*INTERNATIONAL BACCALAUREATE*

***Mathematics: analysis and approaches***

# *MAA*

***EXERCISES [MAA 5.5-5.6]***

***MONOTONY AND CONCAVITY***

*Compiled by Christos Nikolaidis*

## O. Practice questions

**1.** [Maximum mark: 12] ***[without GDC]***

Differentiate each of the functions below and hence determine whether it is **increasing** or **decreasing** on its entire domain.

(i) $f(x)=x^3+x+5$ (ii) $f(x)=5-5x^5$

(iii) $f(x)=5-3\mathrm{e}^{2x}$ (iv) $f(x)=\dfrac{2x+1}{3x+5}$

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**2.** [Maximum mark: 12] ***[without GDC]***

Consider the function $f(x) = x^3 + 3x^2 - 9x$ which passes through the origin.

(a) Find any stationary points and determine their nature. [6]

(b) Find any points of inflexion and justify your answer. [3]

(c) Sketch the graph of the function. (Use your GDC to check the result). [3]

**3.** [Maximum mark: 12] ***[without GDC]***

Consider the function $f(x) = x^3 + 3x^2 + 3x$

(a) Find any stationary points and determine their nature. [6]

(b) Find any points of inflexion and justify your answer. [3]

(c) Sketch the graph of the function. (Use your GDC to check the result). [3]

**4.** [Maximum mark: 5] ***[without GDC]***

It is given that $f'(x) = (x-1)(x-3)(x-4)^2$.

Find the stationary points of $f$ and determine their nature.

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**5.** [Maximum mark: 4] ***[without GDC]***

It is given that $f''(x) = (x-1)(x-3)(x-4)^2$.

Find the points of inflexion of $f$; justify your answer.

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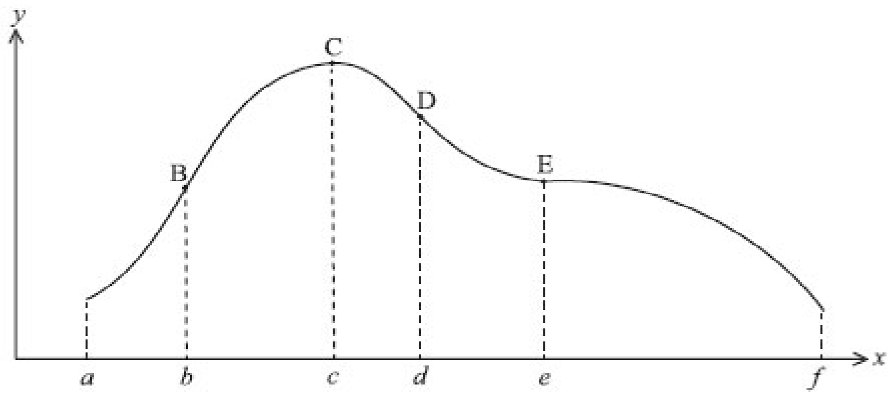
**6.** [Maximum mark: 7] ***[without GDC]***

The graph of a function $g$ is given in the diagram below.




The gradient of the curve has its maximum value at point B and its minimum value at point D. The tangent is horizontal at points C and E.

(a) Complete the table below, by stating, for each iterval, whether

the first derivative $g'$ is **positive** or **negative**;

the second derivative $g''$ is **positive** or **negative**.

| Interval | $g'$ | $g''$ |
|---|---|---|
| $a < x < b$ | **positive** | **positive** |
| $b < x < \mathrm{c}$ | | |
| $c < x < d$ | | |
| $d < x < \mathrm{e}$ | | |
| $e < x < f$ | | |

[4]

(a) Complete the table below, by stating for each point, whether

the first derivative $g'$ is **positive**, **negative** or **zero**,

the second derivative $g''$ is **positive**, **negative** or **zero**.

| | Point | $g'$ | $g''$ |
|---|---|---|---|
| B | $x = b$ | **positive** | **zero** |
| C | $x = c$ | | |
| D | $x = d$ | | |
| E | $x = e$ | | |

[3]

**7.** [Maximum mark: 12] ***[with GDC]***

Let $f(x) = ax^3 + bx^2 + cx$

(a) Find the first and the second derivative of $f(x)$, in terms of $a$, $b$, $c$. [4]

(b) The graph

passes through the point $P(1,4)$

has a local maximum at $P$

has a point of inflexion at $x = 2$.

Write down three linear equations representing this information. [3]

(c) **Hence** find the values of $a$, $b$, $c$. [2]

(d) The function has a local minimum at $x = d$. Find the value of $d$ and justify that it is a minimum. [3]

