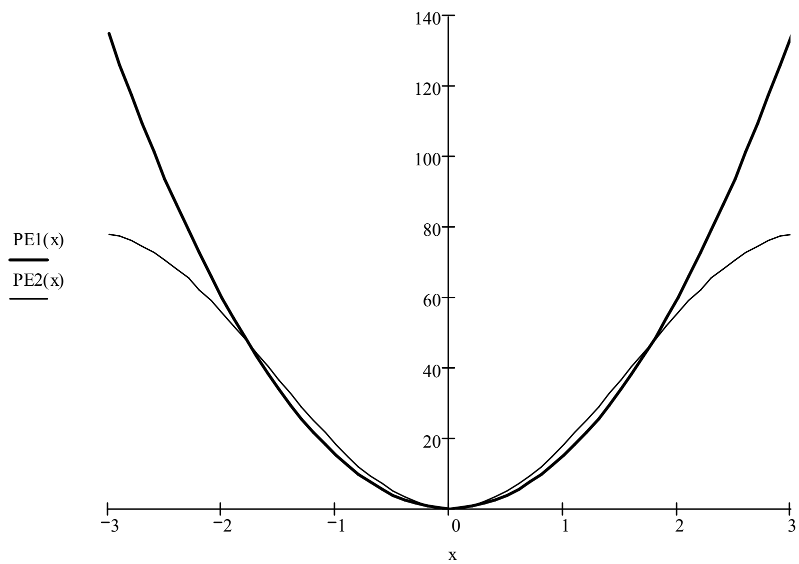


Figure 1 The potential-energy well; thick line is spring-mass, thin line is pendulum.

We will formulate the SHM problem in a general way, and then indicate how several specific physical systems map into that formulation. This will involve differential equations. Damping, proportional to the velocity of the motion, will be included. For analyses with no damping, we can set the damping constant to zero to recover the simpler solutions. One point of these notes is to show how the solution in the text, using a “phase angle” formulation, is related to the differential-equations solution. The latter does not directly produce the former.

DIFFERENTIAL EQUATION SOLUTION

The basic differential equation for the motion of an object experiencing a displacement-dependent force, with a linear, velocity-proportional damping, is

$$\frac{d^2x}{dt^2} + \alpha\,\frac{dx}{dt} + \beta\,x = 0 \tag{1}$$

The constant α includes a damping constant, and may be set to zero in some cases; there are also restrictions on its size, if we are to have periodic motion. The constant β will never be zero, and will change depending on the specific system under consideration. We use x for a generic displacement variable, which also will have various interpretations. More on how these symbols relate to various systems, below.

It is shown in differential equations that the solution to this second-order, linear, ordinary differential equation (ODE) consists of the sum of two terms. There are several ways to find these terms, including Laplace transforms. The text says "trial and error," which is not an acceptable approach. Carrying out the solution, we will find that

$$x(t) \;=\; \exp\left(-\frac{\alpha\, t}{2}\right)\left[c_1 \,\cos\left(\omega\, t\right) \;+\; c_2 \sin\left(\omega\, t\right)\right] \tag{2}$$

with

$$\omega \;=\; \sqrt{\beta \;-\; \left(\frac{\alpha}{2}\right)^2} \tag{3}$$

Note the constants c_1 and c_2 in Eq(2). These constants are determined by the initial conditions of the problem. We will need to specify the initial displacement and initial velocity in order to determine these two constants. Once we have done so, the solution is unique. First, consider the initial position of the object. At time zero,

$$x(0) \;=\; \exp\left(-\frac{\alpha\, 0}{2}\right)\left[c_1 \,\cos\left(0\right) \;+\; c_2 \sin\left(0\right)\right]$$

from which, clearly, $c_1 = x(0)$, or x_0, the object's initial position. For the velocity, we differentiate the solution (2), and again set the time to zero, to obtain

$$c_2 \;=\; \frac{v_0 \;+\; x_0 \dfrac{\alpha}{2}}{\omega}$$

where v_0 is the initial velocity. Then the solution is

$$x(t) \;=\; \exp\left(-\frac{\alpha\, t}{2}\right)\left[x_0 \,\cos\left(\omega\, t\right) \;+\; \frac{v_0 + x_0 \dfrac{\alpha}{2}}{\omega} \sin\left(\omega\, t\right)\right] \tag{4}$$

with ω given by Eq(3). This is a completely acceptable formulation; however, the text wants to express this as a single trig function, with a phase angle. Here's how we do that. Return to Eq(2), and let

$$c_1 \;=\; A \cos\left(\phi\right) \qquad c_2 \;=\; A \sin\left(\phi\right)$$

Then we can write

$$x(t) \;=\; \exp\left(-\frac{\alpha\, t}{2}\right)\left[A \cos\left(\phi\right) \cos\left(\omega t\right) \;+\; A \sin\left(\phi\right) \sin\left(\omega t\right)\right]$$

which is the same thing as

$$x(t) \;=\; A \exp\left(-\frac{\alpha\, t}{2}\right) \cos\left(\omega\, t - \phi\right)$$

