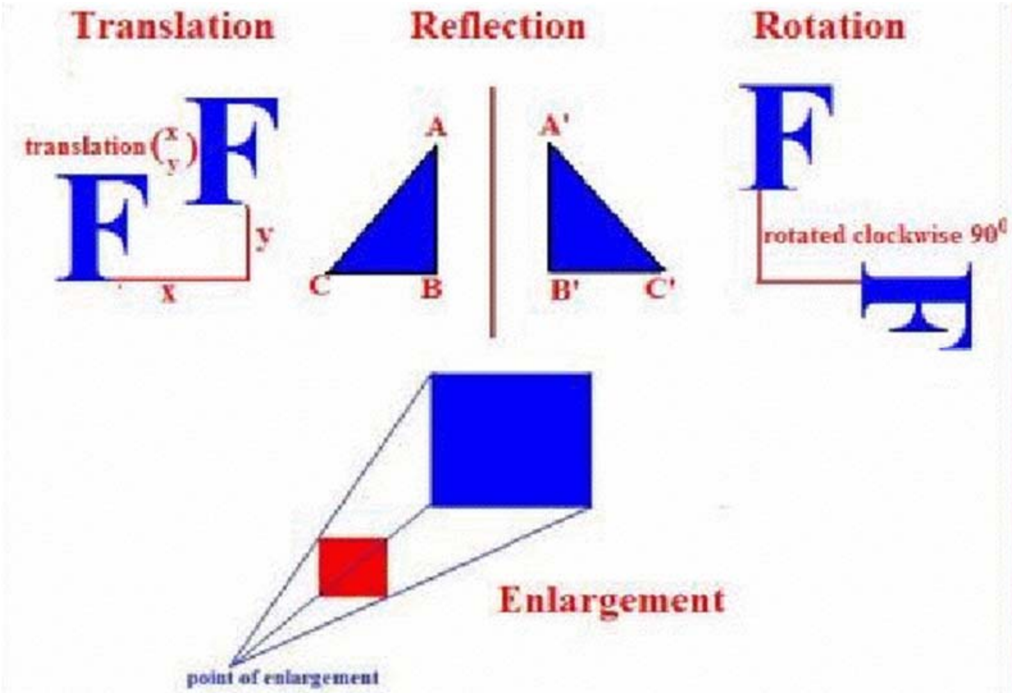
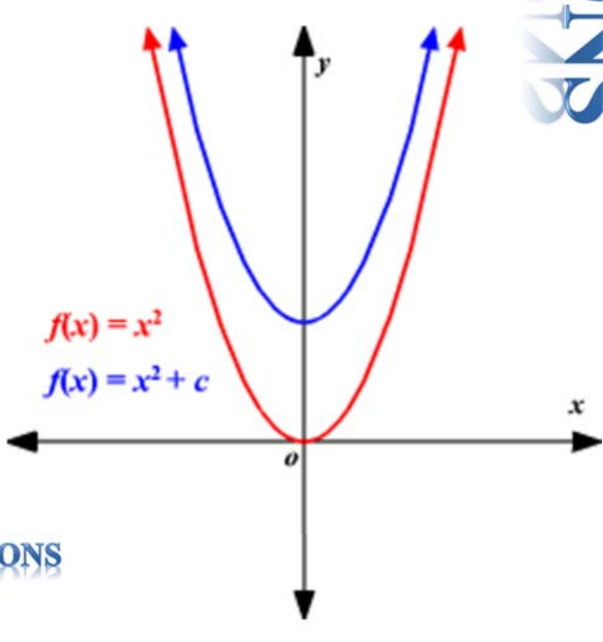
TRANSFORMATIONS



TRANSFORMATIONS

NAME: MR.WAIN



TRANSFORMATIONS

Transformations: $f(x) \to f(x) = af(n(x-b)) + c$ **or** $(x, y) \to \left(\frac{x}{n} + b, ay + c\right)$

- Transformations of a function is one of the following:
  - Dilation (STRETCH) (from the *x*-axis or *y*-axis);
  - Reflection (FLIP) (in *x*-axis or *y*-axis);
  - Translation (SLIDE) (vertically and\or horizontally);
  - Rotation (we don't study these).

- The order to deal with the transformations is **DRT** (alphabetical)
- The **Cartesian Plane** is represented by the set $\boldsymbol{R}^2$ of all ordered pairs of real numbers.

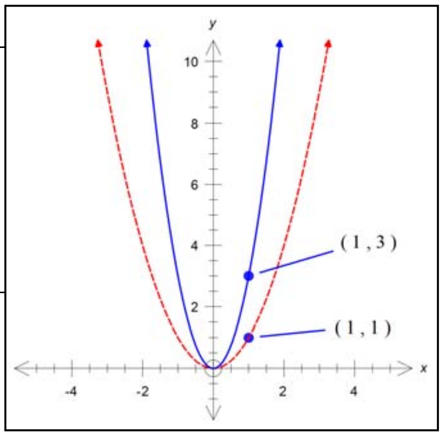
Dilations

- This is a **stretch or contraction** of the graph from the *x*-axis or the *y*-axis
- a causes a dilation of factor a from the *x*-axis $(x, y) \to (x, ay)$
- n causes a dilation of factor $\frac{1}{n}$ from the *y*-axis $(x, y) \to \left(\frac{x}{n}, y\right)$
- We describe the dilations like:
  - The graph is dilated by a factor of a from the *x*-axis, or
  - The graph is dilated by a factor of a parallel to the *y*-axis
  - The graph is dilated by a factor of $\frac{1}{n}$ from the *y*-axis

Example: Sketch the graph of $f(x) = 3x^2$ by comparing it to $f(x) = x^2$

Here $a = 3$
First sketch $f(x) = x^2$
Then multiply each *y* value by 3. ($= 3f(x)$)

The graph is dilated by a factor of 3 from the *x*-axis.



GeoGebra 3a Quadratic Function Transformation.ggb

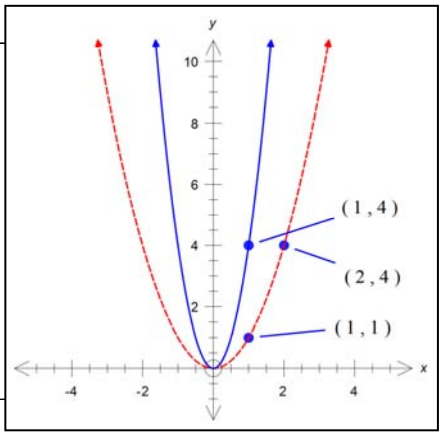
Example: Sketch $f(x) = (2x)^2$

Here $n =$ ___2__

First sketch $f(x) = x^2$
Then multiply each x value by _½_____ .

The graph is __dilated_ by a factor of __½_ from the _Y-axis__.

Could also be a dilation of factor 4 from the *x*-axis. Why?

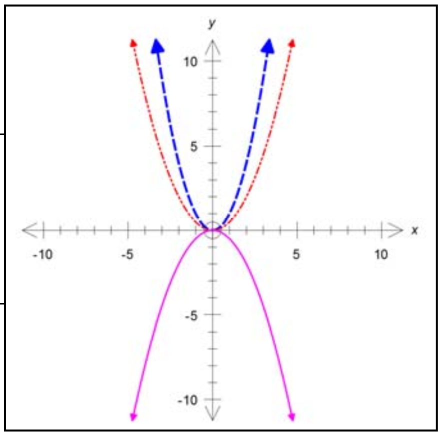


GeoGebra 3a Quadratic Function Transformation.ggb

Reflections

- There are three types of reflections:
  - In the x-axis, $y = -f(x)$, $(x, y) \rightarrow (x, -y)$
  - In the y-axis, $y = f(-x)$, $(x, y) \rightarrow (-x, y)$
  - In the line $y = x$, which we dealt with in **Inverse functions**.

Reflections in the *x*-axis, $y = -f(x)$ or when $a < 0$

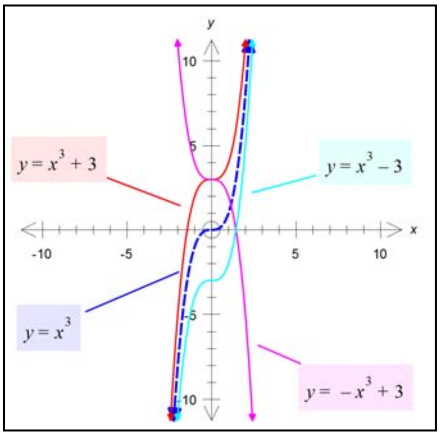


Example: Sketch $f(x) = \dfrac{-x^2}{2}$

Here $a = -\dfrac{1}{2}$

The graph is reflected in the x-axis and dilated by a factor of $\dfrac{1}{2}$ from x-axis.

Reflections in the *y*-axis, $y = f(-x)$



Example: Sketch $f(x) = x^3 + 3$, $f(-x)$ and $-f(-x)$.

$$\boxed{\begin{aligned} f(-x) &= (-x)^3 + 3 \\ &= -x^3 + 3 \end{aligned}}$$

$$\boxed{\begin{aligned} -f(-x) &= -(-x^3 + 3) \\ &= x^3 - 3 \end{aligned}}$$

GeoGebra [2a Cubic Function Transformation.ggb]

- $-f(-x)$ is a reflection in both x & y axes.