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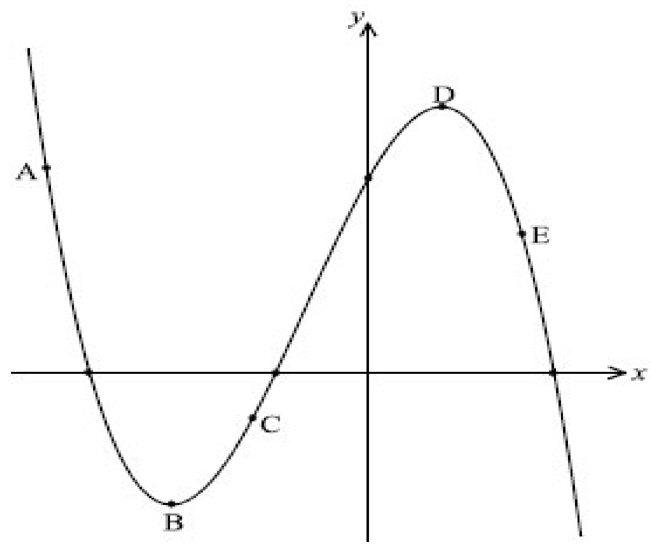
**8.** [Maximum mark: 12] ***[without GDC]***

Consider the function $f : x \mapsto x^2 e^x$. Find any max/min points, points of inflection and hence sketch the graph of the function.

## A. Exam style questions (SHORT)

**9.** [Maximum mark: 7] ***[without GDC]***

The following diagram shows part of the curve of a function $f$. The points A, B, C, D and E lie on the curve, where B is a minimum point and D is a maximum point.




(a) Complete the following table, noting whether $f'(x)$ is positive, negative or zero at the given points.

| | A | B | E |
|---|---|---|---|
| $f'(x)$ | | | |

[2]

(b) Complete the following table, noting whether $f''(x)$ is positive, negative or zero at the given points.

| | A | C | E |
|---|---|---|---|
| $f''(x)$ | | | |

[2]

(c) Complete the following table, noting whether each value is positive, negative or zero.

| $f(0)$ | $f'(0)$ | $f''(0)$ |
|---|---|---|
| | | |

[2]

(d) Write down the number of points of inflexion for this curve. [1]

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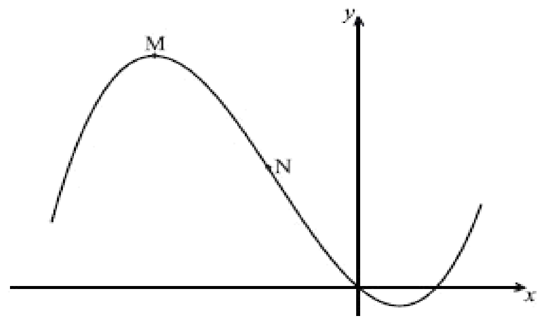
**10.** [Maximum mark: 11] ***[without GDC]***

Consider $f(x) = \frac{1}{3}x^3 + 2x^2 - 5x$. Part of the graph of $f$ is shown below. There is a maximum point at M, and a point of inflexion at N.




(a) Find $f'(x)$ [2]

(b) Find the $x$ - coordinate of M. [3]

(c) Find the $x$ - coordinate of N. [3]

(d) The line $L$ is the tangent to the curve of $f$ at $(3,\ 12)$. Find the equation of $L$ in the form $y = ax + b$. [3]

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**11.** [Maximum mark: 8] ***[without GDC]***

Let $g(x) = x^3 - 3x^2 - 9x + 5$ .

(a) Find the two values of $x$ at which the tangent to the graph of $g$ is horizontal. [5]

(b) For each of these values, determine whether it is a maximum or a minimum. [3]

**12.** [Maximum mark: 5] ***[with GDC]***

Let $f'(x) = -24x^3 + 9x^2 + 3x + 1$ .

(a) There are two points of inflexion on the graph of $f$ . Write down the $x$-coordinates of these points. [3]

(b) Let $g(x) = f''(x)$ . Explain why the graph of $g$ has no points of inflexion. [2]

**13.** [Maximum mark: 7] ***[without GDC]***

Consider the function $f(x)=\dfrac{3x-2}{2x+5}$.

The graph of this function has a vertical and a horizontal asymptote.

(a) Write down the equations of the asymptotes [2]

(b) Find $f'(x)$, simplifying the answer as much as possible. [2]

(c) Write down the number of stationary points of the graph. Justify your answer. [2]

(d) Write down the number of points of inflexion of the graph. [1]

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**14.** [Maximum mark: 5] ***[without GDC]***

A function $f$ has its first derivative given by $f'(x)=(x-3)^3$.

(a) Find the second derivative. [2]

(b) Find $f'(3)$ and $f''(3)$. [1]

(c) Explain why the point P on the graph with $x$-coordinate 3 is not a point of inflexion. [2]

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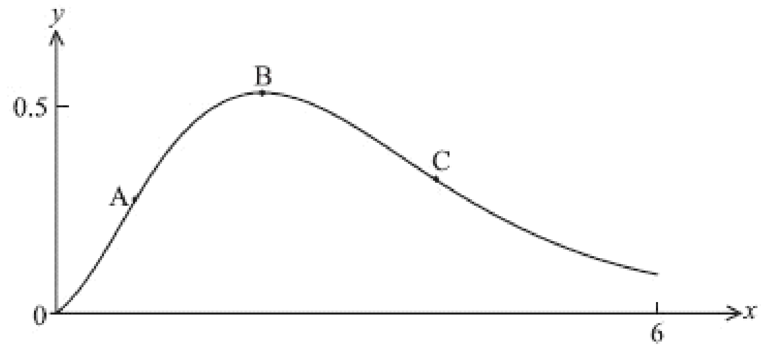
**15.** [Maximum mark: 8] ***[without GDC]***

The diagram below shows the graph of $f(x)=x^2e^{-x}$ for $0\le x\le 6$. There are points of inflexion at A and C and there is a maximum at B.