Recalling that c_1 and c_2 have been determined (from the initial conditions), and so are known constants, then

$$c_1^2 + c_2^2 \;=\; A^2 \left[\sin^2\left(\phi\right) \;+\; \cos^2\left(\phi\right)\right]$$

from which we find the initial amplitude ("initial" because the amplitude decreases with time):

$$A \;=\; \sqrt{c_1^2 + c_2^2} \;=\; \sqrt{x_0^2 \;+\; \left(\frac{v_0 \;+\; x_0 \dfrac{\alpha}{2}}{\omega}\right)^2} \tag{5}$$

As for the phase angle ϕ, we can write

$$\frac{c_2}{c_1} = \frac{A\sin(\phi)}{A\cos(\phi)} = \tan(\phi)$$

and then

$$\phi = \tan^{-1}\left(\frac{c_2}{c_1}\right) = \tan^{-1}\left[\frac{v_0 + x_0\dfrac{\alpha}{2}}{\omega\, x_0}\right] \tag{6}$$

Now we have specified the motion of the object, using a single trig function, with parameters that have been completely determined by, and are now defined in terms of, the initial conditions of the problem. This analysis can also be done using the sine function, and it can be shown that the following solution forms are equivalent.

$$\boxed{x(t) = \exp\left(-\frac{\alpha t}{2}\right)\sqrt{x_0^2 + \left(\frac{v_0 + x_0\dfrac{\alpha}{2}}{\omega}\right)^2}\ \cos\left[\omega t - \tan^{-1}\left(\frac{v_0 + x_0\dfrac{\alpha}{2}}{\omega x_0}\right)\right]} \tag{7}$$

$$\boxed{x(t) = \exp\left(-\frac{\alpha t}{2}\right)\sqrt{x_0^2 + \left(\frac{v_0 + x_0\dfrac{\alpha}{2}}{\omega}\right)^2}\ \sin\left[\omega t + \tan^{-1}\left(\frac{\omega x_0}{v_0 + x_0\dfrac{\alpha}{2}}\right)\right]} \tag{8}$$

These are general solutions for the motion of an object undergoing SHM, with linear damping. The initial displacement x_0 and initial velocity v_0 are given; at least one of these must be nonzero (or else there is no motion). The damping constant α may be zero. Damped motion is, strictly, not periodic, since the motion does not follow the rule $f(t+T) = f(t)$ for any t (T is the period). However, the motion does have a "pseudo-period" or the time between successive maxima or minima.

For convenience, here are the simplified versions of the solutions. First, no damping:

$$x(t) = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}\ \cos\left[\omega t - \tan^{-1}\left(\frac{v_0}{\omega x_0}\right)\right] \tag{9}$$

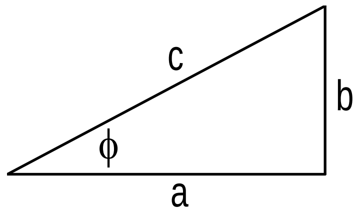
$$x(t) = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}\ \sin\left[\omega t + \tan^{-1}\left(\frac{\omega x_0}{v_0}\right)\right] \tag{10}$$

Next, no damping and no initial velocity:

$$x(t) = x_0\cos(\omega t) \tag{11}$$

$$x(t) = x_0\sin\left(\omega t + \frac{\pi}{2}\right) \tag{12}$$

It is useful to consider the phase angle in triangle form. Consider the sketch below.



$$a = x_0 \qquad b = \frac{v_0 + x_0 \frac{\alpha}{2}}{\omega} \qquad c = \sqrt{a^2 + b^2}$$

From this we see that, if there is no initial velocity, and no damping, the phase angle is zero, and the amplitude of the motion (c) is just the magnitude of the initial displacement. *Neither the amplitude nor the phase angle is arbitrary*- they are determined by the conditions of the motion (initial conditions and damping).