Reflections Worksheet #1

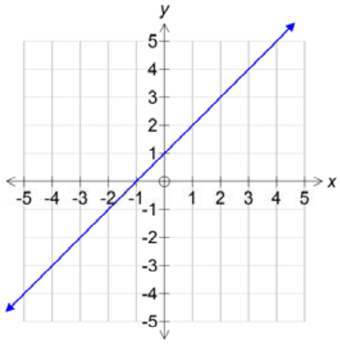
For each of the following graphs of $y = f(x)$, sketch:

(i) $y = -f(x)$

(ii) $y = f(-x)$

(iii) $y = -f(-x)$

(a)



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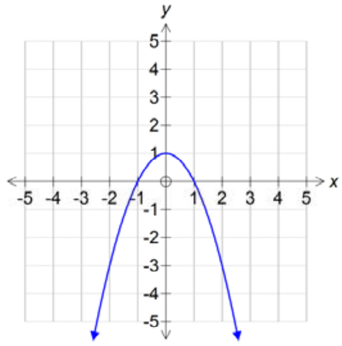

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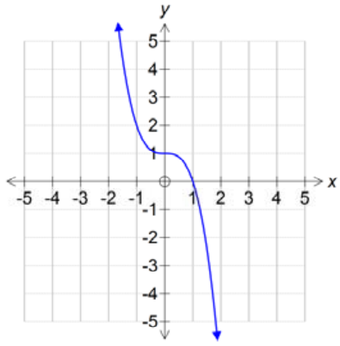
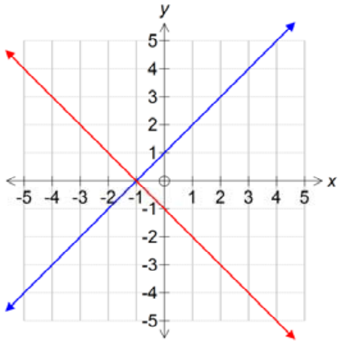
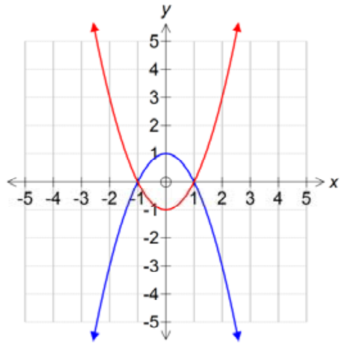


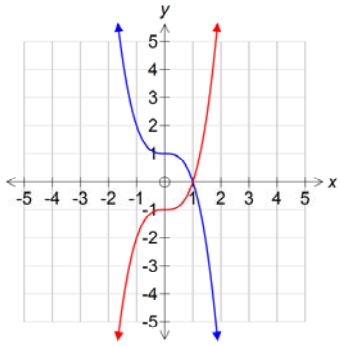
(i)



(i)



(i)



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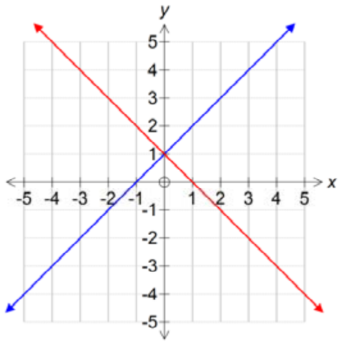

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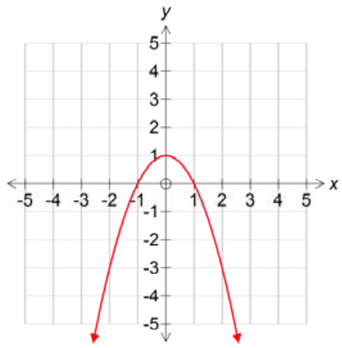

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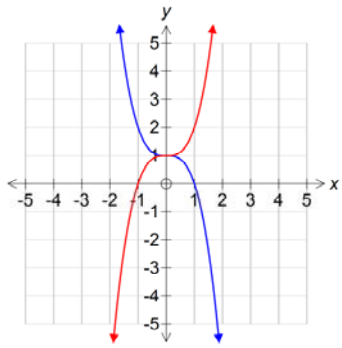

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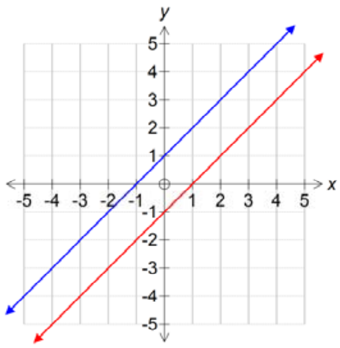

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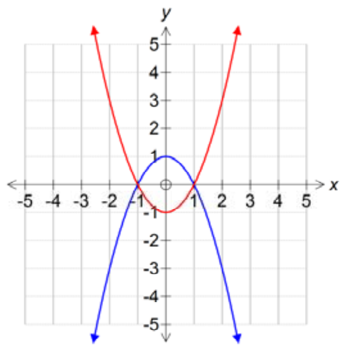

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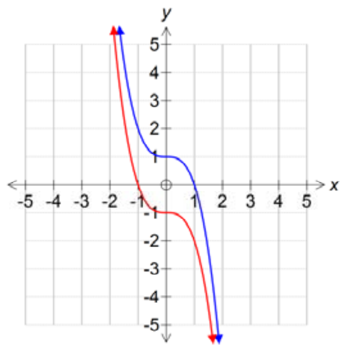


Reflections Worksheet #2

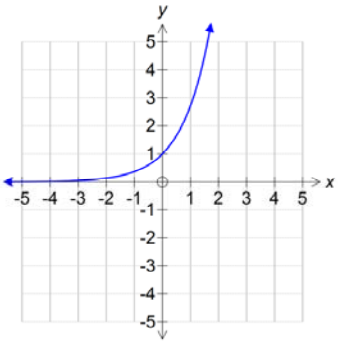
For each of the following graphs of $y = f(x)$, sketch:

(i) $y = -f(x)$

(ii) $y = f(-x)$

(iii) $y = -f(-x)$

(d)



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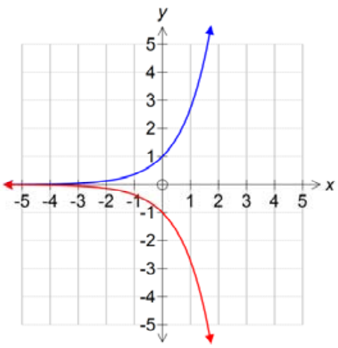

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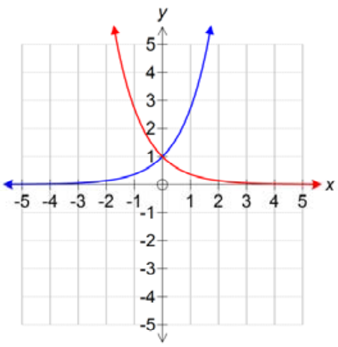

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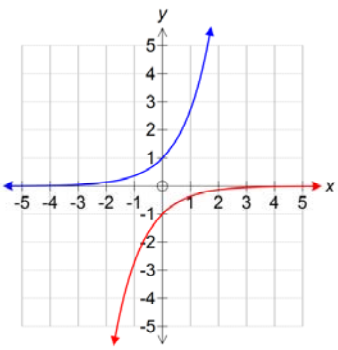

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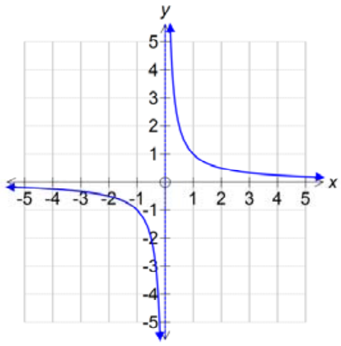

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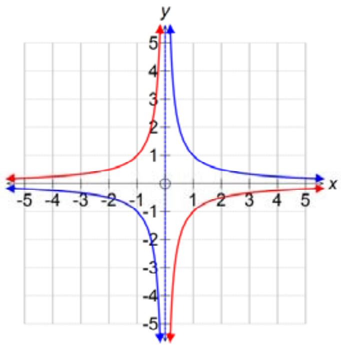

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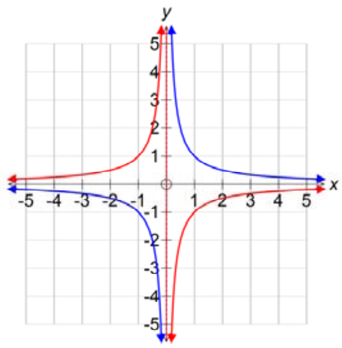

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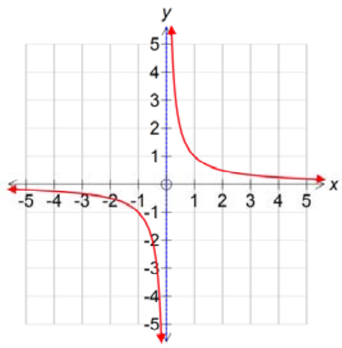

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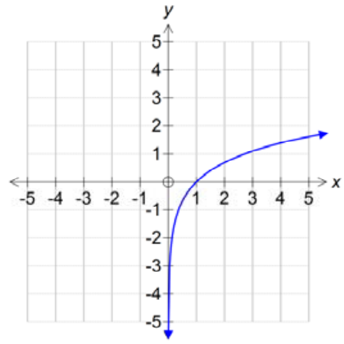
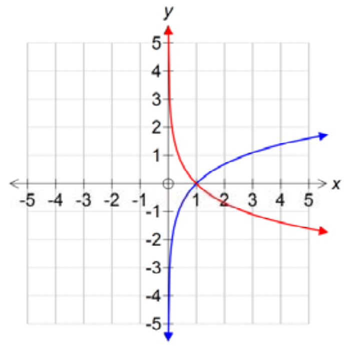


(i)



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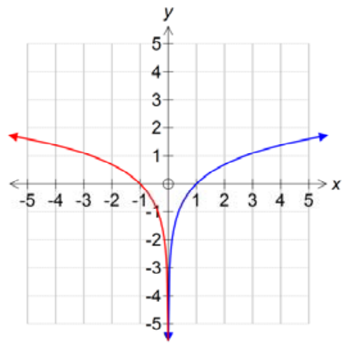

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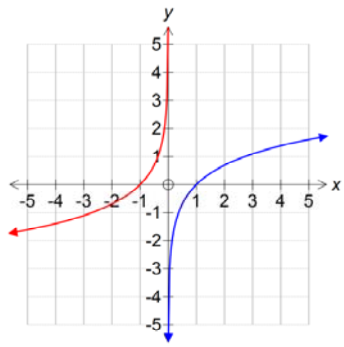
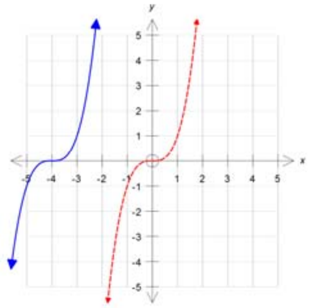


Translations

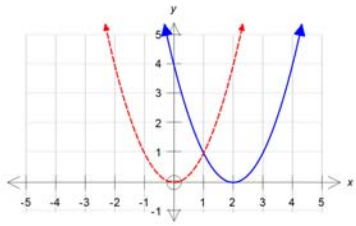
- There are two types of translations:
  - Along the direction of the x-axis : $f(x) = f(x-b)$; $(x, y) \longrightarrow (x + b, y)$
  - Along the direction of the y-axis: $f(x) = f(x)+c$.$(x, y) \longrightarrow (x, y + c)$

1. Along the direction of the *x*-axis : $f(x) = f(x-b)$

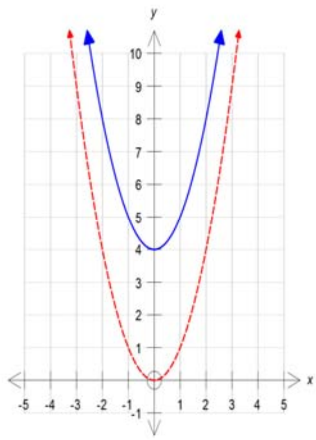


- Sketch the graph of $f(x) = (x+4)^3$.
- Translation of 4 units in the negative direction of the x-axis.