(a) Using the product rule for differentiation, find $f'(x)$. [2]

(b) Find the **exact** value of the ***y*-coordinate** of B. [2]

(c) (i) Show that $f''(x)=(x^2-4x+2)e^{-x}$.

(ii) **Hence**, find the **exact** value of the $x$- coordinate of C. [4]

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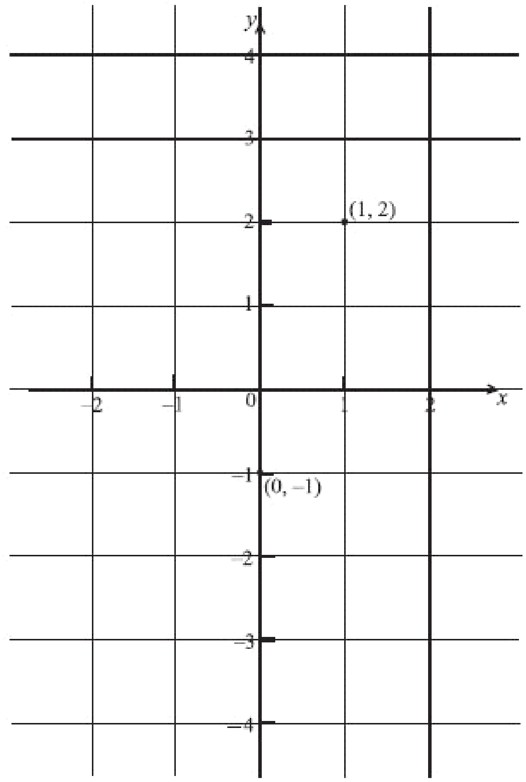
**16.** [Maximum mark: 6] ***[without GDC]***

On the axes below, sketch a curve $y = f(x)$ which satisfies the following conditions.

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ |
|---|---|---|---|
| $-2 \le x < 0$ | | negative | positive |
| $0$ | $-1$ | 0 | positive |
| $0 < x < 1$ | | positive | positive |
| $1$ | $2$ | positive | 0 |
| $1 < x \le 2$ | | positive | negative |




**17.** [Maximum mark: 6] ***[without GDC]***

Let $g(x) = \dfrac{\ln x}{x^2}$, for $x > 0$.

(a) Use the quotient rule to show that $g'(x) = \dfrac{1 - 2\ln x}{x^3}$. [3]

(b) The graph of $g$ has a maximum point at A. Find the $x$-coordinate of A. [3]

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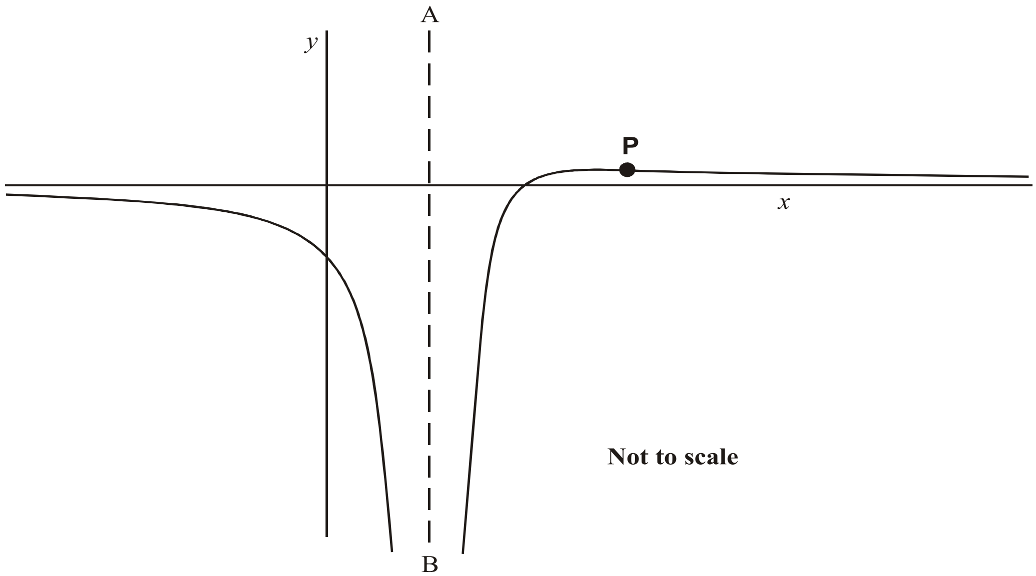
**18.** [Maximum mark: 8] ***[without GDC]***

Consider the function $h: x \mapsto \frac{x-2}{(x-1)^2}, x \neq 1.$

A sketch of part of the graph of *h* is given below.




The line (AB) is a vertical asymptote. The point P is a point of inflexion.