DERIVATIVES

The time derivatives of the motion are of interest. Using just the cosine form, Eq(7), we can write

$$v(t) = \frac{dx}{dt} = -A \exp\left(-\frac{\alpha t}{2}\right)\left[\omega \sin(\omega t - \phi) + \frac{\alpha}{2}\cos(\omega t - \phi)\right] \tag{13}$$

$$a(t) = \frac{d^2x}{dt^2} = -A \exp\left(-\frac{\alpha t}{2}\right)\left[\left(\omega^2 - \frac{\alpha^2}{4}\right)\cos(\omega t - \phi) - \omega\alpha \sin(\omega t - \phi)\right] \tag{14}$$

with A given by Eq(5) and ϕ given by Eq(6). Using zero for the time in Eq(13), and using the phase angle triangle, we will find that $v(0) = v_0$, as expected.

EXAMPLE PLOTS

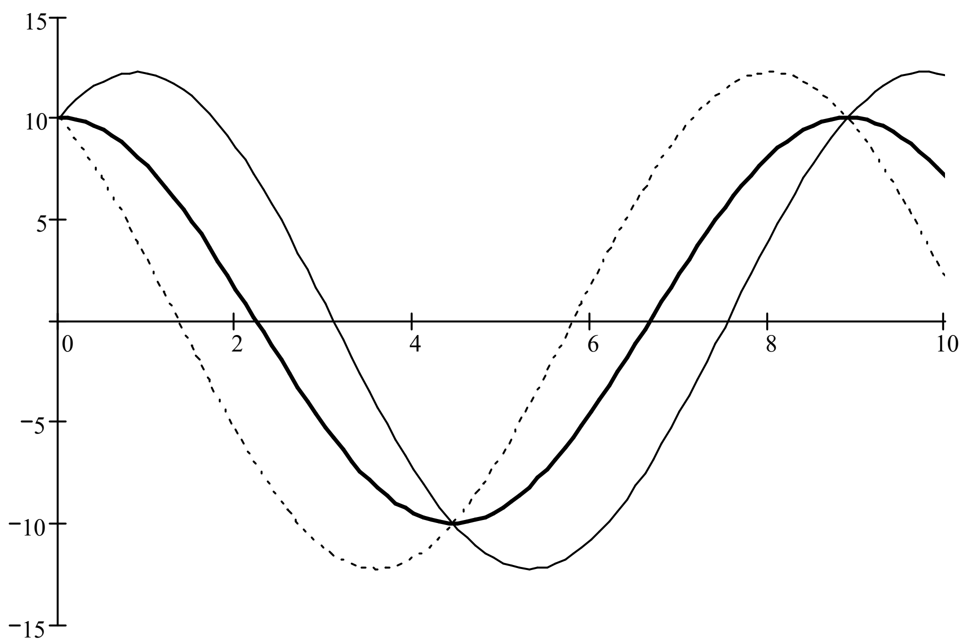


Figure 2. Example solution from Eq(7) for positive (thin line), negative (dotted line), and zero (thick line) initial velocity. Initial displacement 10 units for all three. Positive and negative magnitudes equal. No damping. Note increased amplitude for nonzero initial velocity cases, and the "phase angle" (shifting) effect.