- Sketch the graph of $f(x) = (x-2)^2$
- _translation_ of _2_ units in the _positive_ direction of the X-axis.

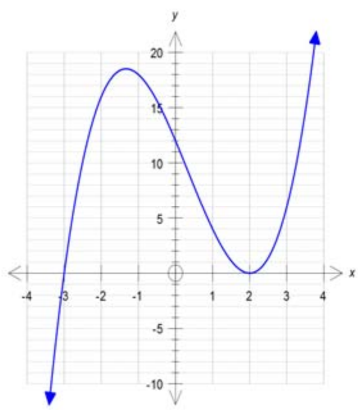


2. Along the direction of the *y*-axis: $f(x) = f(x)+c$



- Sketch the graph of $f(x) = x^2 + 4$
- translation of 4_ units in the positive direction of the Y-axis.

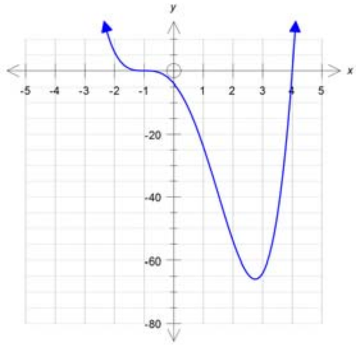
"Repeated Factor Squared"



- Consider the function $f(x) = (x+3)(x-2)^2$
- The X – intercepts are -3 and 2
- (2, 0) is also a Turning Point
- "A repeated factor squared is both an X –Intercept and a Turning Point"

Repeated Factor Cubed”

- Consider the function $f(x) = (x+1)^3(x-4)$
- The X – intercepts are -1 and 4
- (-1, 0) is also a Point of inflexion
- “A repeated factor cubed is both an X –Intercept and a Point of Inflecti



- GeoGebra
  - **Ex4A** Q 1, 3; **Ex 3A** Q 7 ab**;** **Ex 3B** Q 11a;
  - **Ex 3C** Q 1; **Ex 3D** Q 1a; **Ex 3E** Q 2 abe, 3 bc

Transformations Summary $f(x) \to af(n(x-b))+c$ **or** $(x,y) \to \left(\frac{x}{n}+b, ay+c\right)$

Example 1: State the transformations from $f(x)$ to $y = -2f(3(x+4))-1$.

A dilation of factor of 2 from the x – axis and a factor of $\frac{1}{3}$ from the y - axis, followed by a reflection in the x – axis, then a translation of 4 units in the negative direction of the x – axis and a translation of 1 unit in the negative direction of the y – axis.

Example 2: Describe the transformations undergone by $y = \log_e x$ to $y = 1-3\log_e(2x-8)$.

$y = 1-3\log_e(2x-8) = -3\log_e(2(x-4))+1$

A dilation of factor of 3 from the x – axis and a factor of $\frac{1}{2}$ from the y - axis, then a reflection in the x- axis, followed by a translation of 4 units in the positive direction of the x – axis and a translation of 1 unit in the positive direction of the y – axis.

Example 3: Write the equation of the rule when $y = x^2$ is transformed by:

- a translation of 1 unit in the positive direction of the x – axis and 2 units in the positive direction of the y – axis, followed by,
- a dilation of factor of 2 from the y – axis, followed by,
- a reflection in the x – axis.

$$\Rightarrow \left((x-1)^2+2\right) \Rightarrow \left(\frac{x}{2}-1\right)^2+2 \Rightarrow -\left(\left(\frac{x}{2}-1\right)^2+2\right) \therefore y = -\left(\frac{x}{2}-1\right)^2-2 \quad or \quad y = -\left(\frac{1}{2}(x-2)\right)^2-2$$

$$y = -\frac{1}{4}(x-2)^2-2 \quad or \quad f(x) \to f(x-1)+2 \to f\left(\frac{x}{2}-1\right)+2 \to -\left(f\left(\frac{x}{2}-1\right)+2\right) \to -f\left(\frac{x}{2}-1\right)-2$$

Exercise on Sequence of Transformations

1. State the sequence of transformations that each of the following functions have undergone from $y = f(x)$.

(a) $y = 3f(-2(x+3))+4$. **(b)** $y = 0.5f(3(x-2))+1$

(c) $y = 2f(-0.4(x+3))-0.2$ **(d)** $y = 2-3f(2x+1)$

2. Describe the transformations undergone by each of the following functions to produce the second function.

(a) $y = \log_e x$ to $y = 4\log_e 2(x+3)-5$ (b) $y = \sqrt{x}$ to $y = 2\sqrt{3x+4}+5$

(c) $y = \cos x$ to $y = -3\cos\left(2x+\frac{\pi}{4}\right)+1$ (d) $y = x^6$ to $y = 3(2x+5)^6-2$

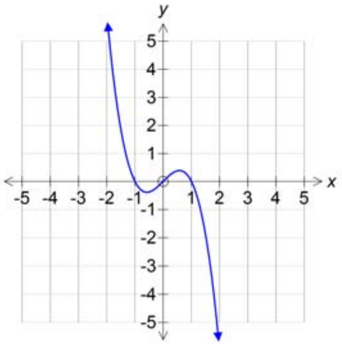
(e) $y = \sin x$ to $y = 2\sin\pi(3x-4)$

Ex4E Q 1,2, 3, 4; **Ex 3A** Q 7 d**; Ex 3B** Q 4; **Ex 3C** Q 2b, 4a; **Ex 3D** Q 4d; **Ex 3E** Q 1a; **Ex4F** Q 1, 2, 3, 4, 5, 6

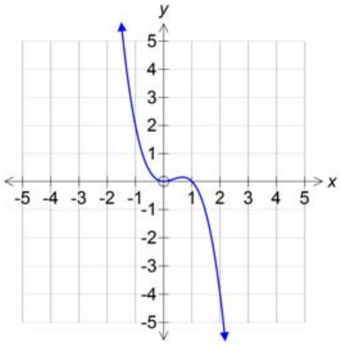
Determining a Rule for a Function from a Graph

- **Worksheet** – Matching Graphs to their rules
  Match the following graphs with the correct equation:

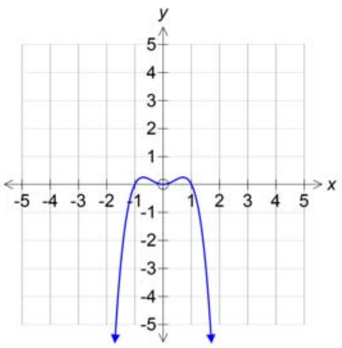
(a)



(b)



(c)



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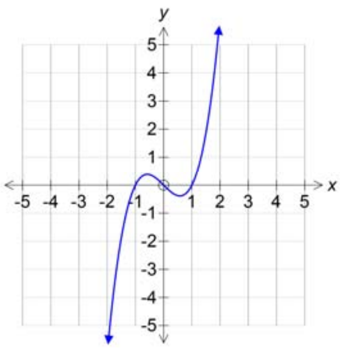

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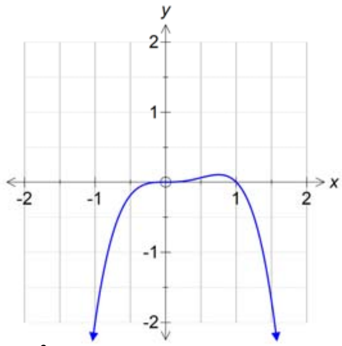

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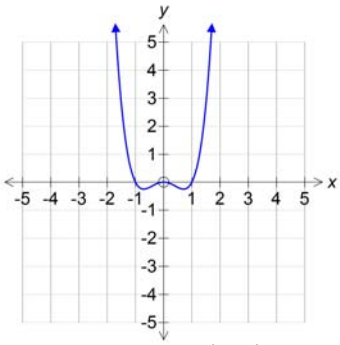


A: $y = x^3(1-x)$ B: $y = x(1-x^2)$ C: $y = x^2(x^2-1)$

D: $y = x^2(1-x)$ E: $y = x(x^2-1)$ F: $y = x^2(1-x^2)$

((a)=B, (b) = D, (c) = F, (d)= E , (e) = A, (f) = C)

- **Example:** Find the rule for:

We know the turning point : $(2,3)\ \therefore\ y = a(x-b)^2 + c$

$$y = a(x-2)^2 + 3$$

$$y - \text{int}: (0,11) \Rightarrow 11 = a(0-2)^2 + 3$$

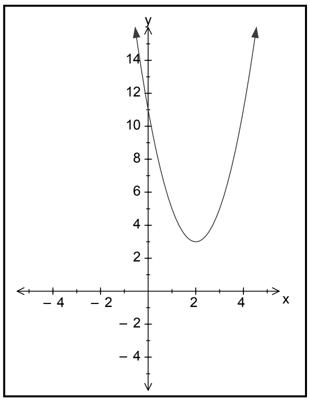
$$11 = 4a + 3$$

$$8 = 4a$$

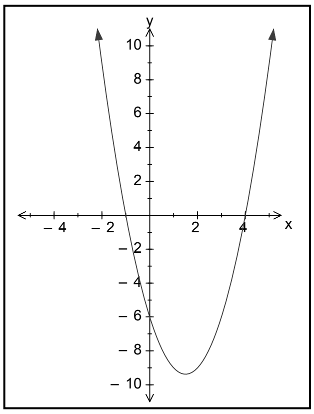
$$2 = a$$

$$\therefore y = 2(x-2)^2 + 3$$

$$or\ y = 2x^2 - 8x + 11$$

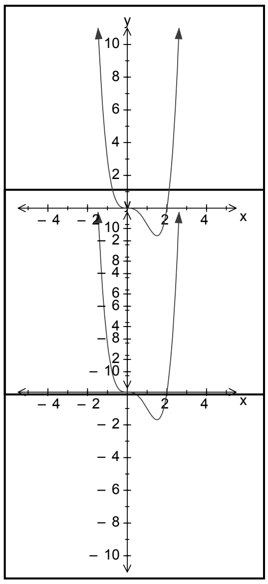


- **Example:** Find the rule for:



$$\textit{We know the } x-\text{ints}: (-1,0),(4,0) \therefore y = a(x+1)(x-4)$$
$$\textit{Use } (0,-6) \Rightarrow -6 = a(0+1)(0-4)$$
$$-6 = a \cdot 1 \cdot -4$$
$$-6 = -4a$$
$$\frac{3}{2} = a$$
$$\therefore y = \frac{3}{2}(x+1)(x-4)$$

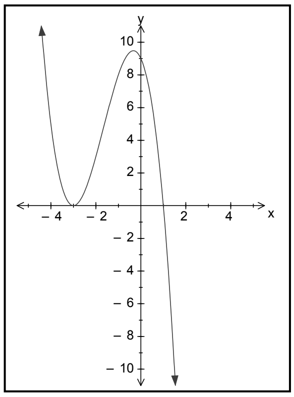
- *Graphical Calculator can be used for this example.*
  - *Insert Lists & Spreadsheet*
  - *X-values – List1*
  - *Y-values – List2*
  - *Regression – Menu – Statistics - Calculations*