(a) Write down the **equation** of the vertical asymptote. [1]

(b) Find $h'(x)$ writing your answer in the form $\frac{a-x}{(x-1)^n}$ [4]

(c) Given that $h''(x) = \frac{2x-8}{(x-1)^4}$, calculate the coordinates of P. [3]

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**19.** [Maximum mark: 8] ***[without GDC]***

The function $g(x)$ is defined for $-3 < x < 3$. The behaviour of $g'(x)$ and $g''(x)$ is given in the tables below.

| $x$ | $-3 < x < -2$ | $-2$ | $-2 < x < 1$ | $1$ | $1 < x < 3$ |
|---|---|---|---|---|---|
| $g'(x)$ | negative | 0 | positive | 0 | negative |

| $x$ | $-3 < x < -\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2} < x < 3$ |
|---|---|---|---|
| $g''(x)$ | positive | 0 | negative |

Use the information above to answer the following. In each case, justify your answer.

(a) Write down the value of $x$ for which $g$ has a maximum. [1]

(b) On which intervals is the value of $g$ decreasing? [2]

(c) Write down the value of $x$ for which the graph of $g$ has a point of inflexion [1]

(d) Given that $g(-3) = 1$, sketch the graph of $g$. On the sketch, clearly indicate the position of the maximum point, the minimum point, and the point of inflexion. [4]

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**20.** [Maximum mark: 7] ***[without GDC]***

Consider the function

$$f(x)=\frac{e^x-1}{e^x+1}$$

(a) Find $f'(x)$ and deduce the monotony of the function. [3]

(b) Explain why $f^{-1}(x)$ exists and find its expression. [4]

**21.** [Maximum mark: 7] ***[without GDC]***

Let $f(x)=(x-1)^2(x-4)^3$. Find the stationary points and determine their nature.

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**22.** [Maximum mark: 6] ***[with GDC]***

Let $f(x)=(x-1)^2(x-4)^3$. Find the points of inflexion; justify your answer.

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**23.** [Maximum mark: 6] ***[without GDC]***

Consider $f(x) = x^2 + \frac{p}{x}$, $x \neq 0$, where $p$ is a constant.

(a) Find $f'(x)$. [2]

(b) There is a minimum value of $f(x)$ when $x = -2$. Find the value of $p$. [4]

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**24.** [Maximum mark: 6] ***[without GDC]***

The function $f$ is defined by $f(x) = \frac{2x}{x^2 + 6}$, for $x \geq b$, where $b \in R$.

(a) Show that $f'(x) = \frac{12 - 2x^2}{(x^2 + 6)^2}$. [3]

(b) Hence find the smallest **exact** value of $b$ for which the inverse function $f^{-1}$ exists. Justify your answer. [3]

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**25.** [Maximum mark: 5] ***[without GDC]***

If $f(x) = x - 3x^{\frac{2}{3}}$, $x > 0$,

(a) find the $x$-coordinate of the point P where $f'(x) = 0$;

(b) determine whether P is a maximum or minimum point.

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**26.** [Maximum mark: 6] ***[with / without GDC]***

Find the $x$-coordinate of the point of inflexion on the graph of $y = xe^x$, $-3 < x \leq 1$.

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**27*.** [Maximum mark: 7] ***[without GDC]***

Let $f: x \mapsto e^{\sin x}$.

(a) Find $f'(x)$. [2]

(b) Show that an equation that would allow you to find the values of $x$ at the points of inflexion is $\sin x = \frac{\sqrt{5}-1}{2}$. [5]

**28.** [Maximum mark: 7] ***[with GDC]***

Consider the curve with equation $f(x) = e^{-2x^2}$ for $x < 0$.

Find the cordinates of the point of inflexion and justify that it is a point of inflexion.

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**29.** [Maximum mark: 6] ***[with / without GDC]***

The function $f$ is given by $f(x) = \frac{x^5 + 2}{x}$, $x \neq 0$. There is a point of inflexion on the graph of $f$ at the point P. Find the coordinates of P.

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**30.** [Maximum mark: 6] ***[without GDC]***

The curve $y = \frac{x^3}{3} - x^2 - 3x + 4$ has a local maximum point at P and a local minimum point at Q. Determine the equation of the straight line passing through P and Q, in the form $ax + by + c = 0$, where $a, b, c \in R$.

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**31.** [Maximum mark: 6] ***[without GDC]***

A function $f$ is defined by $f(x)=ax^3+bx^2+30x+c$ where $a,\ b$ and $c$ are constants. The graph has a maximum at (1,7) and a point of inflexion when $x=3$. Find the value of $a,$ of $b$ and of $c$.

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**32.** [Maximum mark: 8] ***[without GDC]***

A cubic function has a maximum at A(0,5) and a point of inflexion at B(1,1). Find

(a) an expression of the cubic function. [6]

(b) the coordinates of the minimum point; justify that it is a minimum. [2]

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## B. Exam style questions (LONG)

**33.** [Maximum mark: 12] ***[with / without GDC]***

The function $f$ is given by $f(x)=\dfrac{\ln 2x}{x}$, $x>0$ .

(a) (i) Show that $f'(x)=\dfrac{1-\ln 2x}{x^2}$.