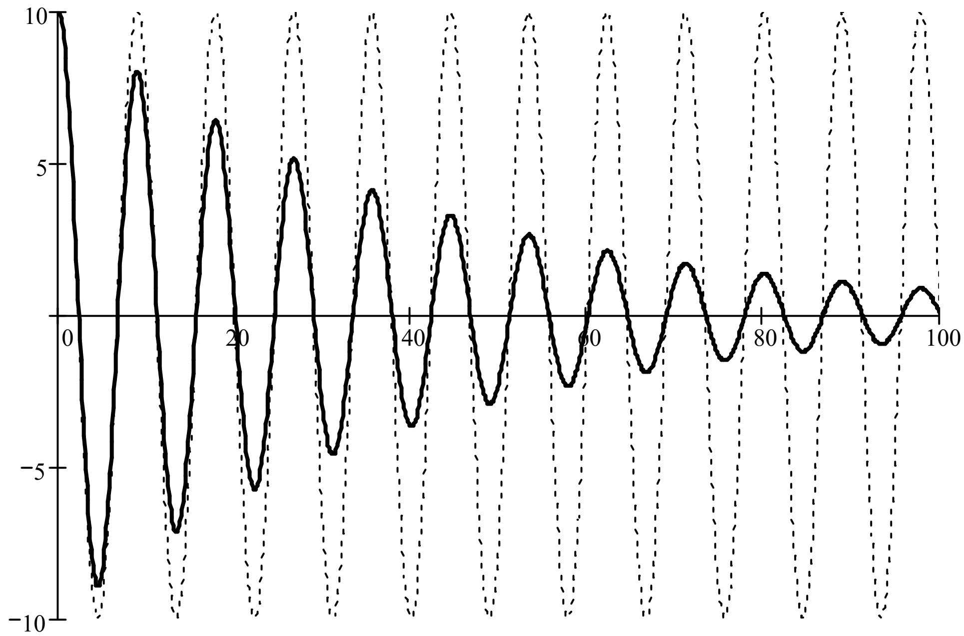


Figure 3. Example of damped motion, with undamped (dotted) motion for same initial displacement.

ENERGY BALANCE

We can analyze the motion of SHM, *when no damping is present*, using an energy balance. Damping is a dissipative loss, and so energy is lost from the system during the motion when damping is present (that's why the motion eventually stops). Consider a spring-mass system, with no damping. The total energy in the system will be

$$E_T = \frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2$$

which is the sum of the potential energy due to the initial displacement x_0 and the kinetic energy due to the initial velocity v_0. At any time, the sum of the kinetic and potential energies must equal this value. We can see from the displacement solutions, and their derivatives, that the kinetic energy is minimized at the "turning points" of the motion. That is, the mass stops (zero velocity) as it reaches an extreme of its displacement, since it will reverse its direction there. Thus, the potential energy is maximized at these points. Similarly, the potential energy is minimized when the mass is at the zero-displacement position, so that the kinetic energy must be maximized there. These relations can be seen from the equations of motion, but it is easier to just reason from conservation of energy. Similar arguments hold for the pendulum systems.

SPECIFIC SYSTEMS

Horizontal spring-mass (no friction)

$$F = m a = -k x - q \frac{dx}{dt} \;; \qquad \frac{d^2 x}{dt^2} + \alpha \frac{dx}{dt} + \frac{k}{m} x = 0$$

Here q is a damping constant, so that α is q / m, and β is k / m. For the more realistic case of sliding friction, rather than the artifice used here of linear damping, a solution has been developed. The ODE for that case is nonlinear, but is piecewise linear, so that piecewise solutions can be found.

Vertical spring-mass

It can be shown that this case is identical to the frictionless, but damped, horizontal case. The downforce due to gravity acts only to reset the rest (equilibrium) position, and the motion about that position is SHM, with the same parameters as the horizontal case. Here we might use y or z as the displacement variable.

Simple Pendulum

The force balance for a mass at the end of a rigid, massless bar can be shown to be

$$F = m\,a = -m\,g\sin(\theta)$$

where θ is the displacement angle, positive counterclockwise from the vertical. The negative indicates that the force is always directed toward the origin (zero angle, or the vertical). This balance leads to a nonlinear second-order ODE, and *the resulting motion is not SHM*. Solutions can be developed for this, and they involve some sophisticated mathematics (elliptic integrals). However, if the displacement angle θ is "small" then the sine of the angle is approximately equal to the angle (in radians, of course), so that we can write the linear ODE

$$m\,L\,\frac{d^2\theta}{dt^2} + q\,L\,\frac{d\theta}{dt} + m\,g\,\theta = 0$$

where L is the length of the bar. The factor L appears with the derivatives so that the motion in angle is converted to motion along an arc length. The displacement variable x is now θ; we will have $\alpha = q\,/\,m$ and $\beta = g\,/\,L$ and the rest of the math follows as before.

Physical Pendulum

In this case, the mass is not concentrated at a point, and we need to use the moment of inertia I. The ODE will be

$$I\,\frac{d^2\theta}{dt^2} + q\,\frac{d\theta}{dt} + m\,g\,d\,\sin(\theta) = 0$$

which as for the simple pendulum is nonlinear. The parameter d is the distance from the CG to the pivot point. Again we restrict the motion to "small" angles, so that we can linearize the ODE. The solution follows as before, and we will have the angular frequency

$$\omega = \sqrt{\frac{m\,g\,d}{I}}$$

which can be used to estimate the moment of inertia, given observation of the period.