- **Example:** Find the rule for: (1, -1) on curve.
- This is a quartic.
- Point of inflection at the origin, therefore a repeated factor cubed, x^3
- Also an *x*-intercept at (2,0)

$$y = ax^3(x-2)$$
$$(1,-1)$$
$$-1 = a(1)^3(1-2)$$
$$-1 = -a$$
$$1 = a$$
$$\therefore y = x^3(x-2)$$

- **Example:** Find the rule for:



$$y = a(x-1)(x+3)^2$$
$$(0,-9) \therefore -9 = a(0-1)(0+3)^2$$
$$-1 = a \Rightarrow y = -(x-1)(x+3)^2$$

- **Ex4A** Q 8, 9; **Ex4B** Q 1, 2, 3, 4, 7, 8 , 9; **Ex4G** Q 1, 3, 4, 5, 6, 7, 8; **Ex3G** 3, 4, 5, 6, 7ab, 8, 9

Transformations of $f(x)=x^{p};\, p=-1,-3,\ldots$

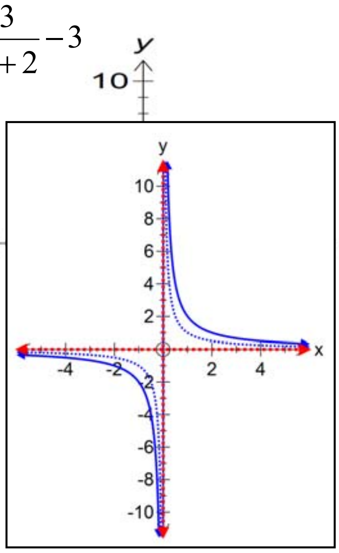
- $\dfrac{1}{x^{p}} \rightarrow \dfrac{a}{\left(n(x-b)\right)^{p}}+c \quad or \quad af(n(x-b))+c$

GeoGebra [8 Hyperbola Function Transformation.ggb]

Examples: Sketch the graph of:

- *(i)* $f(x)=\dfrac{2}{x}$ *(ii)* $f(x)=\dfrac{1}{2x^{3}}$ *(iii)* $f(x)=\dfrac{4}{x-2}$

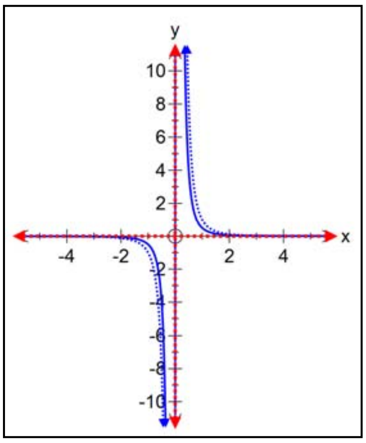
  (iv) $f(x)=\dfrac{4}{2-x}$ *(v)* first show that $f(x)=\dfrac{-3x-3}{x+2}$ is equal to $f(x)=\dfrac{3}{x+2}-3$



(i)

dilation of factor of 2 from the x-axis:

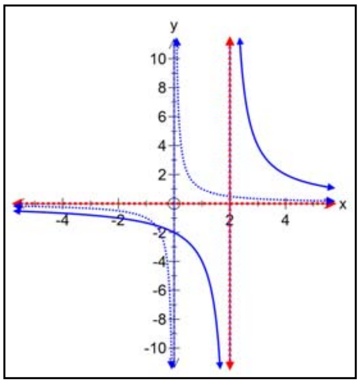
$\boxed{a=2,\rightarrow 2f(x)}$



(ii)

dilation of factor $\dfrac{1}{2}$ from the x-axis:

$\boxed{a=\dfrac{1}{2},\rightarrow \dfrac{1}{2}f(x)}$



(iii)

A dilation of factor 4 from the x-axis and a translation of 2 units to the right (positive direction of the x-axis):
<u>Note:</u> the vertical asymptote also moves 2 units to the right.

Y-intercept: $\boxed{\begin{array}{l} x=0 \\ y=\dfrac{4}{0-2}=\dfrac{4}{-2}=-2 \end{array}}$

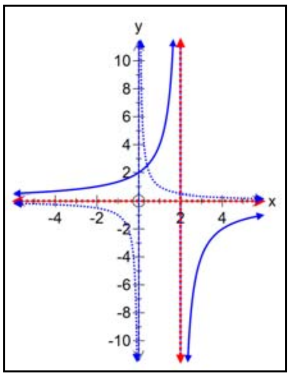
$\boxed{a=4\,\&\,b=2,\rightarrow 4f\left(x-2\right)}$

(iv)

re-write the function: $f(x)=\dfrac{4}{-x+2}=\dfrac{4}{-(x-2)}$

dilation of factor 4, a reflection in the *y*-axis and a translation of 2 units to the right (positive direction of the *x*-axis):

Note: the vertical asymptote also moves 2 units to the right. *Y*-Intercept: $y = 2$



$\boxed{a=4, b=2\ \&\ n=-1 \rightarrow 4f\left(-(x-2)\right)}$

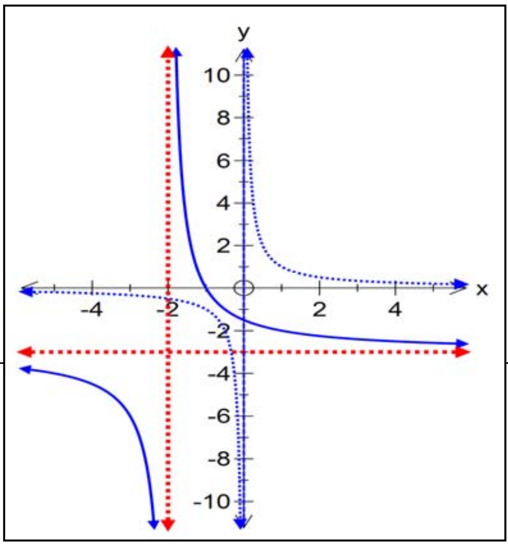
(v) $\boxed{f(x)=\frac{-3x-3}{x+2}=\frac{-3(x+2)+3}{x+2}=\frac{-3(x+2)}{x+2}+\frac{3}{x+2}=\frac{3}{x+2}-3}$

A dilation of factor 3 from the *x*-axis, a translation of three units in the negative direction of the *y*-axis and a translation of 2 units in the negative direction of the *x*-axis:

Intercepts:

$$\begin{aligned} y&=0\\ \frac{3}{x+2}-3&=0\\ \frac{3}{x+2}&=3\\ 3&=3(x+2)\\ 1&=x+2\\ -1&=x \end{aligned}$$

$$\begin{aligned} x&=0\\ y&=\frac{3}{0+2}-3\\ y&=\frac{3}{2}-3\\ y&=\frac{-3}{2} \end{aligned}$$



$\boxed{a=3, b=-2\ \&\ c=-3 \rightarrow 3f(x+2)-3}$

- **Ex3A** Q 2, 3 aefghk, 4, 5 be, 8b; **Ex3B** Q 1, 5 ab, 7 **Ex3D** Q 4 c; **Ex3E** Q 4af; **Ex3F** 1 abef, 2 dgij, 3, 4, 5 ab
-
-
- Hint Ex 3F Q4
- $y=\frac{4x+5}{2x+3}=\frac{2(2x+3)-1}{2x+3}=2-\frac{1}{2x+3}$
- Or synthetic division
- $y=\frac{4x+5}{2x+3}=\frac{2x+\frac{5}{2}}{x+\frac{3}{2}}$ first (divide all terms by 2)

- **Transformations of** $f(x)=x^p; p=-2,-4,\ldots$

- $\dfrac{1}{x^p} \rightarrow \dfrac{a}{(n(x-b))^p}+c$

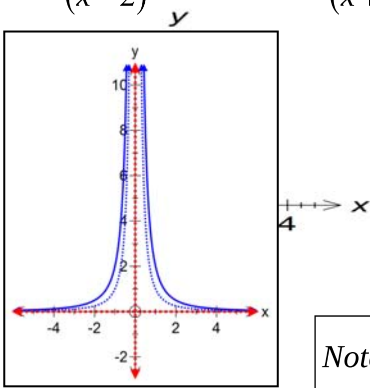
GeoGebra [9 Truncus Function Transformation.ggb]

- **Examples:** Sketch the graph of:
- *(i)* $f(x)=\dfrac{2}{x^2}$ *(ii)* $f(x)=\dfrac{1}{2x^4}$ *(iii)* $f(x)=\dfrac{4}{(x-2)^2}$ *(iv)* $f(x)=\dfrac{-3}{(x+2)^2}$ *(v)* $f(x)=\dfrac{3}{(x+2)^2}-3$

(i)

dilation of factor of 2 from the x-axis:

$if \quad g(x)=\dfrac{1}{x^2} \quad \rightarrow f(x)=2g(x)$



$Note: g(x)=\dfrac{1}{x^2}$

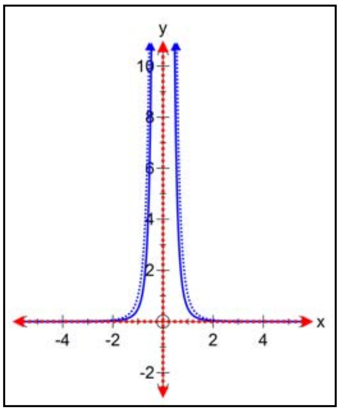
$\dfrac{1}{2}g(x)=\dfrac{1}{2x^2}; \quad dilation \ \dfrac{1}{2} \quad from \quad x-axis$

$g(2x)=\dfrac{1}{(2x)^2}; \quad dilation \ \dfrac{1}{2} \quad from \quad y-axis$

(ii)

dilation of factor $\dfrac{1}{2}$ from the x-axis:

$if \quad h(x)=\dfrac{1}{x^4} \quad \rightarrow f(x)=\dfrac{1}{2}h(x)$



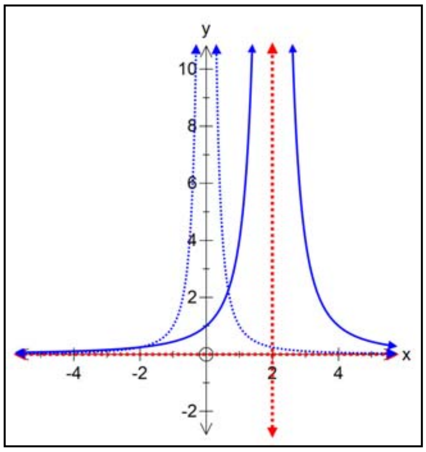
(iii)

dilation of factor 4 and a translation of 2 units to the right (positive direction of the x-axis):

Note: the vertical asymptote also moves 2 units to the right.