Hence

(ii) prove that the graph of $f$ can have only one local max or min point;

(iii) find the coordinates of the maximum point on the graph of $f$. [6]

(b) By showing that the second derivative $f''(x)=\dfrac{2\ln 2x-3}{x^3}$ or otherwise,

find the coordinates of the point of inflexion on the graph of $f$. [6]

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**34.** [Maximum mark: 18] ***[without GDC]***

The function $f$ is defined by $f(x) = xe^{2x}$.

(a) By considering the first and the second derivatives of $f$, show that there is one minimum point P on the graph of $f$, and find the coordinates of P. [7]

(b) Show that $f$ has a point of inflexion Q at $x = -1$. [5]

(c) Determine the intervals on the domain of $f$ where *f* is

(i) concave up; (ii) concave down. [2]

(d) Sketch $f$, clearly showing any intercepts, asymptotes and the points P and Q. [4]

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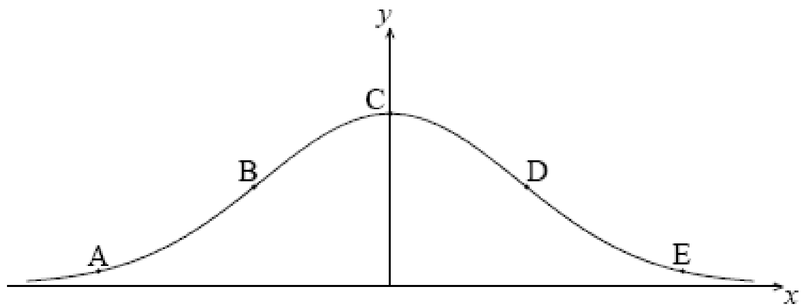
**35.** [Maximum mark: 13] ***[with / without GDC]***

The following diagram shows the graph of $f(x) = \mathrm{e}^{-x^2}$ .




The points A, B, C, D and E lie on the graph of $f$ . Two of these are points of inflexion.

(a) Identify the **two** points of inflexion. [2]

(b) (i) Find $f'(x)$.

(ii) Show that $f''(x) = (4x^2 - 2)\mathrm{e}^{-x^2}$ . [5]

(c) Find the $x$-coordinate of each point of inflexion. [4]

(d) Use the second derivative to show that one of these points is a point of inflexion. [2]

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**36.** [Maximum mark: 10] ***[with GDC]***

Let the function $f$ be defined by $f(x)=\dfrac{2}{1+x^3}$, $x\neq -1$.

(a) (i) Write down the equation of the vertical asymptote of the graph of $f$.

(ii) Write down the equation of the horizontal asymptote of the graph of $f$.

(iii) Sketch the graph of $f$ in the domain $-3\leq x\leq 3$. [4]

(b) (i) Using the fact that $f'(x)=\dfrac{-6x^2}{(1+x^3)^2}$, show that $f''(x)=\dfrac{12x\left(2x^3-1\right)}{(1+x^3)^3}$.

(ii) Find the $x$-coordinates of the points of inflexion of the graph of $f$. [6]

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**37.** [Maximum mark: 14] ***[with GDC]***

The function $f$ is defined as $f(x) = (2x+1)e^{-x}$, $0 \leq x \leq 3$. The point P(0, 1) lies on the graph of $f(x)$, and there is a maximum point at Q.

(a) Sketch the graph of $y = f(x)$, labelling the points P and Q. [3]

(b) (i) Show that $f'(x) = (1-2x)e^{-x}$.

(ii) Find the **exact** coordinates of Q. [7]

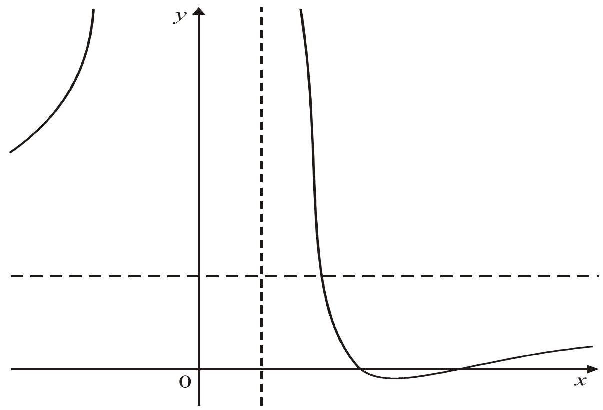
(c) The equation $f(x) = k$, where $k \in \mathbb{R}$, has two solutions. Write down the range of values of $k$. [2]

(d) Given that $f''(x) = e^{-x}(-3+2x)$, show that the curve of $f$ has only one point of inflexion. [2]

**38.** [Maximum mark: 9] ***[without GDC]***

Consider the function $f$ given by $f(x)=\dfrac{2x^2-13x+20}{(x-1)^2}$, $x\neq 1$.

A part of the graph of $f$ is given below (vertical and horizontal asymptotes are shown)




(a) Write down the **equation** of the vertical asymptote. [1]

(b) It is given that $f(100)=1.91$, $f(-100)=2.09$, $f(1000)=1.99$, $f(-1000)=2.01$

Write down the **equation** of the horizontal asymptote. [1]

(c) Show that $f'(x)=\dfrac{9x-27}{(x-1)^3}$, $x\neq 1$. [3]

The second derivative is given by $f''(x)=\dfrac{72-18x}{(x-1)^4}$, $x\neq 1$.

(d) Using values of $f'(x)$ and $f''(x)$ explain why a minimum must occur at $x=3$. [2]

(e) There is a point of inflexion on the graph. Write down the coordinates of this point. [2]

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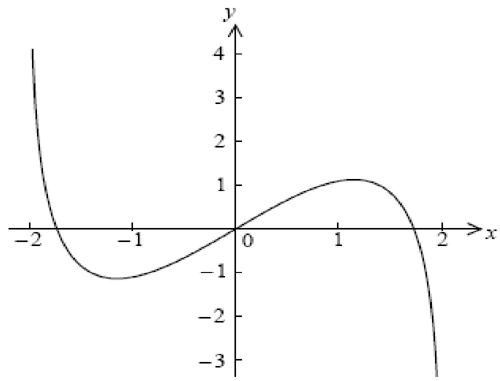
**39.** [Maximum mark: 14] ***[with GDC]***

Consider $f(x) = x\ln(4-x^2)$, for $-2 < x < 2$. The graph of $f$ is given below.




(a) Let P and Q be the stationary points on the curve of $f$.

(i) Find the $x$-coordinate of P and of Q.

(ii) Write down all values of $k$ for which $f(x) = k$ has exactly two solutions. [5]

Let $g(x) = x^3\ln(4-x^2)$, for $-2 < x < 2$.

(b) Show that $g'(x) = \frac{-2x^4}{4-x^2} + 3x^2\ln(4-x^2)$. [4]

(c) Sketch the graph of $g'$. [2]

(d) Write down all values of $w$ for which $g'(x) = w$ has exactly two solutions. [3]

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**40.** [Maximum mark: 17] ***[with / without GDC]***

The function $f$ is defined by $f(x)=\dfrac{\ln x}{x^3}$, $x \geq 1$.

(a) Find $f'(x)$ and $f''(x)$, simplifying your answers. [6]

(b) (i) Find the **exact** value of the $x$- coordinate of the maximum point and justify that this is a maximum.

(ii) Solve $f''(x)=0$, and show that at this value of $x$, there is a point of inflexion on the graph of $f$.

(iii) Sketch the graph of $f$, indicating the maximum point and the point of inflexion. [11]

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**41.** [Maximum mark: 14] ***[with / without GDC]***

Let $f(x) = x^3 - 4x + 1$.

(a) Find $f'(x)$ [2]

(b) The tangent to the curve of $f$ at the point P(1, –2) is parallel to the tangent at a point Q. Find the coordinates of Q. [4]

(c) The graph of $f$ is decreasing for $p < x < q$. Find the value of $p$ and of $q$. [3]

(d) Write down the range of values for the gradient of $f$. [2]

(e) Find the coordinates of the point of inflexion of the graph of $f$. [3]

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**42.** [Maximum mark: 15] ***[without GDC]***

Let $f(x)=3+\frac{20}{x^2-4}$, for $x\neq\pm 2$. The graph of $f$ is given below.

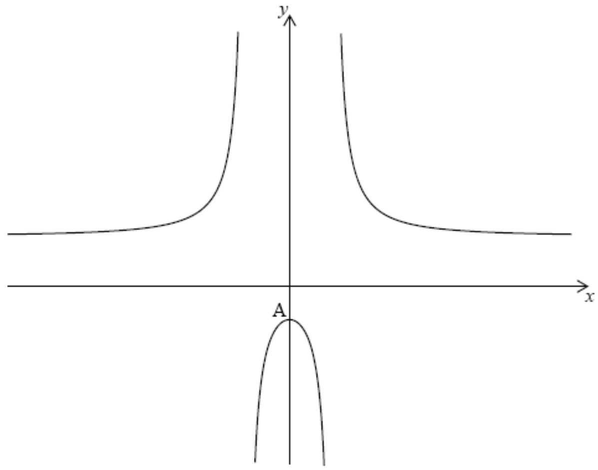


***diagram not to scale***

The $y$-intercept is at the point A.

(a) (i) Find the coordinates of A (ii) Show that $f'(x)=0$ at A. [6]

(b) The second derivative $f''(x)=\frac{40(3x^2+4)}{(x^2-4)^3}$. Use this to

(i) justify that the graph of $f$ has a local maximum at A;

(ii) explain why the graph of $f$ does **not** have a point of inflexion. [6]

(c) Describe the behaviour of the graph of $f$ for large $|x|$. [1]

(d) Write down the range of $f$. [2]

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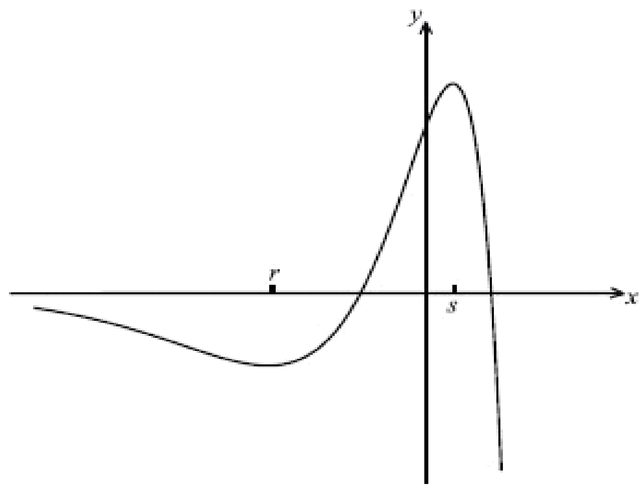
**43.** [Maximum mark: 14] ***[with GDC]***

Let $f(x)=\mathrm{e}^{x}(1-x^{2})$.