$x=0$

Y-intercept: $y=\dfrac{4}{(0-2)^2}=\dfrac{4}{(-2)^2}=\dfrac{4}{4}=1$



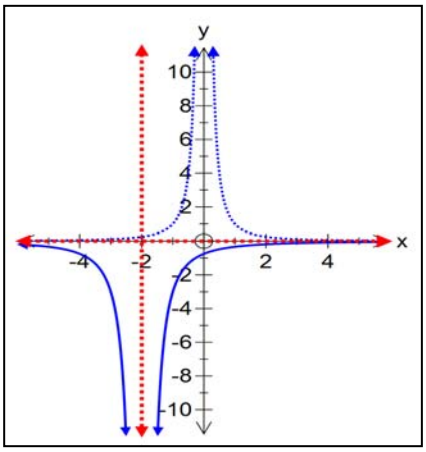
$$if \quad g(x)=\frac{1}{x^2} \quad \rightarrow f(x)=4g(x-2)$$

(iv)

dilation of factor 3, a reflection in the *x*-axis and a translation of 2 units to the left (negative direction of the *x*-axis):
Note: the vertical asymptote also moves 2 units to the left.

Y-Intercept: $y=\frac{-3}{(0+2)^2}=\frac{-3}{4}$

$$if \quad g(x)=\frac{1}{x^2} \quad \rightarrow f(x)=-3g(x+2)$$

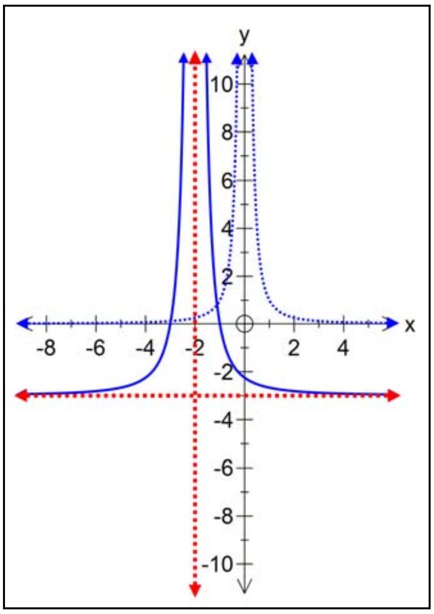


(v) dilation of factor 3, a translation of three units down (negative direction of the *y*-axis) and a translation of 2 units to the left (negative direction of the *x*-axis):

Intercepts:
$$\begin{aligned} &y=0 \\ &\frac{3}{(x+2)^2}-3=0 \\ &\frac{3}{(x+2)^2}=3 \\ &3=3(x+2)^2 \\ &1=(x+2)^2 \\ &\pm 1=x+2 \\ &x=-3,-1 \end{aligned}$$

$$\begin{aligned} &x=0 \\ &y=\frac{3}{(0+2)^2}-3 \\ &y=\frac{3}{4}-3 \\ &y=\frac{-9}{4} \end{aligned}$$

$$if \quad g(x)=\frac{1}{x^2} \quad \rightarrow f(x)=3g(x+2)-3$$



- **Ex3A** Q 3 bcij, 5cd, 7c, 8a; **Ex3B** Q 2, 5d, 6, 10 ae, 11 bde; **Ex3C** Q4 cd; **Ex3D** Q 1c, 4 efg, 6; **Ex3E** Q 1 b, 2 cd, 3a, 4bc; **Ex 3F** 1 cdg, 2h, 5c

Transformations of functions of the form $f(x) = x^{\frac{p}{q}}$

GeoGebra 5b Power Functions Transformation.ggb

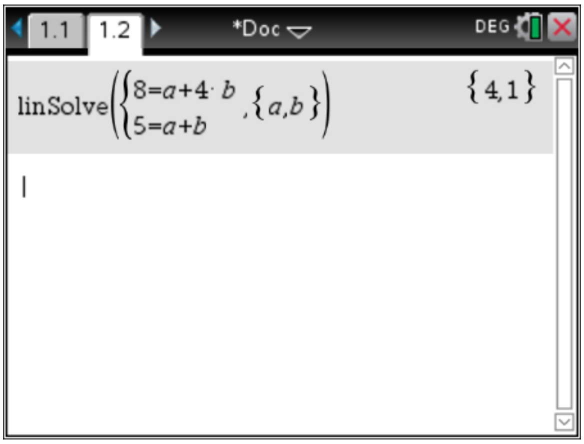
- $x^{\frac{p}{q}} \to a\left(n(x-b)\right)^{\frac{p}{q}} + c \quad OR \quad x^{\frac{p}{q}} \to a\sqrt[q]{\left(n(x-b)\right)^{p}} + c$
- **Ex3A**Q 6, 7e, 8c; **Ex3B** Q 3, 5c, 8, 9, 10 bcd, 11 cfg; **Ex 3C** Q 2a, 3, 4 befg; **Ex3D** Q 1b, 4b, 5, 7; **Ex3E** Q 1, 3 de, 4de ; **Ex3F** Q 2 abcef, 5 def

Determining rules for $f(x)=x^n$

Example: It is known that the points (1, 5) and (4, 2) lie on a curve with the equation $y=\frac{a}{x}+b$. Find the values of a and b.

Solution:

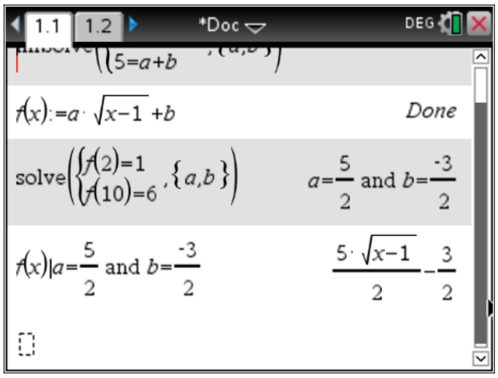
$$\text{When } x=1,\ y=5 \Rightarrow 5=a+b \quad (1)$$
$$\text{When } x=4,\ y=2 \Rightarrow 2=\frac{a}{4}+b \quad (2)$$
$$\text{subtract } (2) \text{ from } (1): \quad 3=\frac{3a}{4}$$
$$\therefore a=4$$
$$\text{substitute in } (1): 5=4+b$$
$$b=1$$
$$y=\frac{4}{x}+b$$



Example 2: It is known that the points (2, 1) and (10, 6) lie on a curve with equation $y=a\sqrt{x-1}+b$. Find the equation.

Solution:

$$(2,1): \quad 1=a\sqrt{2-1}+b$$
$$1=a+b \qquad (1)$$
$$(10,6): \quad 6=a\sqrt{10-1}+b$$
$$6=3a+b \qquad (2)$$
$$\text{Subtract } (1) \text{ from } (2): \quad 5=2a$$
$$\therefore \quad a=\frac{5}{2}$$
$$\text{substitute in } (1): \quad 1=\frac{5}{2}+b$$
$$b=-\frac{3}{2}$$
$$\Rightarrow y=\frac{5}{2}\sqrt{x-1}-\frac{3}{2} \quad or \quad y=\frac{5\sqrt{x-1}-3}{2}$$



- **Ex3H** Q 1, 2, 3, 4, 5, 6, 7, 8

Transformations using Matrices:

- (x', y') is called the image of (x, y).
- the transformations are written as follows:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

2×1 *matrix translations*

2×2 *matrix dilations/reflections*

- You can have more than one dilation/reflection matrix.
- Remember: multiply rows by columns, add/subtract elements in the same position.
- The transformation matrices are:

Reflection in the *x*-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$(x, y) \to (x, -y)$ $f(x) \to -f(x)$
Reflection in the *y*-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$(x, y) \to (-x, y)$ $f(x) \to f(-x)$
Reflection in the line *y*=*x*	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$(x, y) \to (y, x)$ $f(x) \to f(y)$
Dilation of factor *a* from the *x*-axis	$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$	$(x, y) \to (x, ay)$ $f(x) \to af(x)$
Dilation of factor *k* from the *y*-axis (note *n* = 1/*k*)	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	$(x, y) \to (kx, y)$ $f(x) \to f\left(\frac{x}{k}\right)$
Translation Matrix (add)	$\begin{bmatrix} b \\ c \end{bmatrix}$	$(x, y) \to (x+b, y+c)$ $f(x) \to f(x-b)+c$

Example 1: find the image of the point (2, 3) under:

a a reflection in the *x*-axis

b a dilation of factor 4 from the *y*-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$
$$\Rightarrow (2,3) \to (2,-3)$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$
$$\Rightarrow (2,3) \to (8,3)$$

Example 2: Consider a linear transformation such that (1, 0) → (3, -1) and (0, 1) → (-2, 4). Find the image of (-3, 5)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad and \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\Rightarrow a = 3, c = -1 \qquad b = -2, d = 4$$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -19 \\ 23 \end{bmatrix} \Rightarrow (-3,5) \to (19,-23)$$

Example 3: A transformation is defined by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Find the equation of the graph of $y = \sin(x) + x$, under this transformation.

Solution:

1. Write the dilations in terms of matrices	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
2. Multiply matrices	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2\times x + 0\times y \\ 0\times x + 3\times y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$
3. Determine the result in terms of *x'* and *y'* & rearrange to make *x* and *y* the subject of each equation.	$x' = 2x \qquad y' = 3y$ $\Rightarrow \frac{x'}{2} = x \qquad \frac{y'}{3} = y$
4. Sub each into the original equation.	$\frac{y'}{3} = \sin\left(\frac{x'}{2}\right) + \frac{x'}{2}$
5. Rearrange to make *y'* the subject	$y' = 3\sin\left(\frac{x'}{2}\right) + \frac{3x'}{2}$
6. Then drop the '	$y = 3\sin\left(\frac{x}{2}\right) + \frac{3x}{2}$

Example 4: A transformation is described by the matrix equation **A(X + B) = X′**, where

$$A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find the image of the straight line with equation $y = 2x + 5$ under this transformation.