(a) Show that $f'(x)=\mathrm{e}^{x}(1-2x-x^{2})$. [3]

Part of the graph of $y=f(x)$, for $-6\le x\le 2$, is shown below. The $x$-coordinates of the local minimum and maximum points are $r$ and $s$ respectively.




(b) Write down the **equation** of the horizontal asymptote. [1]

(c) Write down the value of $r$ and of $s$. [4]

(d) Show that the normal *L* to the curve of $f$ at P(0, 1) has equation $x+y=1$. [4]

(e) Find the coordinates of the points where the curve and the line *L* intersect. [2]

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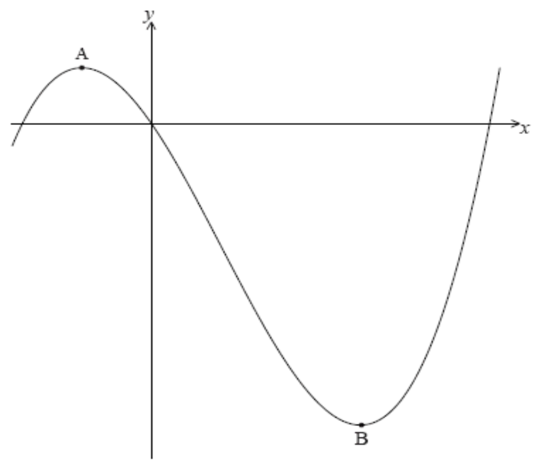
**44.** [Maximum mark: 12] ***[without GDC]***

Let $f(x)=\frac{1}{3}x^3-x^2-3x$. Part of the graph of $f$ is shown below.




There is a maximum point at A and a minimum point at B(3, –9).

(a) Find the coordinates of A. [6]

(b) Write down the coordinates of

(i) the image of B after reflection in the $y$-axis;

(ii) the image of B after translation by the vector $\begin{pmatrix}-2\\5\end{pmatrix}$;

(iii) the image of B after reflection in the $x$-axis followed by a horizontal stretch with scale factor $\frac{1}{2}$. [6]

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**45.** [Maximum mark: 12] ***[without GDC]***

Let $f(x) = \dfrac{\cos x}{\sin x}$, for $\sin x \neq 0$.

(a) Use the quotient rule to show that $f'(x) = \dfrac{-1}{\sin^2 x}$. [4]

(b) Find $f''(x)$. [3]

In the following table, $f'\left(\dfrac{\pi}{2}\right) = p$ and $f'\left(\dfrac{\pi}{2}\right) = q$. The table also gives approximate values of $f'(x)$ and $f''(x)$ near $x = \dfrac{\pi}{2}$.

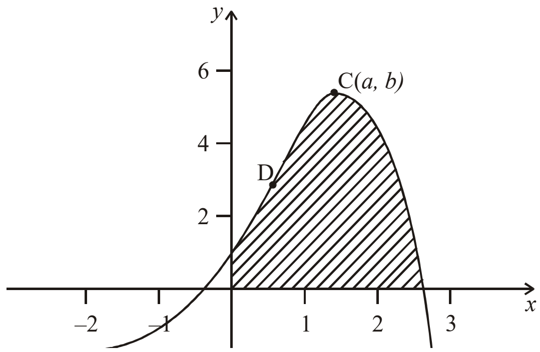
| $x$ | $\dfrac{\pi}{2} - 0.1$ | $\dfrac{\pi}{2}$ | $\dfrac{\pi}{2} + 0.1$ |
|---|---|---|---|
| $f'(x)$ | –1.01 | $p$ | –1.01 |
| $f''(x)$ | 0.203 | $q$ | –0.203 |

(c) Find the value of $p$ and of $q$. [3]

(d) Use information from the table to explain why there is a point of inflexion on the graph of $f$ where $x = \dfrac{\pi}{2}$. [2]

**46.** [Maximum mark: 12] ***[without GDC]***

The diagram shows the graph of $y = e^{x}(\cos x + \sin x)$, $-2 \le x \le 3$. The graph has a maximum turning point at $C(a, b)$ and a point of inflexion at D.




(a) Find $\frac{dy}{dx}$. [3]

(b) Find the **exact** value of $a$ and of $b$. [4]

(c) Show that at D, $y = \sqrt{2}e^{\frac{\pi}{4}}$. [5]

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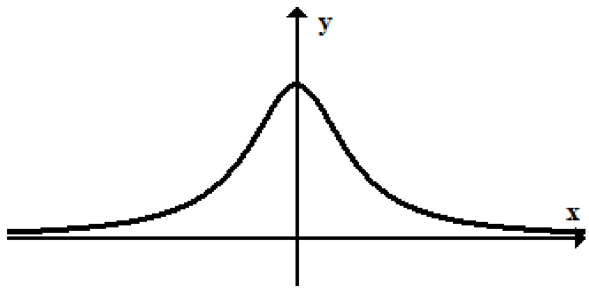
**47.** [Maximum mark: 12] ***[without GDC]***

Part of the graph of $f(x)=\dfrac{1}{1+x^2}$ is shown below.