A(X + B) = X′

$$\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x+1 \\ y+2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} -3(y+2) \\ 2(x+1) \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \Rightarrow x' = -3(y+2) \quad \& \quad y' = 2(x+1)$$

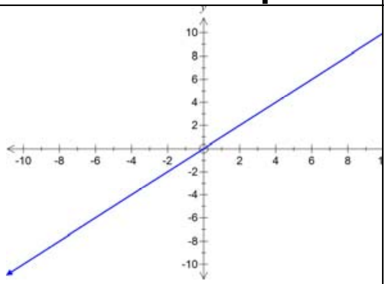
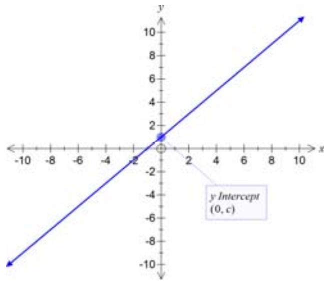
$$y = \frac{-x'}{3} - 2 \quad \& \quad x = \frac{y'}{2} - 1$$

Sub. into $y = 2x + 5$...

$$\frac{-x'}{3} - 2 = 2\left(\frac{y'}{2} - 1\right) + 5$$

$$y' = \frac{-x'}{3} - 5$$

Ex3I Q 1, 3, 4, 6, 7, 8, 9, 10, 11, 13, 15, 16

<table>
<tr><th colspan="4">Transformations of standard graphs</th></tr>
<tr><th>$f(x)$</th><th>Parent Graph</th><th>$af(n(x-b))+c$</th><th>Example</th></tr>
<tr><td>x</td><td></td><td>$ax+c$</td><td>
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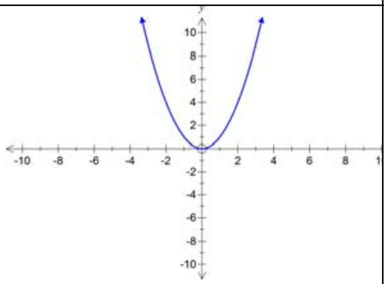
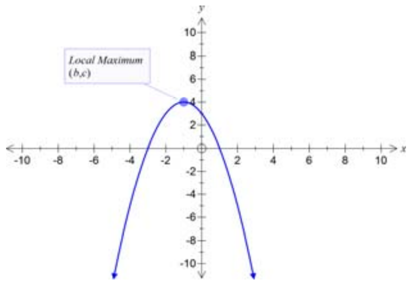
<tr><td>x^2</td><td></td><td>$a(n(x-b))^2+c$</td><td>
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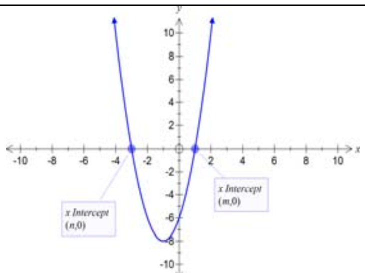
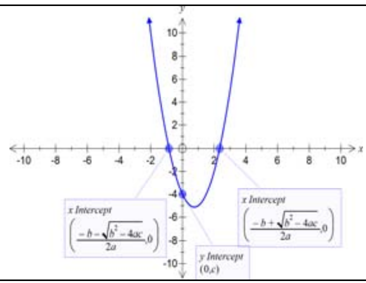
<tr><td>$a(x-m)(x-n)$</td><td>
</td><td>ax^2+bx+c</td><td>
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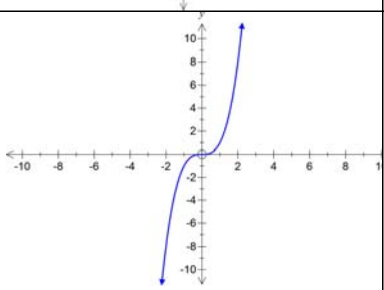
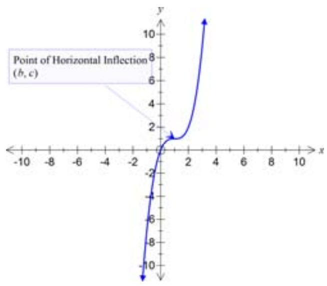
<tr><td>x^3</td><td></td><td>$a(n(x-b))^3+c$</td><td>
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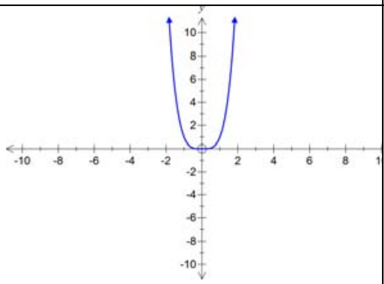
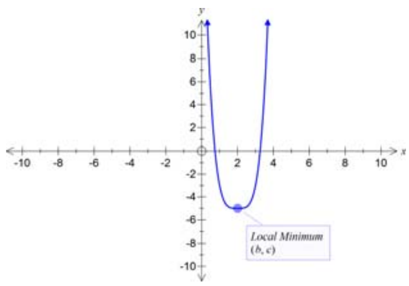
<tr><td>x^4</td><td></td><td>$a(n(x-b))^4+c$</td><td>
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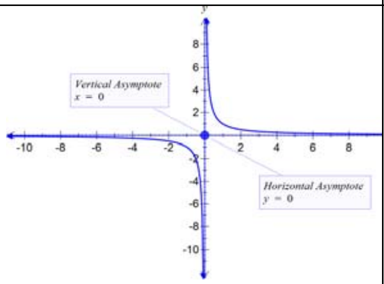
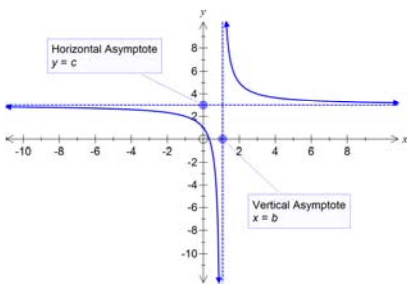
<tr><td>$\frac{1}{x}$ or x^{-1}</td><td>
</td><td>$\frac{a}{n(x-b)}+c$</td><td>
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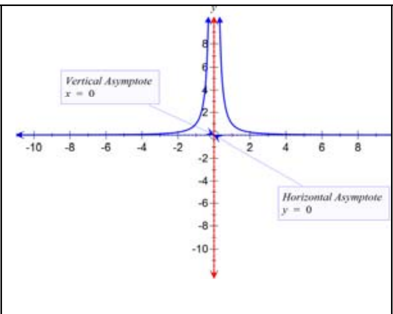
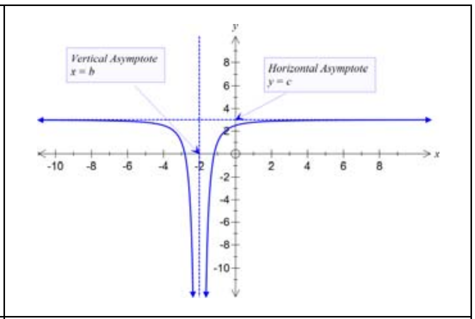
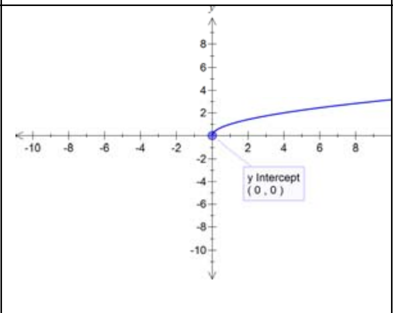
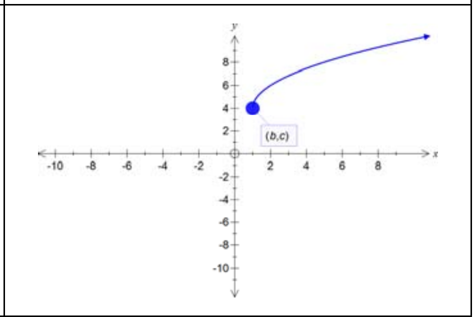
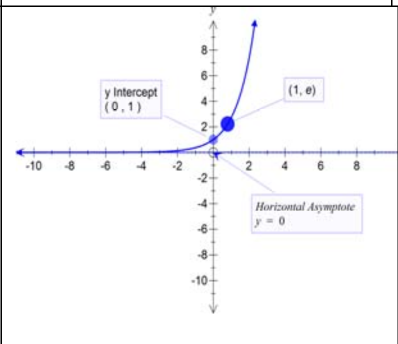
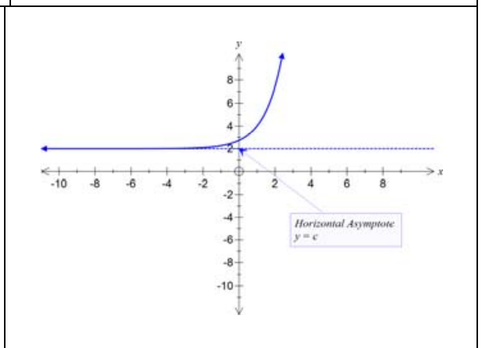
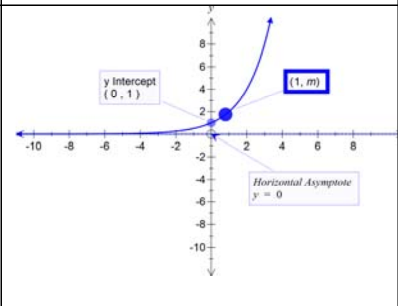
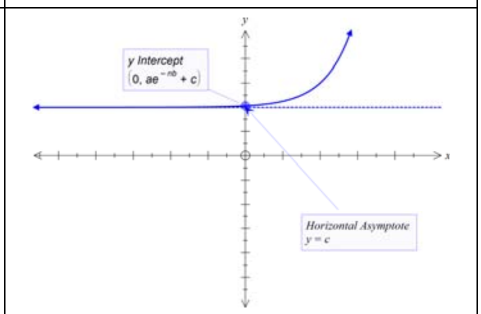
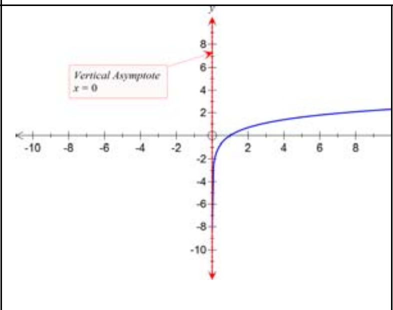
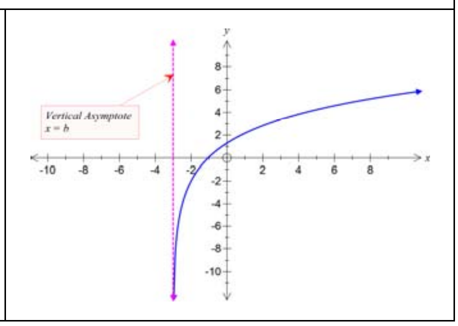
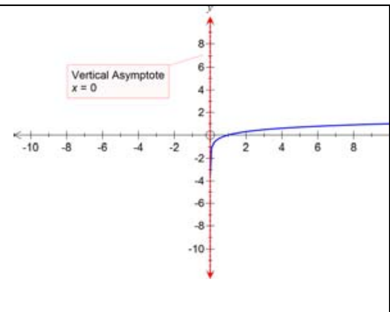
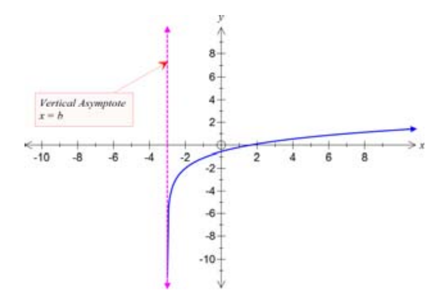
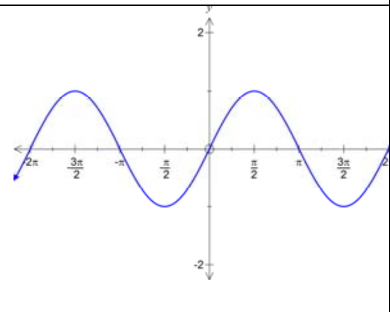
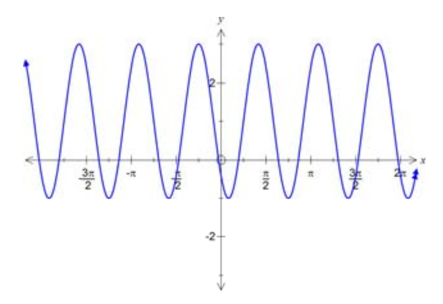
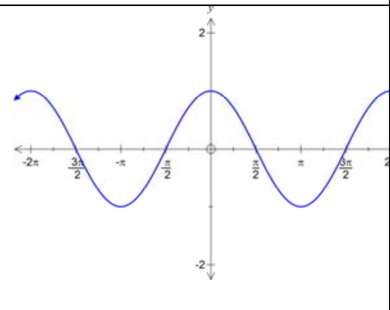
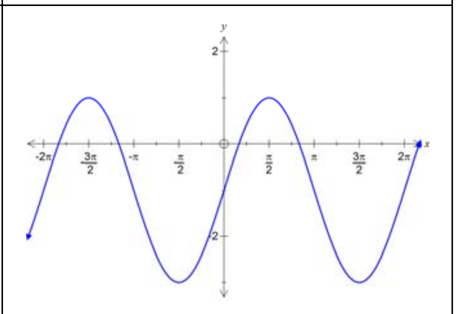
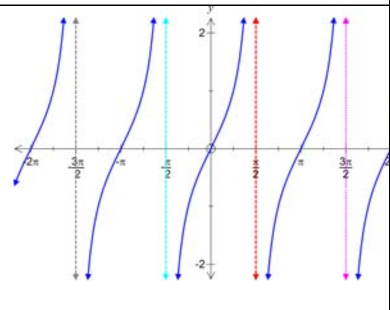
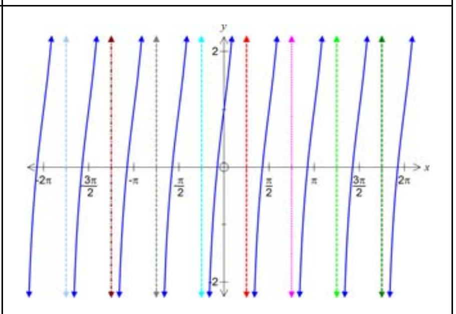
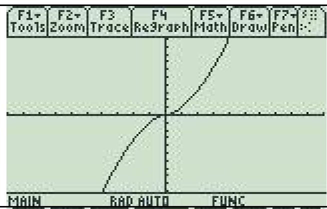
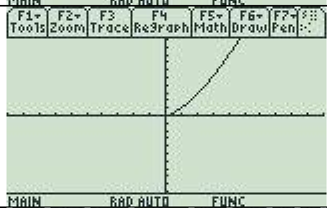
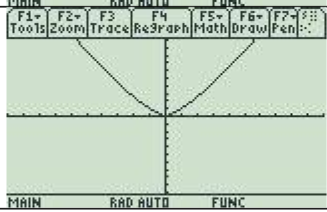
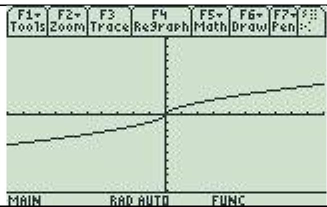
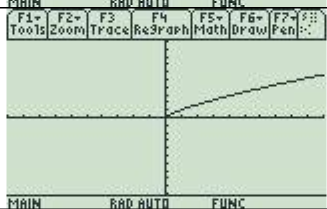
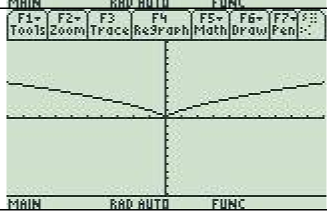
$\frac{1}{x^2}$ or x^{-2}		$\frac{a}{(n(x-b))^2}+c$	
$x^{\frac{1}{2}}$ or $\sqrt{x}$		$a\sqrt{n(x-b)}+c$	
$x^{\frac{p}{q}}$ or $\sqrt[q]{x^p}$	See below	$a\sqrt[q]{n(x-b)^p}+c$	See below
e^x		$ae^{n(x-b)}+c$	
m^x		$am^{(n(x-b))}+c$	
$\log_e x$		$a\log_e(n(x-b))+c$	