(a) Write down the equation of the horizontal asymptote of the graph of $f$. [1]

(b) Find $f'(x)$. [3]

(c) Show that the second derivative is given by $f''(x)=\dfrac{6x^2-2}{(1+x^2)^3}$. [4]

(d) Let A be the point on the curve of $f$ where the gradient of the tangent is a maximum. Find the $x$-coordinate of A. [4]

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**48.** [Maximum mark: 11] ***[with GDC]***

Consider the function $f(x)=\mathrm{e}^{(2x-1)}+\dfrac{5}{2x-1}$, $x\neq\dfrac{1}{2}$.

(a) Sketch the curve of $f$ for $-2\leq x\leq 2$, including any asymptotes. [3]

(b) Write down the equation of the vertical asymptote of $f$. [1]

(c) Find $f'(x)$. [4]

(d) (i) Write down the value of $x$ at the minimum point on the curve of $f$.

(ii) The equation $f(x)=k$ has no solutions for $p\leq k<q$. Write down the value of $p$ and of $q$. [3]

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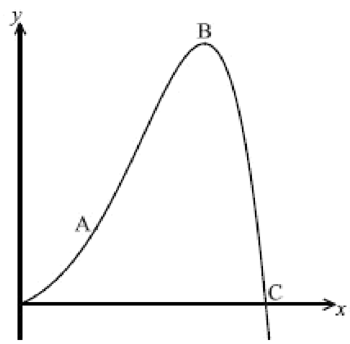
**49.** [Maximum mark: 12] ***[without GDC]***

The function $f$ is defined as $f(x) = e^x \sin x$, where $x$ is in radians. Part of the curve of $f$ is shown below.




There is a point of inflexion at A, and a local maximum point at B. The curve of $f$ intersects the *x*-axis at the point C.

(a) Write down the $x$-coordinate of the point C. [2]

(b) (i) Find $f'(x)$.

(ii) Write down the value of $f'(x)$ at the point B. [4]

(c) Show that $f''(x) = 2e^x \cos x$. [2]

(d) (i) Write down the value of $f''(x)$ at A, the point of inflexion.

(ii) Hence, calculate the coordinates of A. [4]

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**50**.** [Maximum mark: 15] ***[with GDC]***

The function $f$ is defined by $f(x)=\dfrac{x^2-x+1}{x^2+x+1}$

(a) (i) Find an expression for $f'(x)$, simplifying your answer.

(ii) The tangents to the curve of $f(x)$ at points A and B are parallel to the $x$-axis. Find the coordinates of A and of B. [5]

(b) (i) Sketch the graph of $y=f'(x)$.

(ii) Find the $x$-coordinates of the three points of inflexion on the graph of $f$. [5]

(c) Find the range of (i) $f$; (ii) the composite function $f\circ f$. [5]

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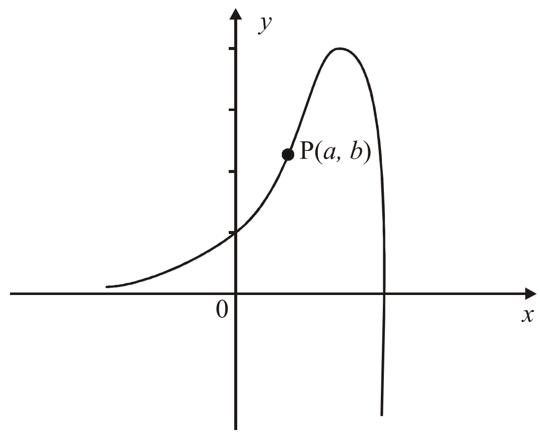
**51**.** [Maximum mark: 14] ***[without GDC]***

The diagram shows part of the graph of the curve with equation $y = e^{2x}\cos x$.




(a) Show that $\frac{dy}{dx} = e^{2x}(2\cos x - \sin x)$ [2]

(b) Find $\frac{d^2y}{dx^2}$. [4]

There is an inflexion point at P$(a, b)$.

(c) Use the results from parts (a) and (b) to prove that:

(i) $\tan a = \frac{3}{4}$; (ii) the gradient of the curve at P is $e^{2a}$. [8]

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**52*.** [Maximum mark: 13] ***[with GDC]***

Let $f(x)=x\left(\sqrt[3]{(x^2-1)^2}\right)$, $-1.4 \le x \le 1.4$

(a) **Sketch** the graph of $f(x)$. (An exact scale diagram is **not** required.)

On your graph indicate the approximate position of

(i) each zero; (ii) each maximum point; (iii) each minimum point. [4]

(b) (i) Find $f'(x)$ clearly stating its domain.

(ii) Find the $x$-coordinates of the maximum and minimum points of $f(x)$, for $-1 < x < 1$. [7]

(c) Find the $x$-coordinate of the point of inflexion of $f(x)$, where $x > 0$, giving your answer correct to **four** decimal places. [2]