$\log_{10} x$		$a\log_{10}(n(x-b))+c$	
$\sin x$		$a\sin(n(x-b))+c$	
$\cos x$		$a\cos(n(x-b))+c$	
$\tan x$		$a\tan(n(x-b))+c$	

$$y = x^{\frac{p}{q}}$$

	p	q	**Domain**	**Example graph**	**Equations**
$p > q$ $\frac{p}{q} > 1$	p odd	q odd	R		$y = x^3$ $y = x^{\frac{5}{3}}$
	p odd	q even	$x \geq 0$		$y = x^{\frac{3}{2}}$
	p even	q odd	R		$y = x^2$ $y = x^{\frac{4}{3}}$
$p < q$ $\frac{p}{q} < 1$	p odd	q odd	R		$y = x^{\frac{1}{3}}$ $y = x^{\frac{3}{5}}$
	p odd	q even	$x \geq 0$		$y = x^{\frac{1}{2}}$ $y = x^{\frac{3}{4}}$
	p even	q odd	R		$y = x^{\frac{2}{3}}$

VCAA EXAM QUESTIONS for TRANSFORMATIONS

2008

Question 8

The graph of the function $f: D \rightarrow R,\ f(x) = \dfrac{x-3}{2-x}$, where D is the maximal domain has asymptotes

A. $x = 3, \quad y = 2$

B. $x = -2, \quad y = 1$

C. $x = 1, \quad y = -1$

D. $x = 2, \quad y = -1$

E. $x = 2, \quad y = 1$

Q8 $f(x) = \dfrac{x-3}{2-x} = -\left(\dfrac{x-3}{x-2}\right) = -\left(\dfrac{x-2-1}{x-2}\right) = -\left(1 - \dfrac{1}{x-2}\right)$

$= \dfrac{1}{x-2} - 1$. Asymptotes: $x = 2$, $y = -1$. D

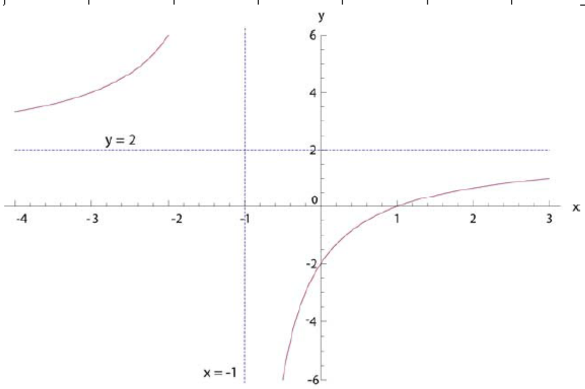
Question 2

On the axes below, sketch the graph of $f: R\backslash\{-1\} \rightarrow R, \quad f(x) = 2 - \dfrac{4}{x+1}$.

Label all axis intercepts. Label each asymptote with its equation.

Question 2

Marks	0	1	2	3	4	Average
%	13	8	7	15	57	3.1



This question was generally well done with most students recognising a hyperbola. Students should label asymptotes and/or axial intercepts. In this question students sometimes gave the wrong intercept ($x = 3$ was common), extra intercepts or no horizontal asymptote.

2009

Question 2

At the point (1, 1) on the graph of the function with rule $y = (x-1)^3 + 1$

A. there is a local maximum.

B. there is a local minimum.

C. there is a stationary point of inflection.

D. the gradient is not defined.

E. there is a point of discontinuity.

Q2 C

Question 21

A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at (2, 0) and (–3, 0). The maximum value of the derivative function is 10.

The value of x for which the graph of $y = f(x)$ has a local maximum is

A. -2

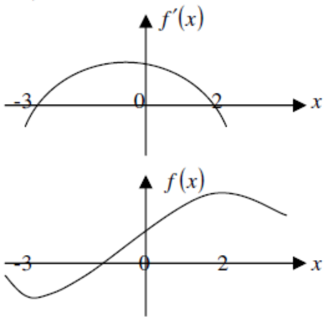
B. 2

C. -3

D. 3

E. $-\dfrac{1}{2}$

Q21 $f'(x) = a(x-2)(x+3)$ is a quadratic function. For it to have a maximum value, $a < 0$.



B

2011

Question 3

a. Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9$.

i. Find $f'(x)$

ii. Explain why $f'(x) \geq 5$ for all x.

$1 + 1 = 2$ marks

b. The cubic function p is defined by $p: R \to R, p(x) = ax^3 + bx^2 + cx + k$, where a, b, c and k are real numbers.

i. If p has m stationary points, what possible values can m have?

ii. If p has an inverse function, what possible values can m have?

$1 + 1 = 2$ marks

c. The cubic function q is defined by $q: R \to R, q(x) = 3 - 2x^3$.

i. Write down an expression for $q^{-1}(x)$.

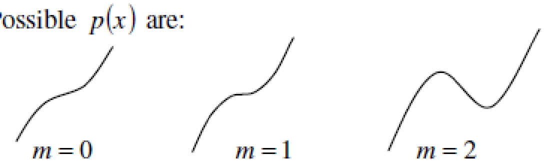
ii. Determine the coordinates of the point(s) of interse

$2 + 2 = 4$ marks

Q3ai $f(x) = 4x^3 + 5x - 9$, $f'(x) = 12x^2 + 5$

Q3aii Since $x^2 \geq 0$ for all x, .: $12x^2 \geq 0$, .: $12x^2 + 5 \geq 0 + 5$, .: $f'(x) \geq 5$ for all x.

Q3bi Possible $p(x)$ are:



Q3bii The first two cases ($m = 0$ and $m = 1$) above are one-to-one functions, .: each has an inverse function.

Q3ci Let $q(x) = 3 - 2x^3 = y$, the equation of the inverse is

$$3 - 2y^3 = x, \therefore y^3 = \frac{3-x}{2}, \quad y = \left(\frac{3-x}{2}\right)^{\frac{1}{3}}, \therefore q^{-1}(x) = \left(\frac{3-x}{2}\right)^{\frac{1}{3}}$$

Q3cii At the intersection $y = x$, .: $x = 3 - 2x^3$, .: $x = 1$ and $y = 1$. The point of intersection is $(1,1)$.

d. The cubic function g is defined by $g: R \to R, g(x) = x^3 + 2x^2 + cx + k$, where c and k are real numbers.

i. If g has exactly one stationary point, find the value of c.

Q3di $g(x) = x^3 + 2x^2 + cx + k$, $g'(x) = 3x^2 + 4x + c$

For $g'(x) = 0$ at exactly one point, $\Delta = 4^2 - 4(3)c = 0$, .: $c = \frac{4}{3}$

Q3dii $g(x) = x^3 + 2x^2 + \frac{4}{3}x + k$. Let $g'(x) = 3x^2 + 4x + \frac{4}{3} = 0$,

.: the stationary is at $x = -\frac{2}{3}$.

At the intersection of $y = g(x)$ and $y = g^{-1}(x)$, $y = x$,

.: $x^3 + 2x^2 + \frac{4}{3}x + k = x$,

$\left(-\frac{2}{3}\right)^3 + 2\left(-\frac{2}{3}\right)^2 + \frac{4}{3}\left(-\frac{2}{3}\right) + k = -\frac{2}{3}$, .: $k = -\frac{10}{27}$

ii. If this stationary point occurs at a ... value of k.

3 + 3 = 6 marks

Total 14 marks

2012

Question 8

The function $f: R \to R, f(x) = ax^3 + bx^2 + cx$, where a is a negative real number and b and c are real numbers.

For the real numbers $p < m < 0 < n < q$, we have $f(p) = f(q) = 0$ and $f'(m) = f'(n) = 0$.

The gradient of the graph of $y = f(x)$ is negative for

A. $(-\infty, m) \cup (n, \infty)$

B. (m, n)

C. $(p, 0) \cup (q, \infty)$

D. $(p, m) \cup (0, q)$

E. (p, q)

Question	% A	% B	% C	% D	% E	% No answer	Comments
8	49	14	18	14	4	1	The gradient of the graph of $y = f(x)$ is negative for $(-\infty, m) \cup (n, \infty)$. There is a local minimum at $x = m$ and a local maximum at $x = n$. Eighteen per cent of students chose option C, $(p, 0) \cup (q, \infty)$. This is when the graph of $y = f(x)$ is negative.

Question 16

The graph of a cubic function f has a local maximum at $(a, -3)$ and a local minimum at $(b, -8)$.

The values of c, such that the equation $f(x) + c = 0$ has exactly one solution, are

A. $3 < c < 8$

B. $c > -3$ or $c < -8$

C. $-8 < c < -3$

D. $c < 3$ or $c > 8$

E. $c < -8$

16	13	17	27	34	8	1	The local maximum will be touching the x-axis when $c = 3$, giving two distinct solutions. So if $c < 3$ there will be one solution. The local minimum will be above the x-axis when $c > 8$. Hence $c < 3$ or $c > 8$.

2014

Question 1

The point P (4, –3) lies on the graph of a function f. The graph of f is translated four units vertically up and then reflected in the y-axis.

The coordinates of the final image of P are

A. (–4, 1)

B. (–4, 3)

C. (0, –3)

D. (4, –6)

E. (–4, –1)

Q1 $(4, -3) \to (4, 1) \to (-4, 1)$ A

Question 12

The transformation $T: R^2 \to R^2$ with rule

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

maps the line with equation $x - 2y = 3$ onto the line with equation

A. $x + y = 0$

B. $x + 4y = 0$

C. $-x - y = 4$

D. $x + 4y = -6$

E. $x - 2y = 1$

Q12 $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x+1 \\ 2y-2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $\therefore$ $x = 1 - x'$ and $y = \dfrac{y'+2}{2}$

$\therefore$ $x - 2y = 3 \to -x' - y' = 4$ C

Question 5 (13 marks)

Let $f: R \to R, f(x) = (x-3)(x-1)\left(x^2+3\right)$ and $g: R \to R, g(x) = x^4 - 8x$.

a. Express $x^4 - 8x$ in the form $x(x-a)\left((x+b)^2 + c\right)$. 2 marks

b. Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$. 1 mark

c. Find the values of d such that the graph of $y = f(x+d)$ has

i. one positive x-axis intercept

ii. two positive x-axis intercepts.

d. Find the value of n for which the equation $g(x) = n$ has one solution.

Q5a $g(x) = x^4 - 8x = x\left(x^3 - 2^3\right) = x(x-2)\left(x^2 + 2x + 4\right)$

$= x(x-2)\left(x^2 + 2x + 1 - 1 + 4\right) = x(x-2)\left((x+1)^2 + 3\right)$

Q5b Translate the graph of $y = f(x)$ to the left by 1.

$g(x) = f(x+1) = (x+1-3)(x+1-1)\left((x+1)^2 + 3\right)$

$= x(x-2)\left((x+1)^2 + 3\right)$

Q5ci $y = f(x+d) = (x+d-3)(x+d-1)\left((x+d)^2 + 3\right)$

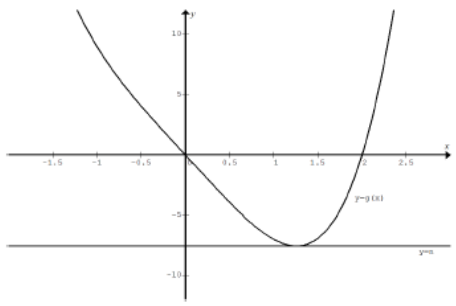
For one positive x-intercept, $d - 1 \geq 0$ and $d - 3 < 0$

$\therefore$ $d \geq 1$ and $d < 3$, i.e. $1 \leq d < 3$

Q5cii For two positive x-intercepts, $d - 1 < 0$ and $d - 3 < 0$,

$\therefore$ $d < 1$ and $d < 3$, $\therefore$ $d < 1$

Q5d



The graphs of $y = g(x) = x^4 - 8x$ and $y = n$ intersect at one point when $y = n$ is a tangent to $y = g(x)$ at its minimum turning point.

$g'(x) = 4x^3 - 8$

Let $g'(x) = 0$, $\therefore$ $x = 2^{\frac{1}{3}}$, $n = g\left(2^{\frac{1}{3}}\right) = -6 \times 2^{\frac{1}{3}}$

e. At the point $(u, g(u))$, the gradient of $y = g(x)$ is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i. Find the value of $u^3 + v^3$.

ii. Find u and v if $u + v = 1$.

Q5ei At $(u, g(u))$, $g'(u) = m$, .: $4u^3 - 8 = m$ where $m \in R^+$

At $(v, g(v))$, $g'(v) = -m$, .: $4v^3 - 8 = -m$

.: $4u^3 + 4v^3 - 16 = 0$, .: $u^3 + v^3 = 4$

Q5eii Solve $u^3 + v^3 = 4$ and $u + v = 1$ simultaneously by CAS.
Given $g'(u) = m$ is positive, and $g'(v) = -m$ is negative, .: $u > v$

Hence $u = \frac{1+\sqrt{5}}{2}$ and $v = \frac{1-\sqrt{5}}{2}$.

f. **i.** Find the equation of the tangent to the graph of $y = g(x)$ at the point $(p, g(p))$. 1 mark

ii. Find the equations of the tangents to the graph of $y = g(x)$ that pass through the point with coordinates $\left(\frac{3}{2}, -12\right)$. 3 marks

Q5fi $g(p) = p^4 - 8p$, $g'(p) = 4p^3 - 8$

Equation of the tangent: $y - g(p) = g'(p)(x - p)$

$y - (p^4 - 8p) = (4p^3 - 8)(x - p)$

.: $y = (4p^3 - 8)x - 3p^4$

Q5fii Equation of the tangents: $y = (4p^3 - 8)x - 3p^4$

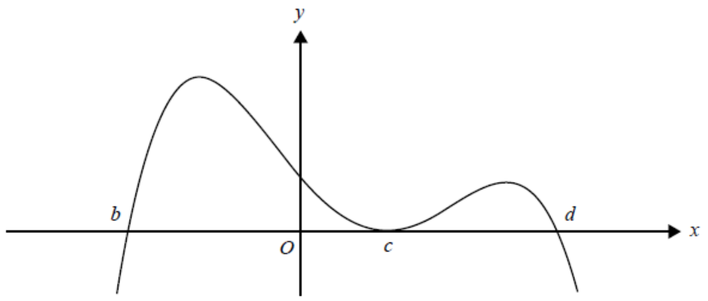
The tangents pass through $\left(\frac{3}{2}, -12\right)$, .: $-12 = (4p^3 - 8)\frac{3}{2} - 3p^4$

.: $3p^3(p - 2) = 0$, .: $p = 0$ or $p = 2$

The tangents are: $y = -8x$ and $y = 24x - 48$

2015

Question 3



The rule for a function with the graph above could be

A. $y = -2(x + b)(x - c)^2(x - d)$

B. $y = 2(x + b)(x - c)^2(x - d)$

C. $y = -2(x - b)(x - c)^2(x - d)$

D. $y = 2(x - b)(x - c)(x - d)$

E. $y = -2(x - b)(x + c)^2(x + d)$

3	61	14	20	2	4	0	The rule for the graph is in the form $f(x) = a(x-b)(x-c)^2(x-d)$, where a is negative and could be -2. $f(x) = -2(x-b)(x-c)^2(x-d)$ b is negative; for example if $b = -2$, the factor is $(x-(-2)) = (x+2)$. Most students chose option A, $y = -2(x+b)(x-c)^2(x-d)$, but the factor $(x+b)$ is incorrect.

Question 11

The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A. dilation by a factor of 2 from the y-axis.

B. dilation by a factor of 2 from the x-axis.

C. dilation by a factor of $\frac{1}{2}$ from the x-axis.

D. dilation by a factor of 8 from the y-axis.

E. dilation by a factor of $\frac{1}{2}$ from the y-axis.

11	24	7	7	32	29	0	$y_1 = \sqrt{8x^3+1}$, $y_2 = \sqrt{8\left(\frac{x}{2}\right)^3 + 1} = \sqrt{x^3+1}$ The graph of y_1 has been dilated by a factor of 2 from the y-axis to get the graph of y_2. This can be shown by sketching the graphs of both functions. For example, the point with coordinates (1, 3) is transformed to (2, 3).

Question 17

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts.

The set of all possible values of c is

A. R

B. R^+

C. $\{0, 4\}$

D. $(0, 4)$

E. $(-\infty, 4)$

D

Question 20

If $f(x - 1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A. $x^2 - 2$

B. $x^2 + 2$

C. $x^2 - 2x + 2$

D. $x^2 - 2x + 4$

E. $x^2 - 4x + 6$

B